L2

4800

Karatsuba, Recurrences

Sep 13 2016

shelat

warmup

Simplify $(1 + a + a^2 + \cdots + a^L)(a-1) =$

$$\begin{array}{l} a + a^2 + a^3 + \cdots + a^L + a^{L+1} \\ -1 - a - a^2 - a^3 - \cdots - a^L \\ \hline a^{L+1} - 1 \end{array}$$

$$\boxed{1 + a + a^2 + \cdots a^L = \frac{a^{L+1} - 1}{a - 1}}$$

when $a \neq 1$

warmup

$$\sum_{i=0}^{L} a^i = \frac{a^{L+1} - 1}{a - 1}$$

1. stand
2. set your “number” to one
3. greet a neighbor (pause if odd person out)
4. if you are older, give your “number” to young and sit
   if you are younger, add “numbers”
5. if you are standing & you have a neighbor, goto 3

1	2	3	4	5
stand	set	greet	sit/add	repeat

how fast does it work:

$$T(n)$$ steps to finish in a room of size n

1 2

stand set

3 4 5

greet sit/add repeat

how fast does it work:

$$T(n) = 1 + 1 + T(\lceil n/2 \rceil)$$

$$T(1) = 3$$

recurrence?

$$T(n) = T(\lceil n/2 \rceil) + 2$$

$$T(1) = 3$$

solve a simpler case when n is a power of 2.

$$T(2^k) = 2 + T(2^{k-1})$$

$$
\begin{aligned}
T(2^k) &= 2 + T(2^{k-1}) \\
&= 2 + 2 + T(2^{k-2}) \\
&= \overbrace{2 + 2 + \cdots + 2}^{k} + T(2^0) \\
&= 2k + T(1)
\end{aligned}
$$

Asymptotic notation

set of functions

$O(g)$ at most within const of g for large n

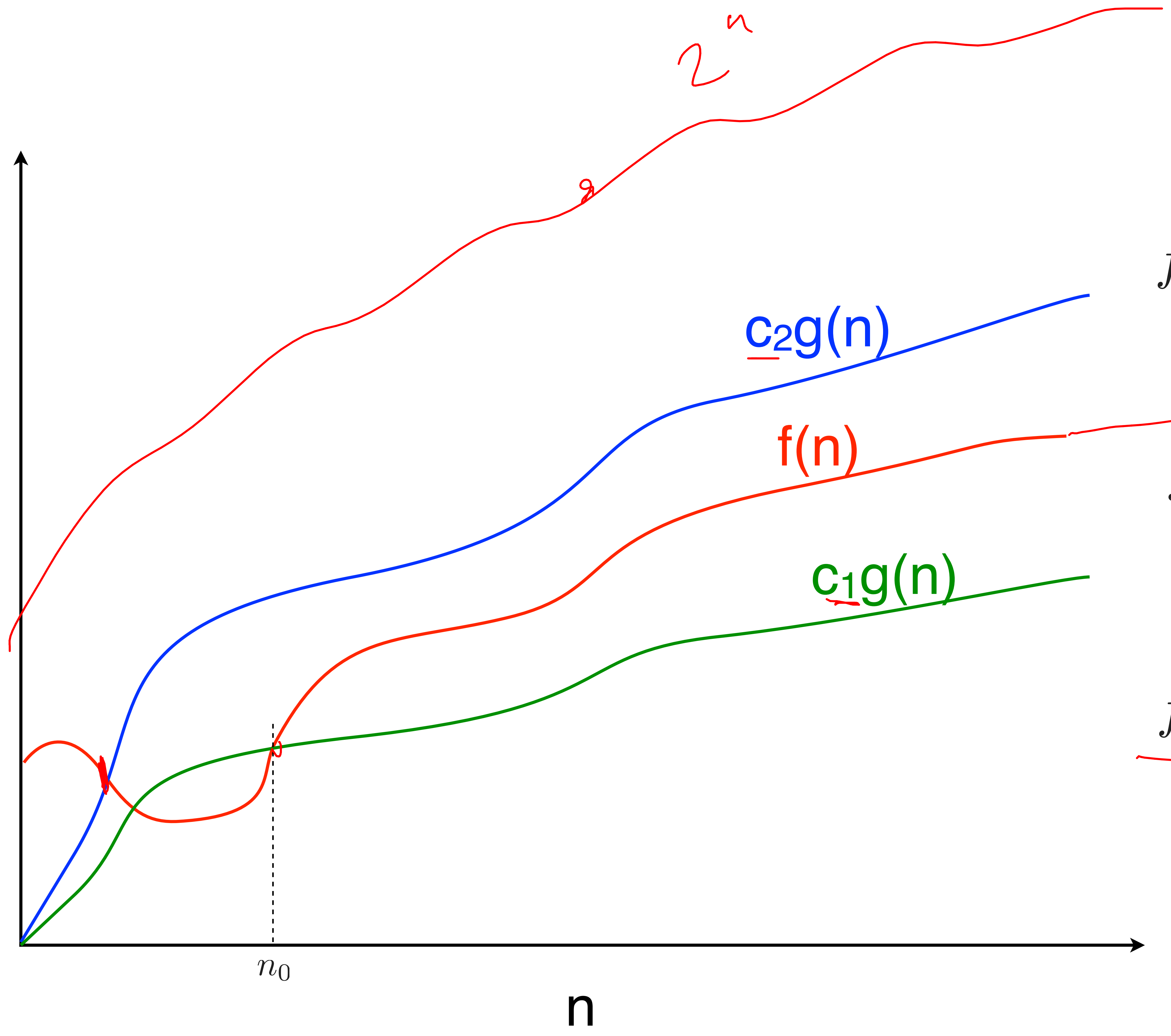
$= \{ f \mid \exists$ constants $c, n_0 > 0$ such that

$\forall$ $n > n_0$ $|c \cdot g(n)| > |f(n)|$ $\}$

"forall"

Asymptotic notation

$O(g)$	at most within const of g	for large n
$\Omega(g)$	at least within const of g "lower bound"	for large n
$\Theta(g)$	within a const of g "tight bound"	for large n



$$f(n) = O(g(n))$$

→ running time of some alg.

$$f(n) = \Theta(g(n))$$

$$f(n) = \Omega(g(n))$$

"intuition here"

$$T(2^k) = 2 + T(2^{k-1})$$

$$= 2 + 2 + T(2^{k-2})$$

$$= \overbrace{2 + 2 + \cdots + 2}^{k} + T(2^0)$$

$$= 2k + T(1) = O(\log(2^k)) = \Theta(\log(n)) \text{ for powers of } 2$$

$T(n) \leq T(m)$ if $n \leq m$. (Show)

$T(n) \leq T(2^{\lceil \log n \rceil}) \Rightarrow T(n) = O(\lceil \log n \rceil)$

Similar arguments to show $\Omega(\log(n)) \Rightarrow \Theta(\log n)$

“intuition here”

$$T(2^k) = 2 + T(2^{k-1})$$

$$= 2 + 2 + T(2^{k-2})$$

$$= \overbrace{2 + 2 + \cdots + 2}^{k} + T(2^0)$$

$$= 2k + T(1) \; = O(\log(2^k))$$

$$\forall 0 < n < m, T(n) \leq T(m)$$

$$T(m) \leq T(2^{\lceil \log(m) \rceil}) = 2\lceil \log(m) \rceil + 2$$

$$T(m) = \Omega(\log(m))$$

$$= \Theta(\log(m))$$

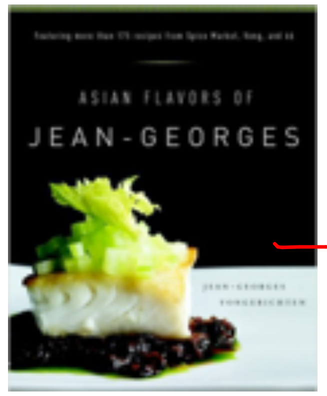
main ideas:

1. Alg, solved it by dividing problem in half each step
2. Analyzed running time using a recurrence
3. Asymptotic notation to simplify

How to solve recurrence relations

 - tree method

?-✓ - guess & check (INDUCTION)



→ cookbook (masters thm)

http://www.drblank.com/law301.jpg

→ substitution method

Multiplication

1

1 7 8 9 → n digit

× 1 4 3 2 → n digit

n rows $\quad \underline{\underline{\Theta(n^2)}}$

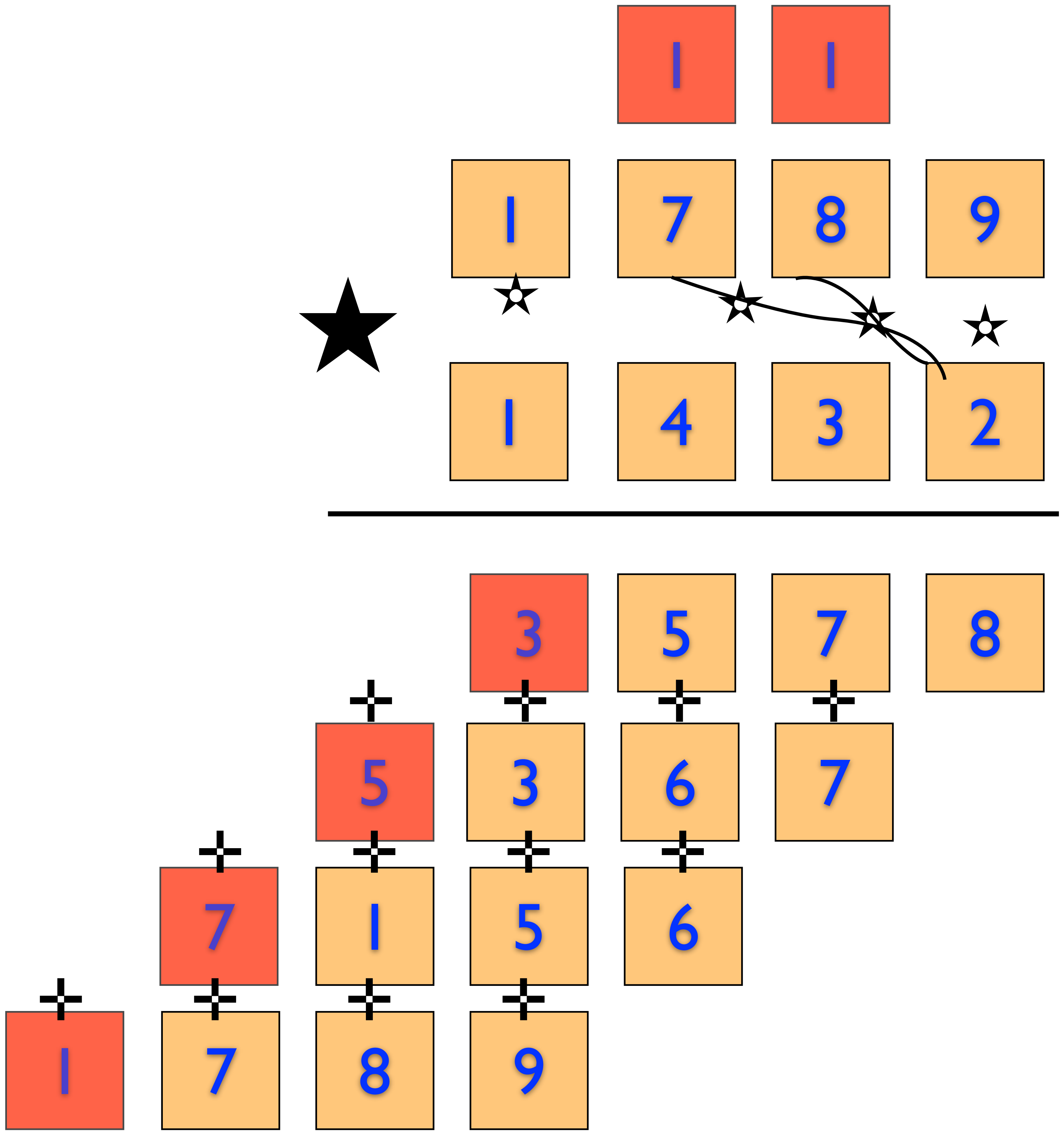
2 5 7 8 → n mult $\quad$ n additions

5 3 6 7 → n mult $\quad$ "

7 1 5 6 → n mult $\quad$ "

1 7 8 9 → n " $\quad$ "

2 5 6 0 8 4 8 $\quad$ 2n several additions



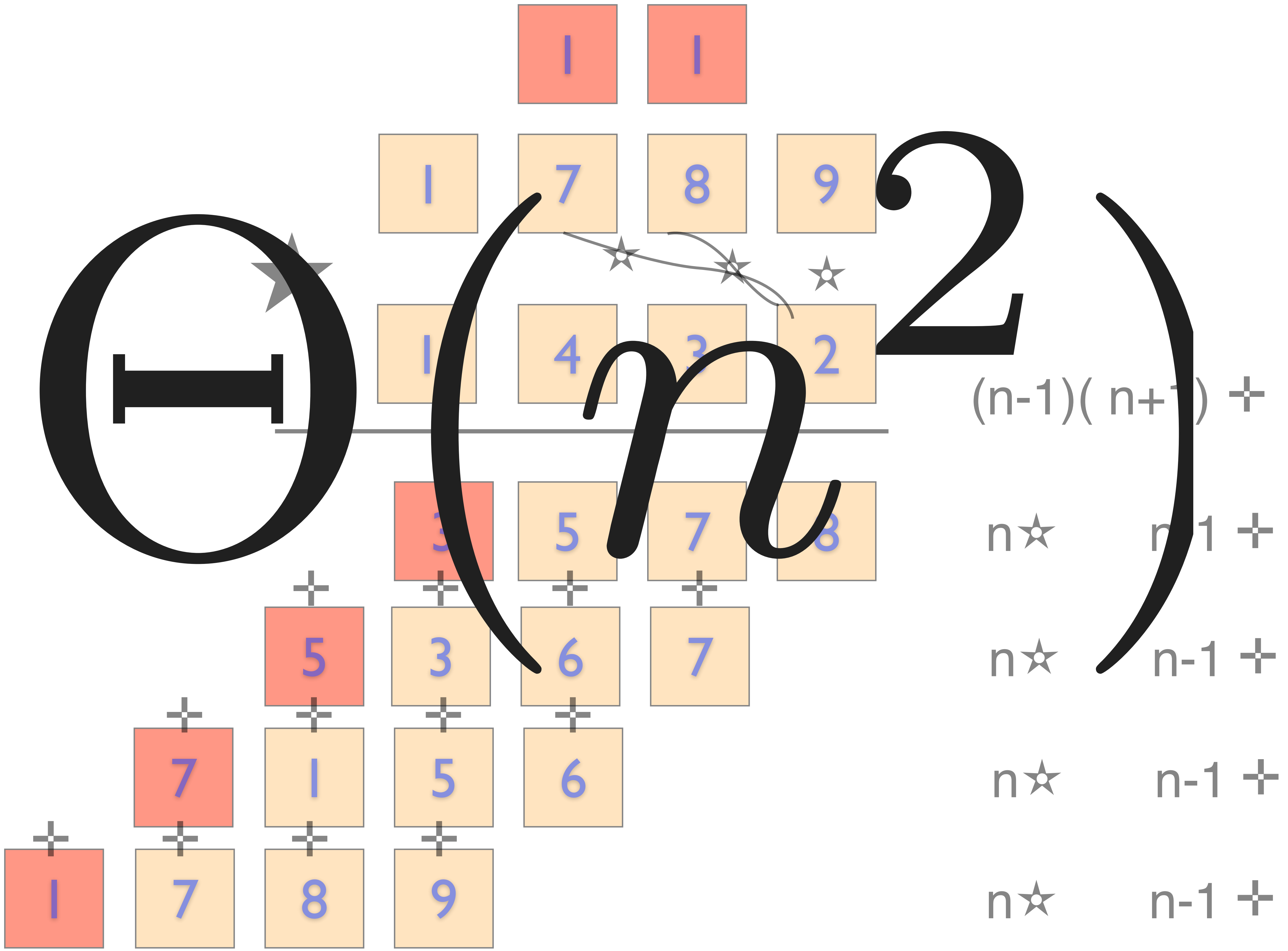
(n-1)(n+1) ✛

n✫ n-1 ✛

n✫ n-1 ✛

n✫ n-1 ✛

n✫ n-1 ✛



Θ(n²)
(n-1)(n+1) ✢
n☆ n-1 ✢
n☆ n-1 ✢
n☆ n-1 ✢
n☆ n-1 ✢

Theme 1

can we do better?

Yes

how to break n-digit mult into a

smaller problem

$$\big(\underbrace{17}_{a}\cdot 100 + \underbrace{89}_{b}\big) \ast \big(\underbrace{14}_{c}\cdot 100 + \underbrace{32}_{d}\big)$$

$\underbrace{1\ 7}_{a}\ \underbrace{8\ 9}_{b}$ ★ $1\ 4\ \ 3\ 2$

n/2 digit

$$(a\cdot c)(100^2) + (a\cdot d + b\cdot c)\cdot 100 + bd$$

1	7	8	9	★	1	4	3	2

a	b		c	d

n/2

a | c

ab+bc

b · d

$$ac100^2 + (ad + bc)100 + bd$$

n/2 digit mult

n/2

n/2

$4 \ \frac{n}{2}$ digit mults

and 3 n digit additions

n-digit inputs

Mult(ab, cd)

1 7 8 9 1 4 3 2

a b c d

$$ac100^2 + (ad + bc)100 + bd$$

Base case: return b*d if inputs are 1-digit

Else:

Mult(ab, cd)

Base case: return b*d if inputs are 1-digit

Else: Compute x = Mult(a,c)
Compute y = Mult(a,d)
Compute z = Mult(b,c)
Compute w = Mult(b,d)

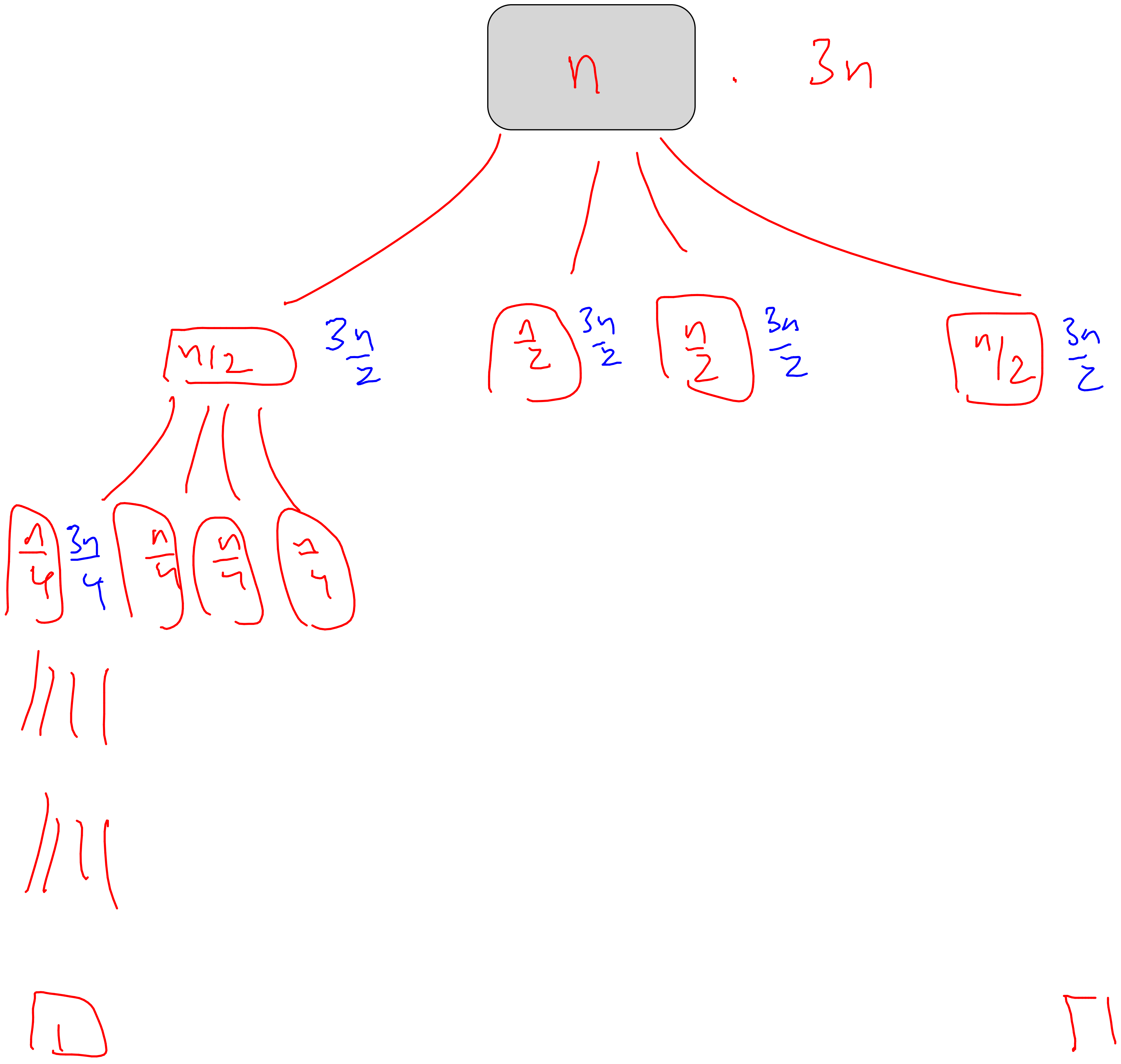
$4T(\frac{n}{2})$

Return r = x*100^2 + (y+z)100 + w

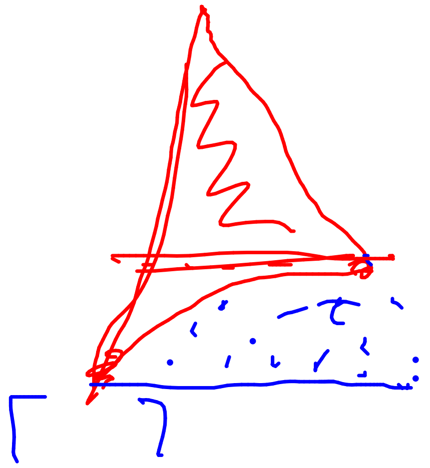
$3n$

$T(n) = 4T(\frac{n}{2}) + 3n$

$$T(n) = 4T(n/2) + 3O(n)$$



0	$3n$
1	$4 \cdot \frac{3n}{2} = 2^1 \cdot 3n$
2	$16 \cdot \frac{3n}{4} = 2^2 \cdot 3n$
3	$64 \cdot \frac{3n}{8} = 2^3 \cdot 3n$
$\vdots$	
$\lceil \log n \rceil$	$= 2^{\lceil \log n \rceil} \cdot 3n$



calculations:

$$T(n) = 3n + 3n\cdot 2 + 3n\cdot 2^2 + \dots 3n\cdot 2^{\lceil \log n \rceil}$$

$$1 + a + \dots + a^L = \frac{a^{L+1} - 1}{a - 1}$$

$$= 3n\left(\underbrace{1 + 2 + 2^2 + \dots + 2^{\lceil \log n \rceil}}\right)$$

$$= 3n\left[\frac{2^{\lceil \log n \rceil + 1} - 1}{2 - 1}\right] = 3n\left[2\cdot \underbrace{2^{\lceil \log n \rceil}}_{n} - 1\right]$$

$$= 3n\left[2\cdot n - 1\right]$$

$$= 6n^2 - 3n = O(n^2)$$

by similar arguments $\Omega(n^2)$

$$\Rightarrow \Theta(n^2)$$

$$\boxed{2^{\log_2 n} = n}$$

$$10^{\log_{10} n} = n$$

$\log_2 n = x$ such that $2^x = n$

$$\log(2^x) = n$$

$$\Rightarrow n = x$$

$$\underbrace{\log_{10} n}_{} = x \quad \text{s.t} \quad \underline{10^x = n}$$

$$2^y = 10 \qquad \underline{y = \log_2 10}$$

$$\log_{10}\left(2^{\log_2 10}\right)^x = \log_{10}(n)$$

$$\log_{10}(n) = x \cdot \log_{10} 2^{\log_2 10} = \underline{x \cdot \log_2 10 \cdot \log_{10} 2}$$

Karatsuba

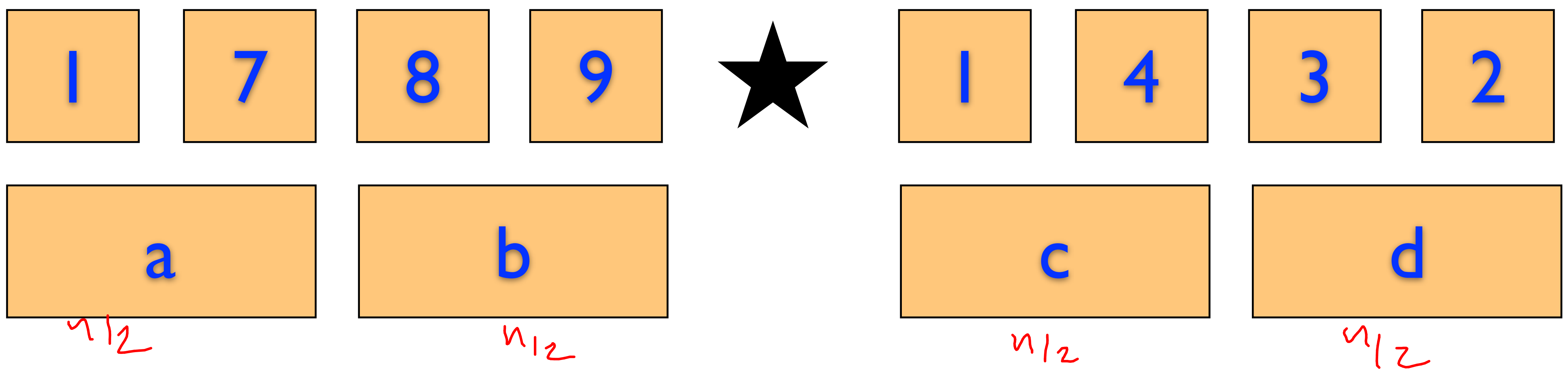
1	7	8	9	★	1	4	3	2
a		b			c		d	

$$ac100^2 + (ad + bc)100 + bd$$

$$(a+b)(c+d) = ac + ad + bc + bd$$

$$ad + bc = (a+b)(c+d) - ac - bd$$

Karatsuba algorithm



Recursively compute

❶ $ac, bd, (a+b)(c+d)$ $3\,T(\frac{n}{2})$ * not exactly

2 additions

❷ $ad + bc = (a+b)(c+d) - ac - bd$ 4n subtraction

❸ $ac100^2 + (ad+bc)100 + bd$ 4n additions.

Karatsuba(ab, cd)

Base case: return b*d if inputs are 1-digit

ac = Karatsuba(a,c)

bd = Karatsuba(b,d)

t = Karatsuba((a+b),(c+d))

$$3T(n/2) + 2n$$

Ignoring issue of carries

mid = t - ac - bd

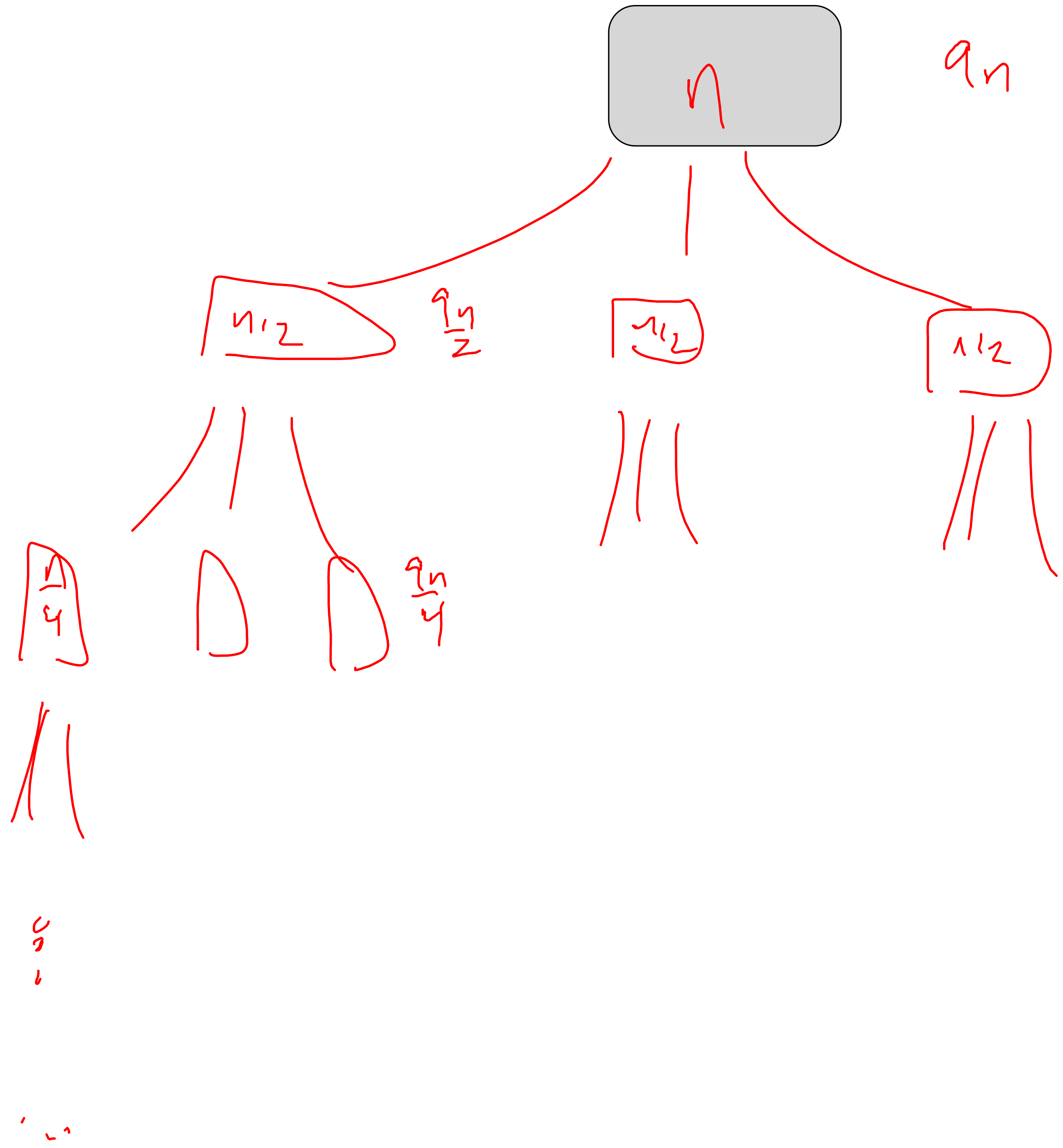
$$4n$$

RETURN ac*100^2 + mid*100 + bd

$$4n$$

$T(n) = 3T(n/2) + 10n$

$$T(n) = 3T(n/2) + 9n$$



0	$9n$
1	$3 \cdot \frac{9n}{2} = \left(\frac{3}{2}\right) \cdot 9n$
2	$9 \cdot \frac{9n}{4} = \left(\frac{3}{2}\right)^2 \cdot 9n$
3	$27 \cdot \frac{9n}{8} = \left(\frac{3}{2}\right)^3 \cdot 9n$
$\lceil \log_2 n \rceil$	$= \left(\frac{3}{2}\right)^{\lceil \log_2 n \rceil} \cdot 9n$

calculations:

$$T(n) = 9n + \left(\frac{3}{2}\right)\cdot 9n + \left(\frac{3}{2}\right)^2 \cdot 9n + \dots + \left(\frac{3}{2}\right)^{\lceil \log_2 n \rceil} \cdot 9n$$

$$= 9n\left[1 + \left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 + \dots + \left(\frac{3}{2}\right)^{\log_2 n}\right] = 9n\left[\frac{\left(\frac{3}{2}\right)^{\lceil \log_2 n \rceil + 1} - 1}{\underline{\left(\frac{3}{2}\right) - 1}}\right]$$

$$= (9n)(2)\left[\left(\frac{3}{2}\right)^{\log_2 n + 1} - 1\right]$$

$$3 = 2^{\log_2 3}$$

$$= (9n)(2)\left(\frac{3}{2}\right)\left[\frac{3^{\log_2 n}}{\underbrace{2^{\log_2 n}}_{n}}\right] - 18n$$

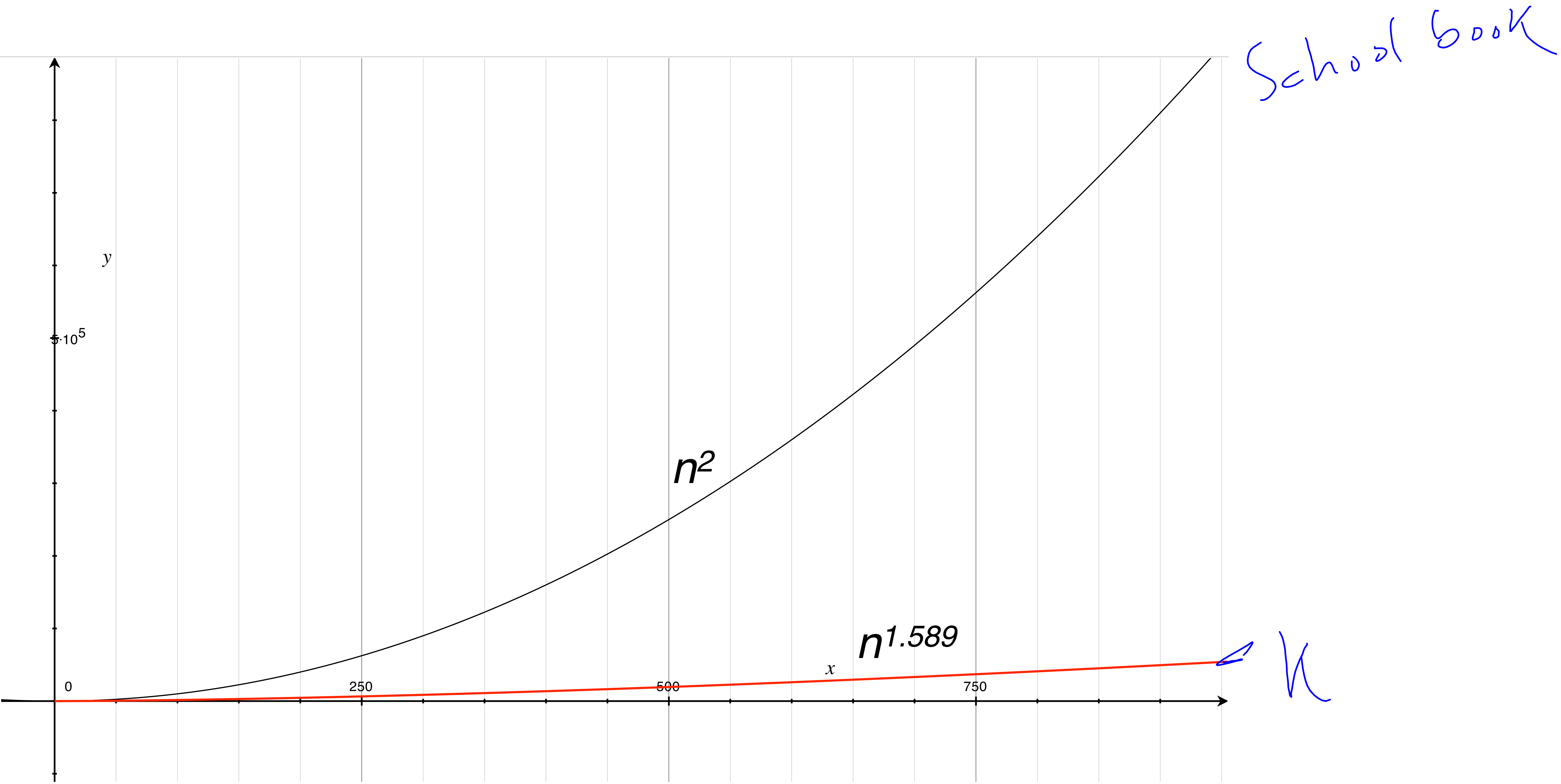
$$= 27 \cdot \frac{3^{\log_2 n}}{\text{lanka}} - 18n = 27 \cdot n^{\log_2 3} - 18n = O\left(n^{\log_2 3}\right)$$

$$= \left(2^{\log_2 3}\right)^{\log_2 n} = \left(2^{\log_2 n}\right)^{\log_2 3} = \left(n^{\log_2 3}\right)$$

calculations:

$$T(n) = 3T(n/2) + 9n$$

$$O(n^{\log_2(3)}) \quad O(n^{1.589})$$



School book
y
5·10^5
n^2
n^1.589
K
x
0
250
500
750

1	7	8	9	★	1	4	3	2
a		b			c		d	

$$T(n) = 3T(n/2) + 9n$$

$$T(n) = 4T(n/2) + 3n$$

simpler proof technique?

1

induction redux

classic

goal: prove that some property P(k) is true for all k

$\forall k, P(k)$ holds

❶ classic

one long proof...

goal: prove that some property P(k) is true for all k

$\forall k, P(k)$ holds

❶

Induction

classic
base case: $P(1)$

classic
inductive
step:

$$\boxed{\left.\begin{matrix} P(1) \\ \vdots \\ P(k) \end{matrix}\right\}} \text{ implies } P(k+1) \text{ true}$$

❷ induction redux asymptotic style

base case: $P(n^*)$

inductive step: $\left.\begin{matrix} P(n^*) \\ \cdots \\ P(k) \end{matrix}\right\}$ implies $P(k+1)$ true

simpler proof (guess +chk)

$$T(n) = 3T(n/2) + 9n$$

$T(n) = O(n^{1.6})$

simpler proof

Proof $T(n) = O(n^{1.6})$ i.e $T(n) < 3000 \cdot n^{1.6}$

Base case: holds for $T(2)$.

Inductive hypothesis: Spse this holds for all $k < n_0$.

Consider $T(n+1) = 3T\left(\frac{n+1}{2}\right) + 9(n+1)$

$$< 3 \cdot \left(3000 \cdot \left(\frac{n+1}{2}\right)^{1.6}\right) + 9(n+1)$$

by Ind. hypothesis b/c $\frac{n+1}{2} < n$

$$= \frac{3}{2^{1.6}} \left[3000 (n+1)^{1.6}\right] + 9(n+1)$$

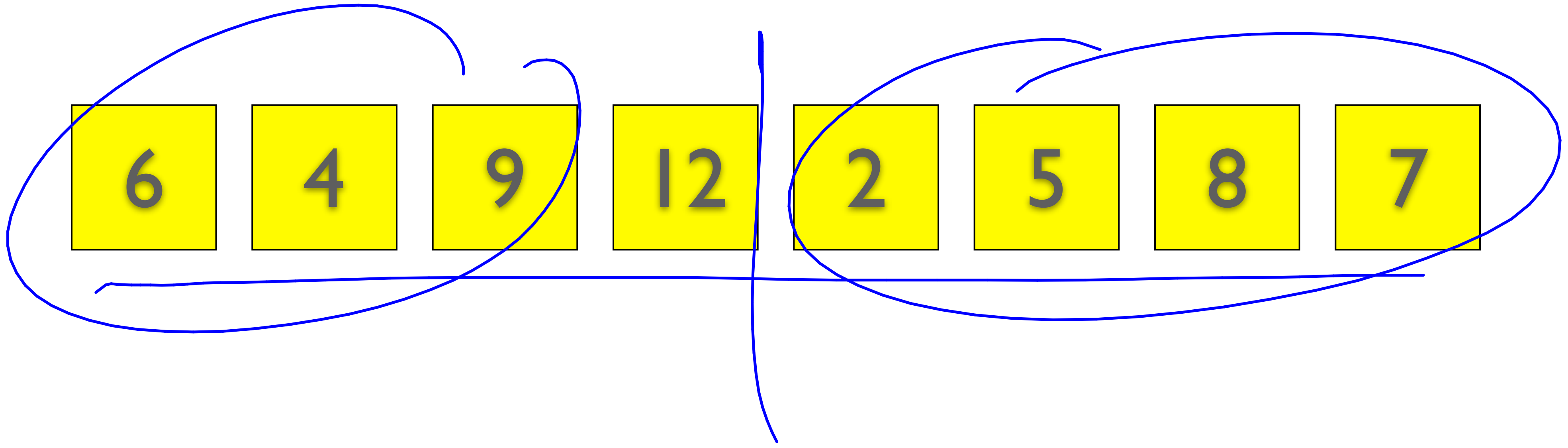
$$< 0.997 (3000)(n+1)^{1.6} + 9(n+1)$$

$$= 1 \cdot 3000 (n+1)^{1.6} \underbrace{- \overline{(0.003)(3000)}^{9} (n+1)^{1.6} + 9(n+1)}_{\text{in negative}} < 3000 \cdot (n+1)^{1.6}$$

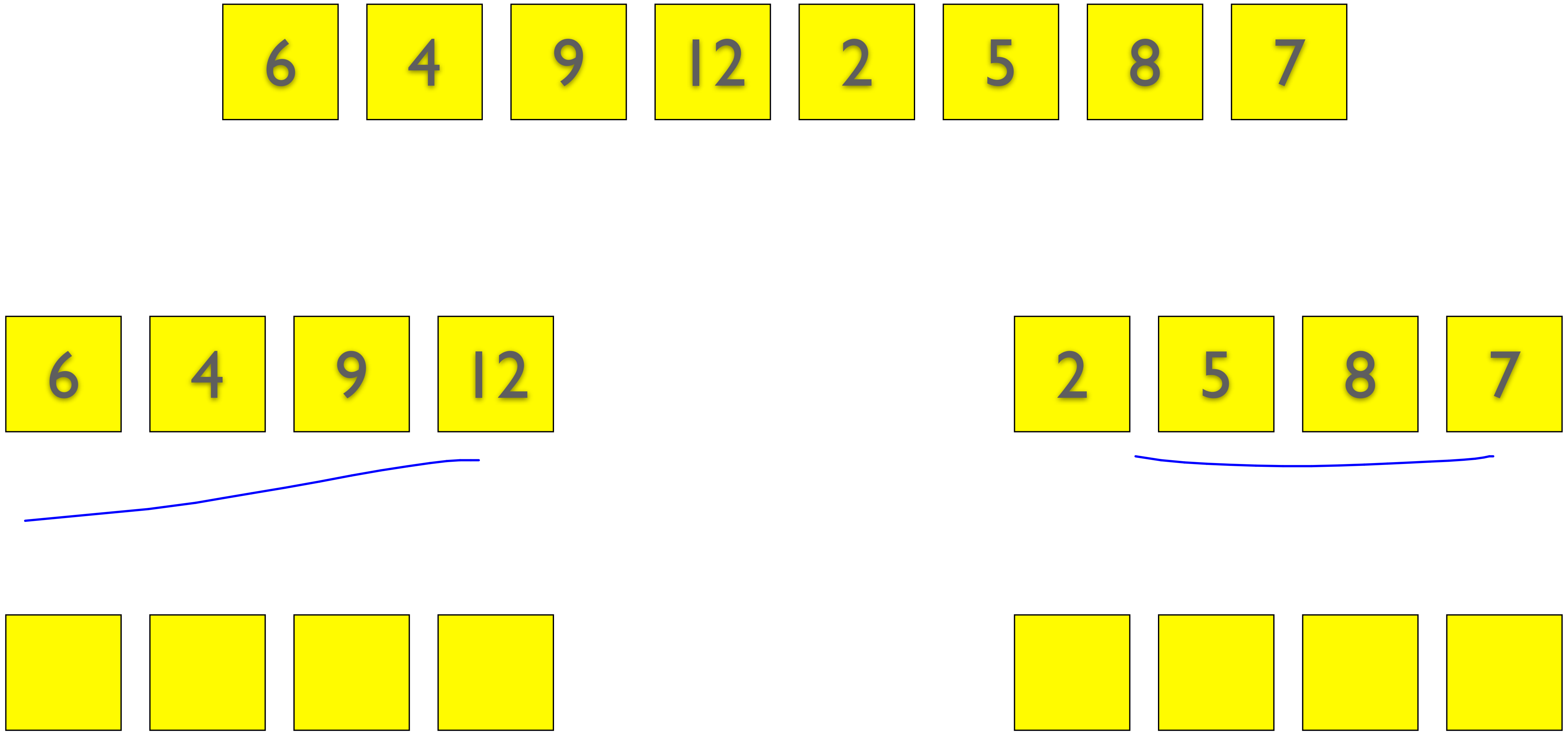
mergesort

goal:

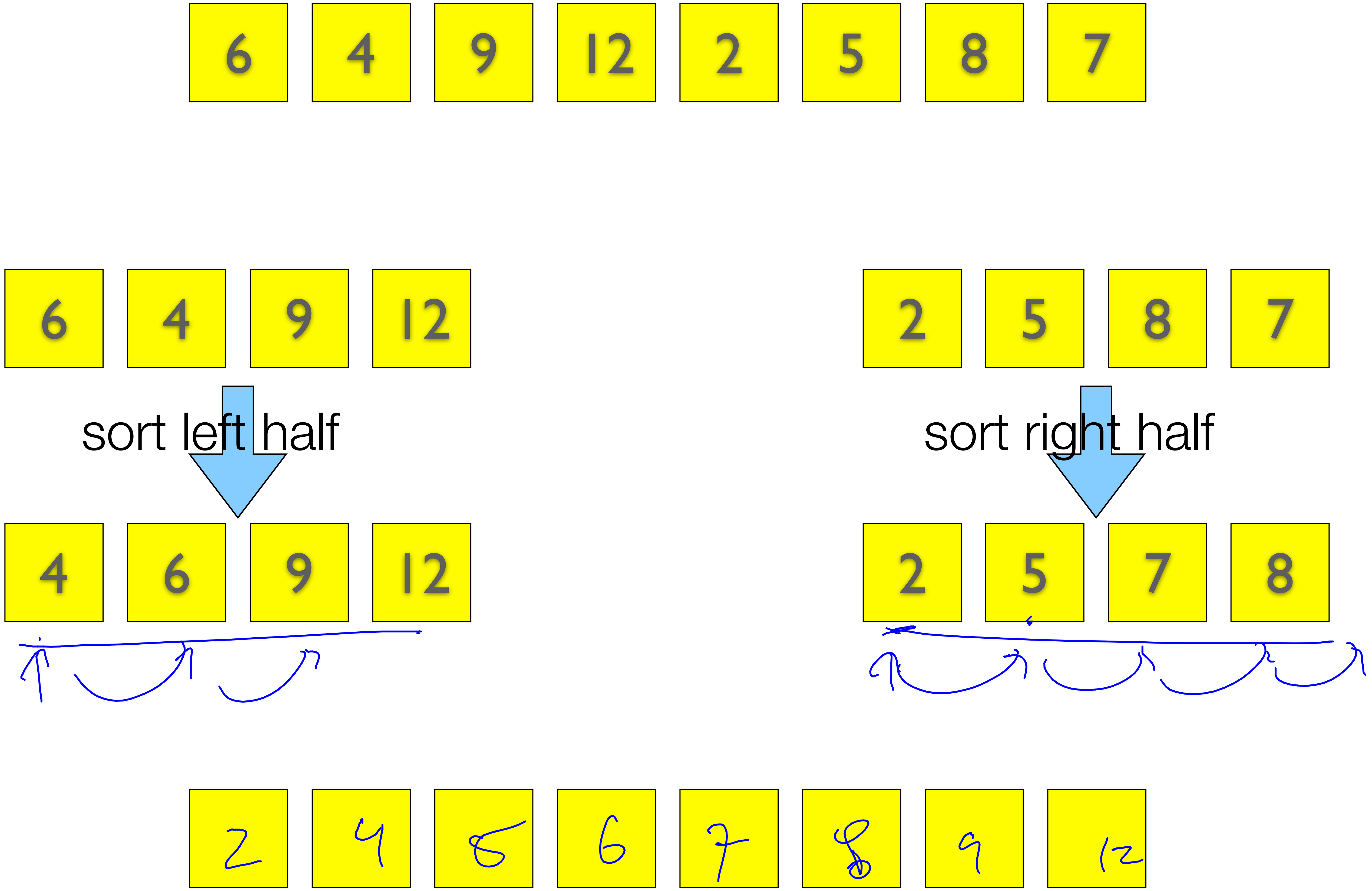
technique:



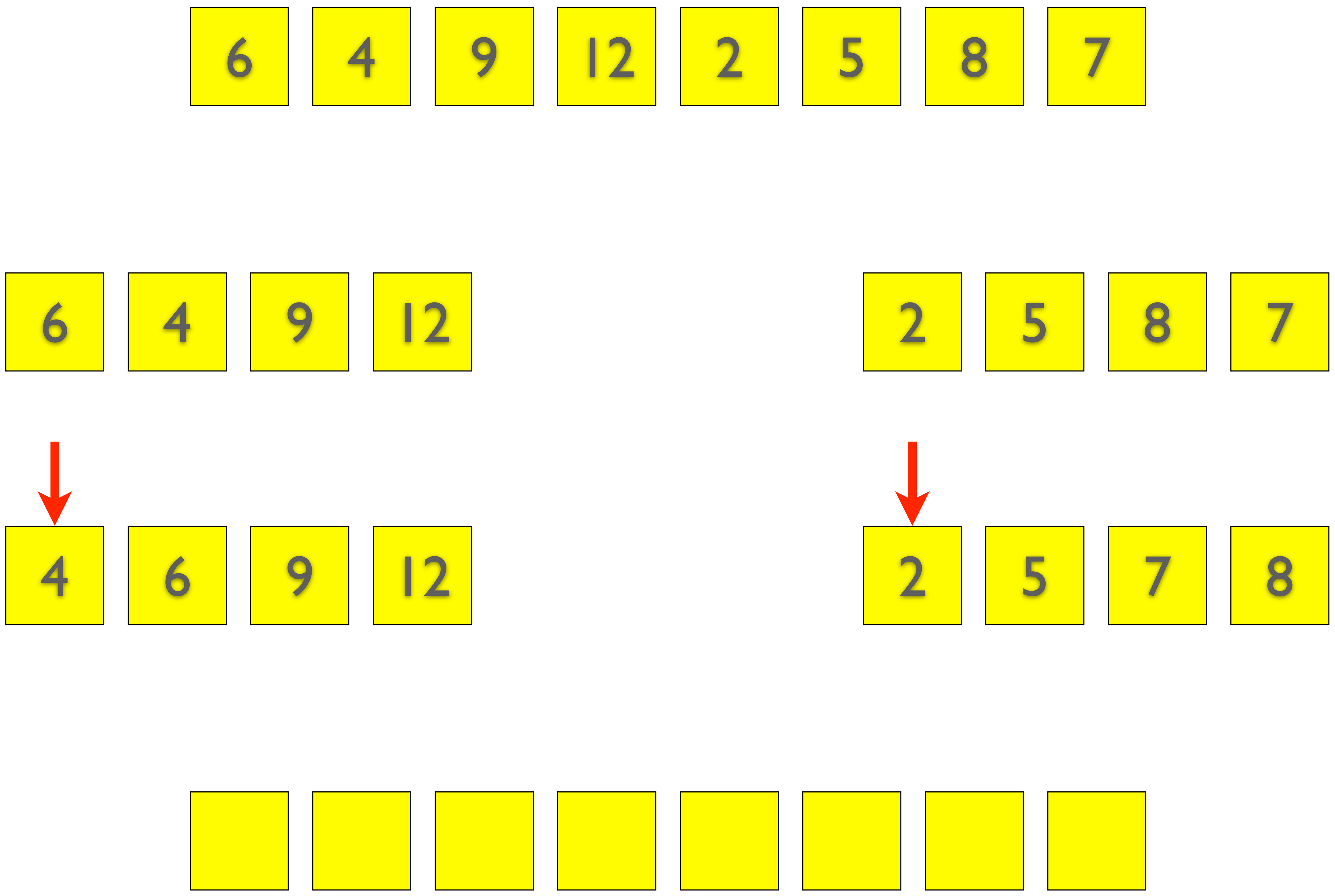
mergesort



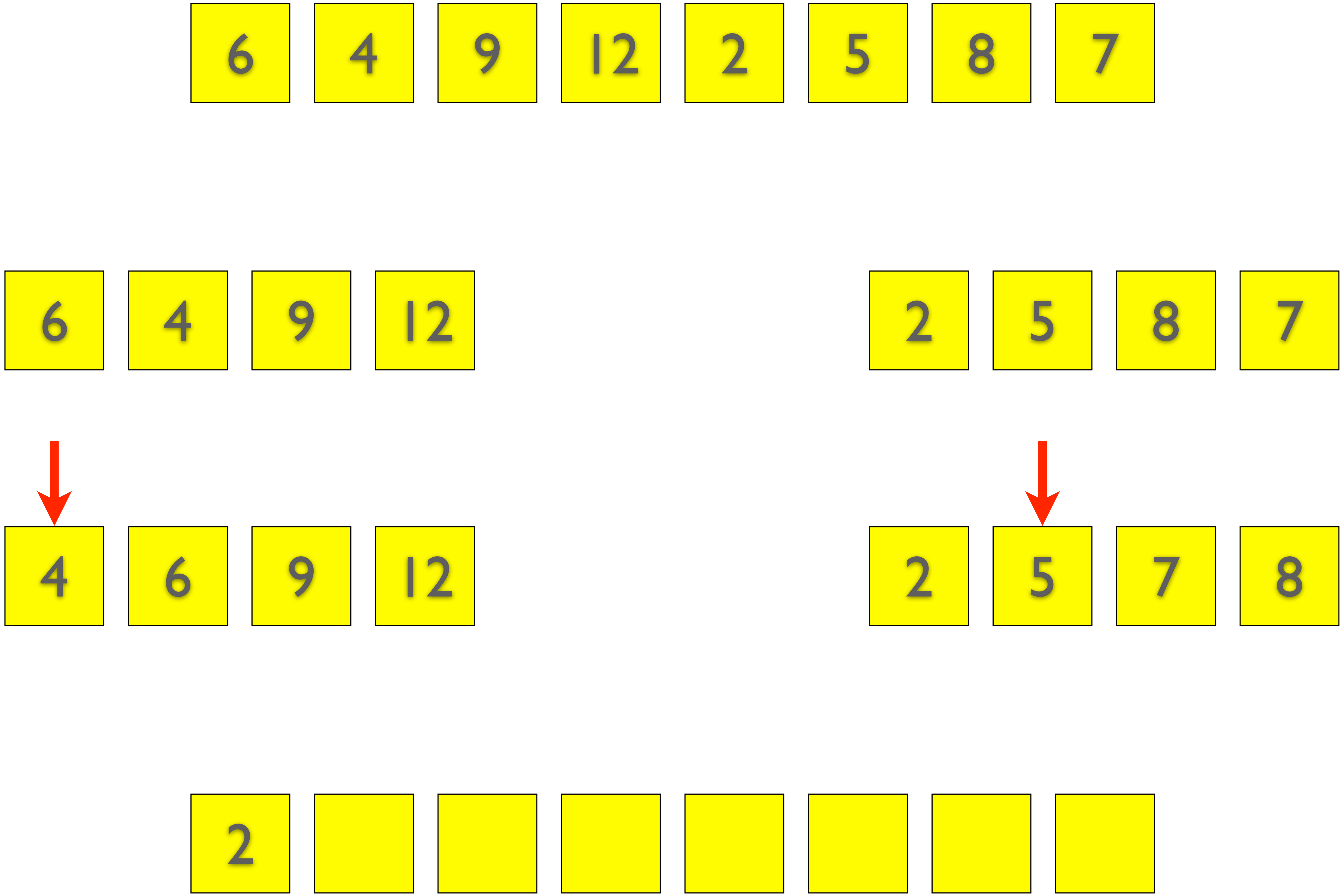
mergesort


6 4 9 12 2 5 8 7
6 4 9 12
2 5 8 7
sort left half
sort right half
4 6 9 12
2 5 7 8
2 4 5 6 7 8 9 12

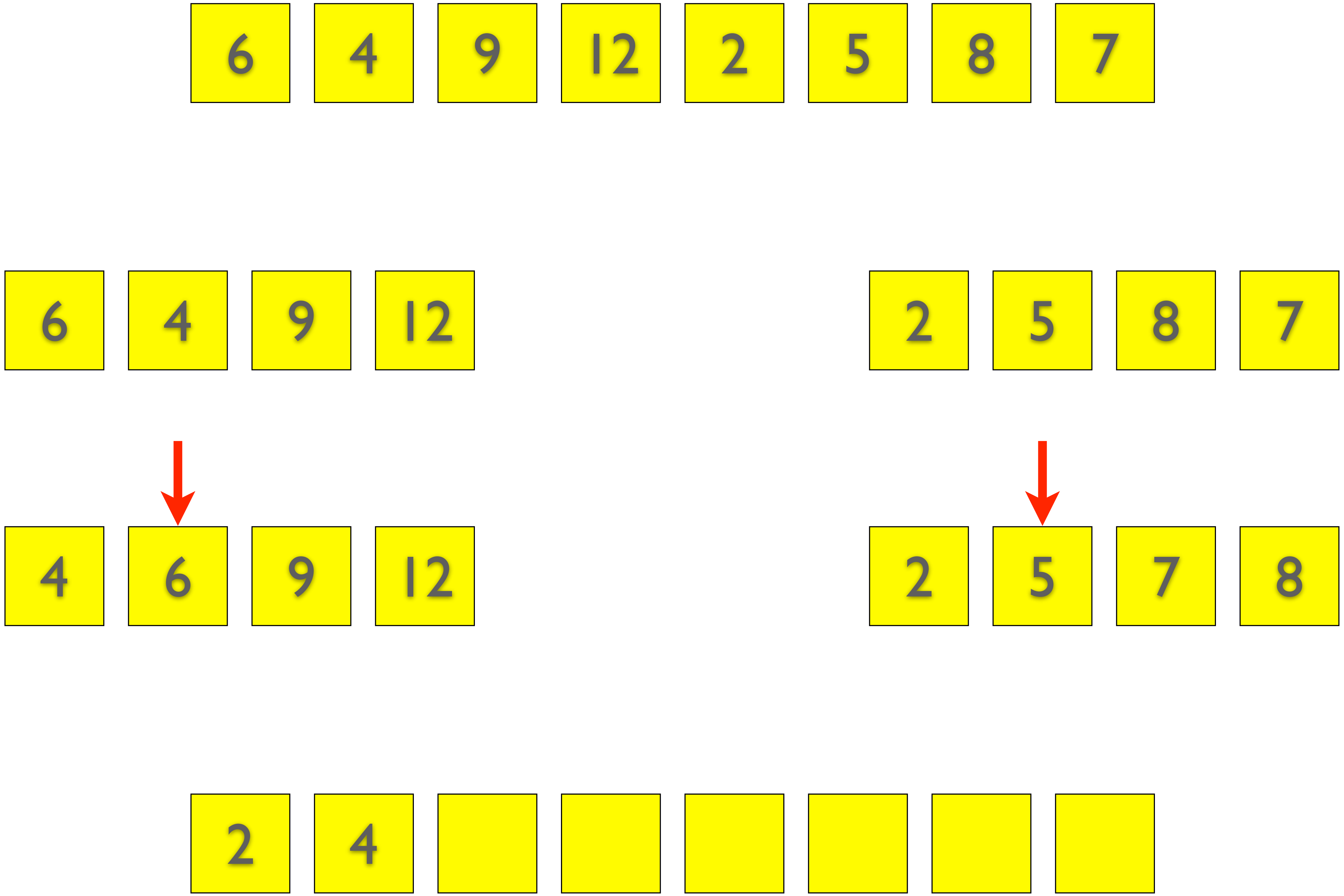
mergesort



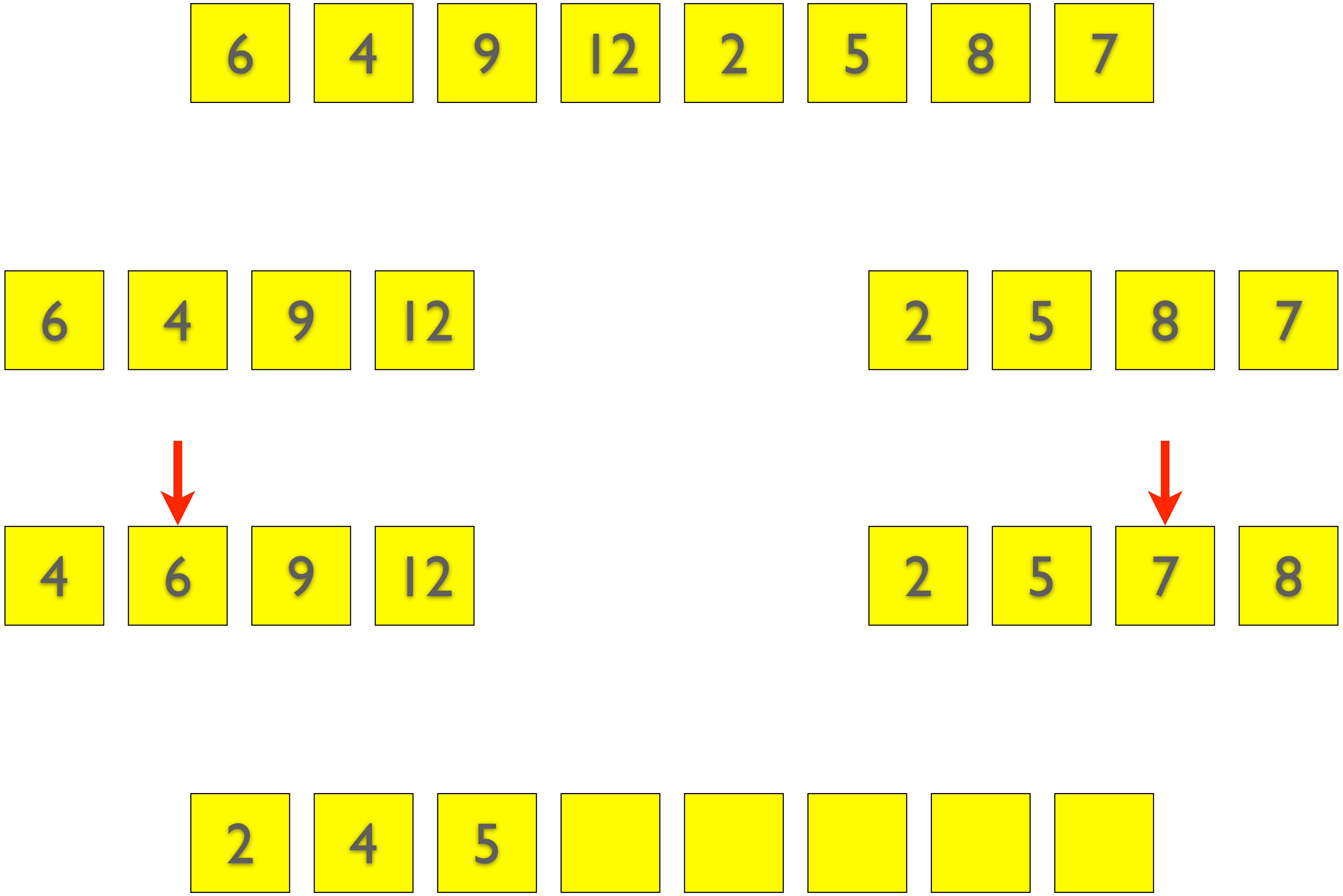
mergesort



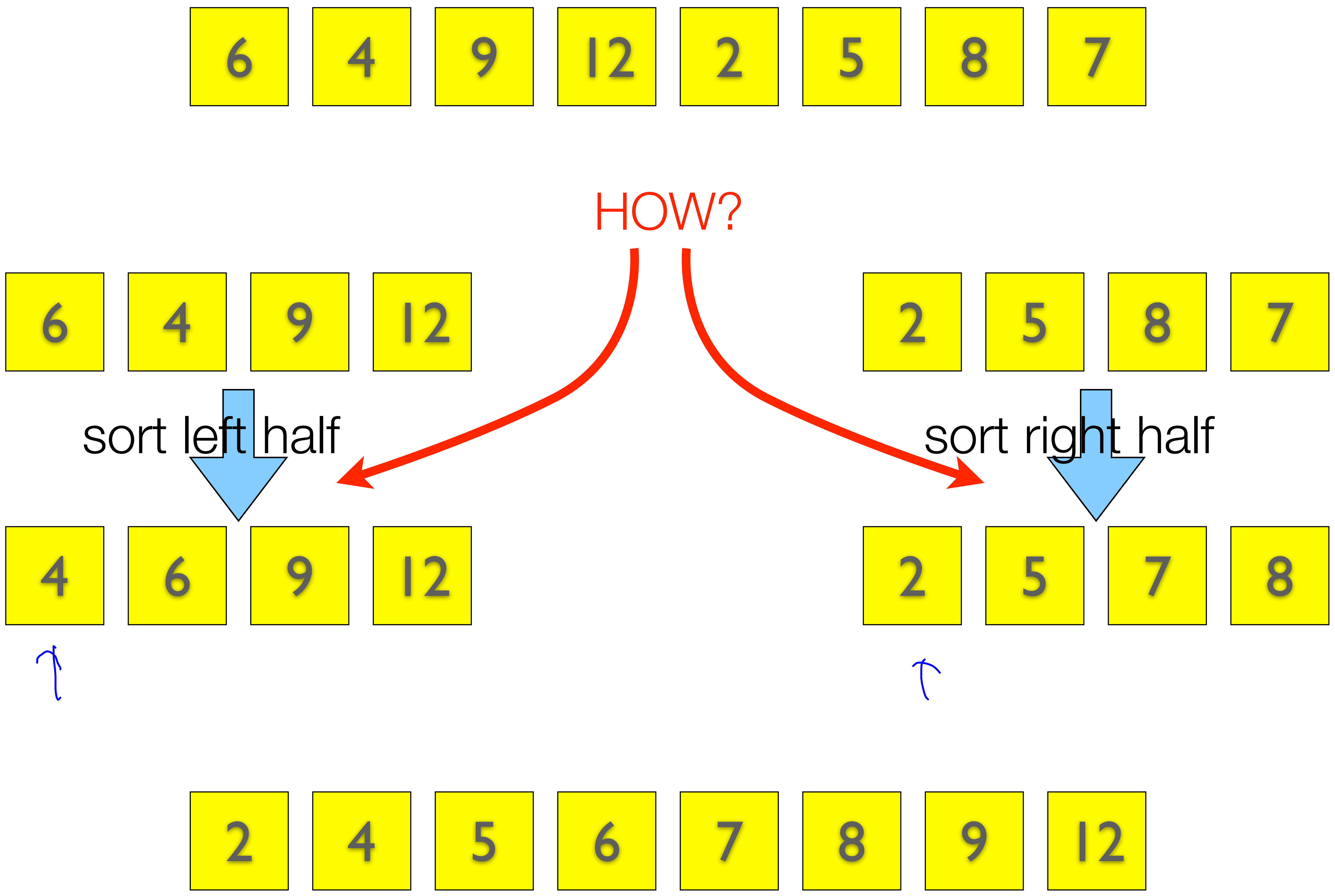
mergesort



mergesort



mergesort


6 4 9 12 2 5 8 7
HOW?
6 4 9 12
2 5 8 7
sort left half
sort right half
4 6 9 12
2 5 7 8
2 4 5 6 7 8 9 12

mergesort(`A,start,end`)

❶

❷

❸

❹

❺

mergesort(A,start,end)

1. `if start < end`
2. $q \leftarrow \lfloor (\text{start} + \text{end})/2 \rfloor$
3. `mergesort(A,start,q)`
   `mergesort(A,q+1,end)`
4. `merge(A,start,q,end)`
5. `else ...`

mergesort(`A,start,end`)

1.
```
if start < end
```

2. $q \leftarrow \lfloor(\text{start} + \text{end})/2\rfloor$

3.
```
mergesort(A,start,q)
mergesort(A,q+1,end)
```

4.
```
merge(A,start,q,end)
```

5.
```
else ...
```

MERGE($A[1..n], m$):

```
i ← 1; j ← m+1
for k ← 1 to n
    if j > n
        B[k] ← A[i]; i ← i+1
    else if i > m
        B[k] ← A[j]; j ← j+1
    else if A[i] < A[j]
        B[k] ← A[i]; i ← i+1
    else
        B[k] ← A[j]; j ← j+1

for k ← 1 to n
    A[k] ← B[k]
```

jeff erickson

mergesort(A,start,end)

running time?

1. `if start < end`

2. $q \leftarrow \lfloor (\text{start} + \text{end})/2 \rfloor$

3. ```
   mergesort(A,start,q)
   mergesort(A,q+1,end)
   ```

4. `merge(A,start,q,end)`

5. `else ...`
   ```

$$T(n) = 2T(n/2) + n$$

show:

$$T(n) = 2T(n/2) + n$$

prove:

hypothesis:

base case:

inductive step:

$$T(n) = 2T(n/2) + n$$

prove: $T(n) = O(n \log n)$

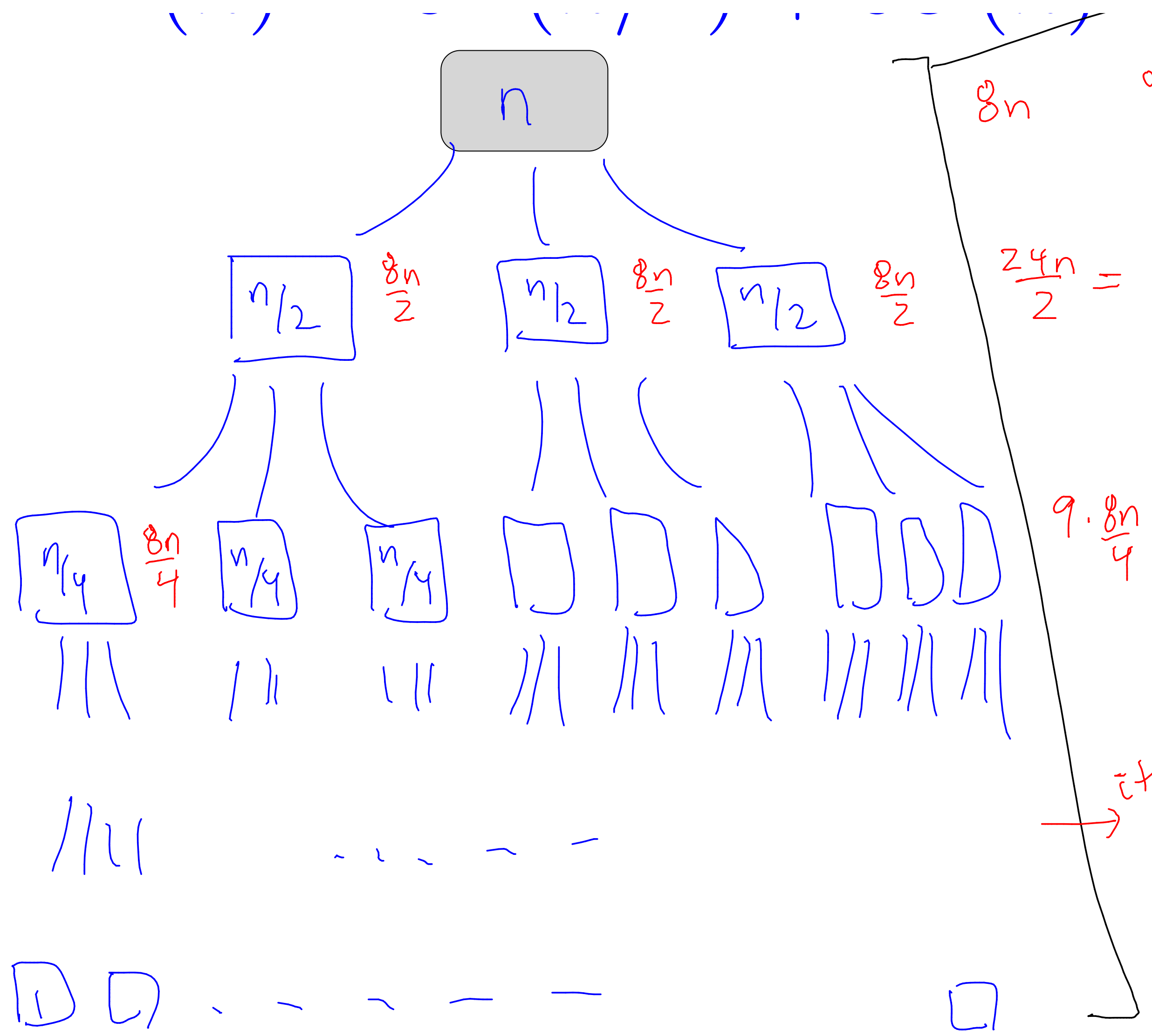
property: $T(n) < cn \log n$ for c>1

base case:

inductive step:

$$T(n) = 3T(n/2) + 9n$$

$$O(n^{1.589}) \qquad\qquad O(n^{\log_2(3)})$$



$\lceil \log_2 n \rceil$

$8n$ 0

$\frac{24n}{2} = \left(\frac{3}{2}\right) 8n$ 1

$9 \cdot \frac{8n}{4} = \left(\frac{9}{4}\right) \cdot 8n$ 2

ith level amount of work $= \left(\frac{3}{2}\right)^i \cdot 8n$

$$T(n) = 3T(n/2) + 9n$$ (guess +chk)

❶ classic

one long proof...

goal: prove that some property P(k) is true for all k

$\forall k, P(k)$ holds

$$T(n) = 3T(n/2) + 9n$$ (guess +chk)

show: $T(k) = O(n^{\log 3})$

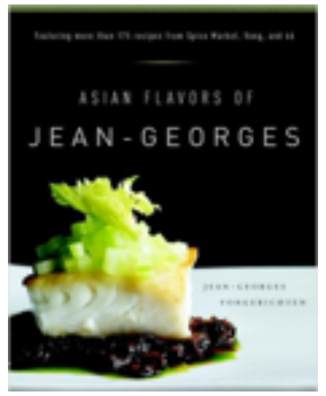
property:

base case: (handled by constants d' and d")

inductive step

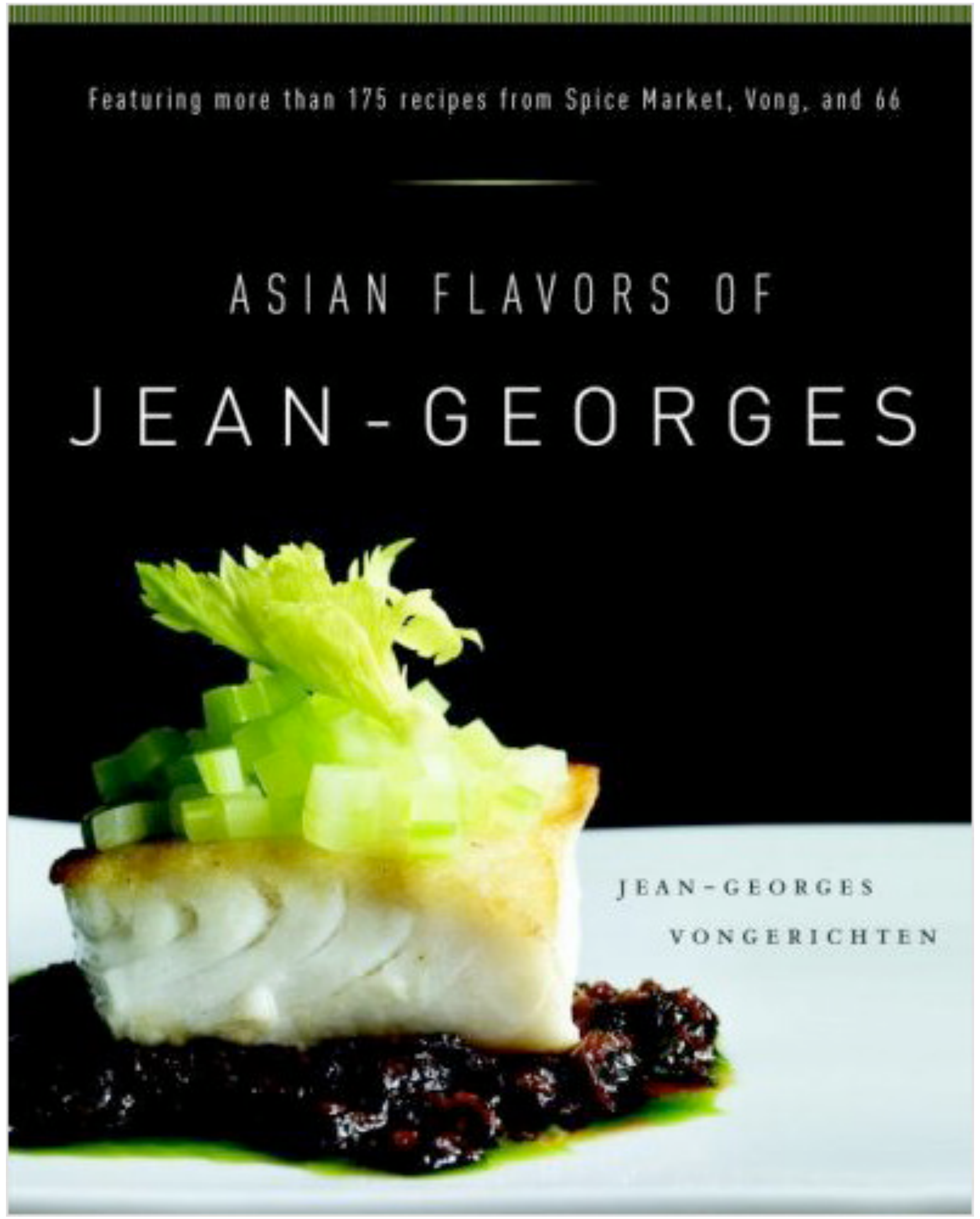
$$T(n) = 8T(n/2) + \Theta(n^2)$$ (guess +chk)

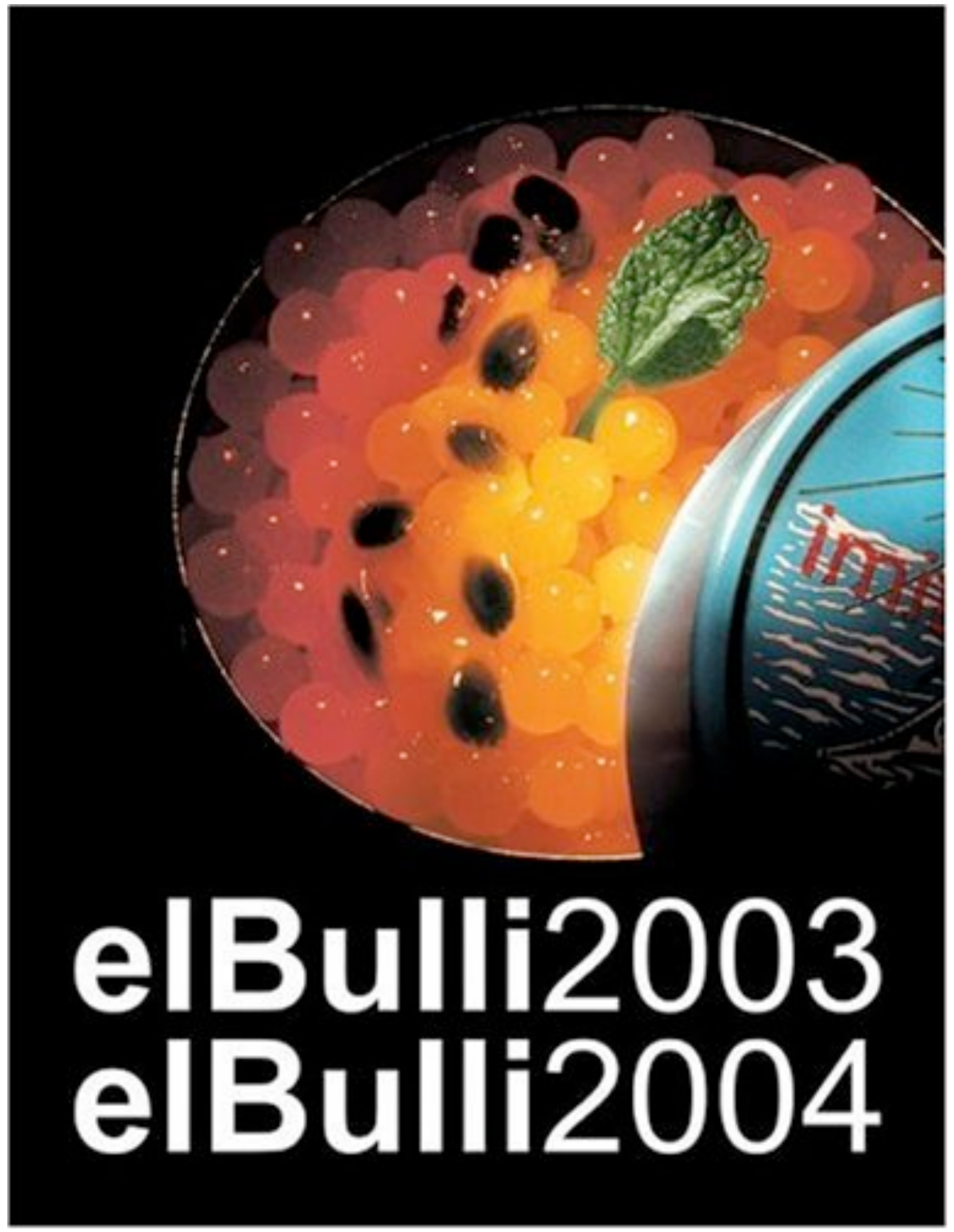
?-✓



http://www.drblank.com/law301.jpg

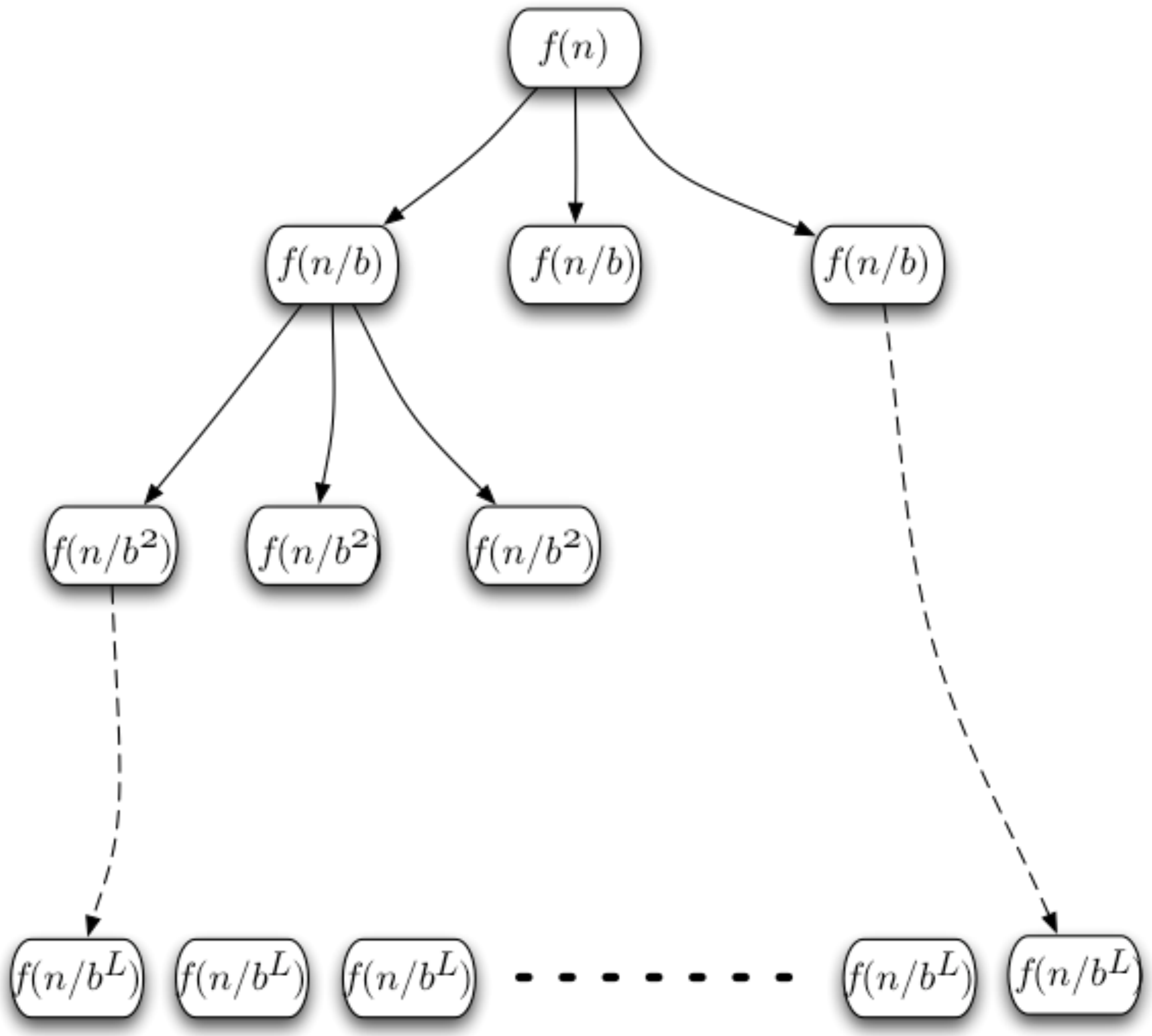
cookbook


Featuring more than 175 recipes from Spice Market, Vong, and 66
ASIAN FLAVORS OF
JEAN-GEORGES
JEAN-GEORGES
VONGERICHTEN


elBulli2003
elBulli2004

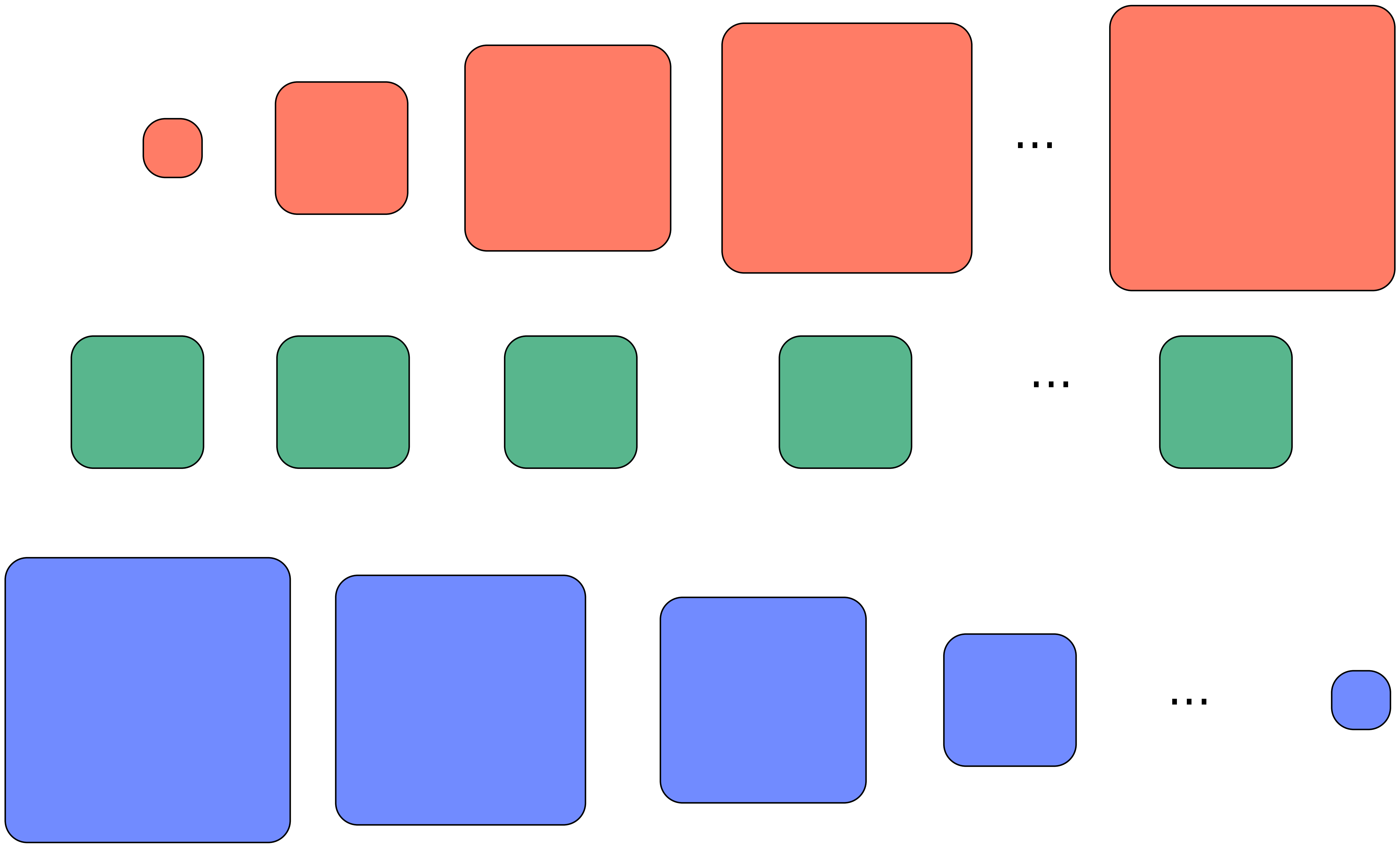
$$T(n) = aT(n/b) + f(n)$$

$$T(n) = aT(n/b) + f(n)$$



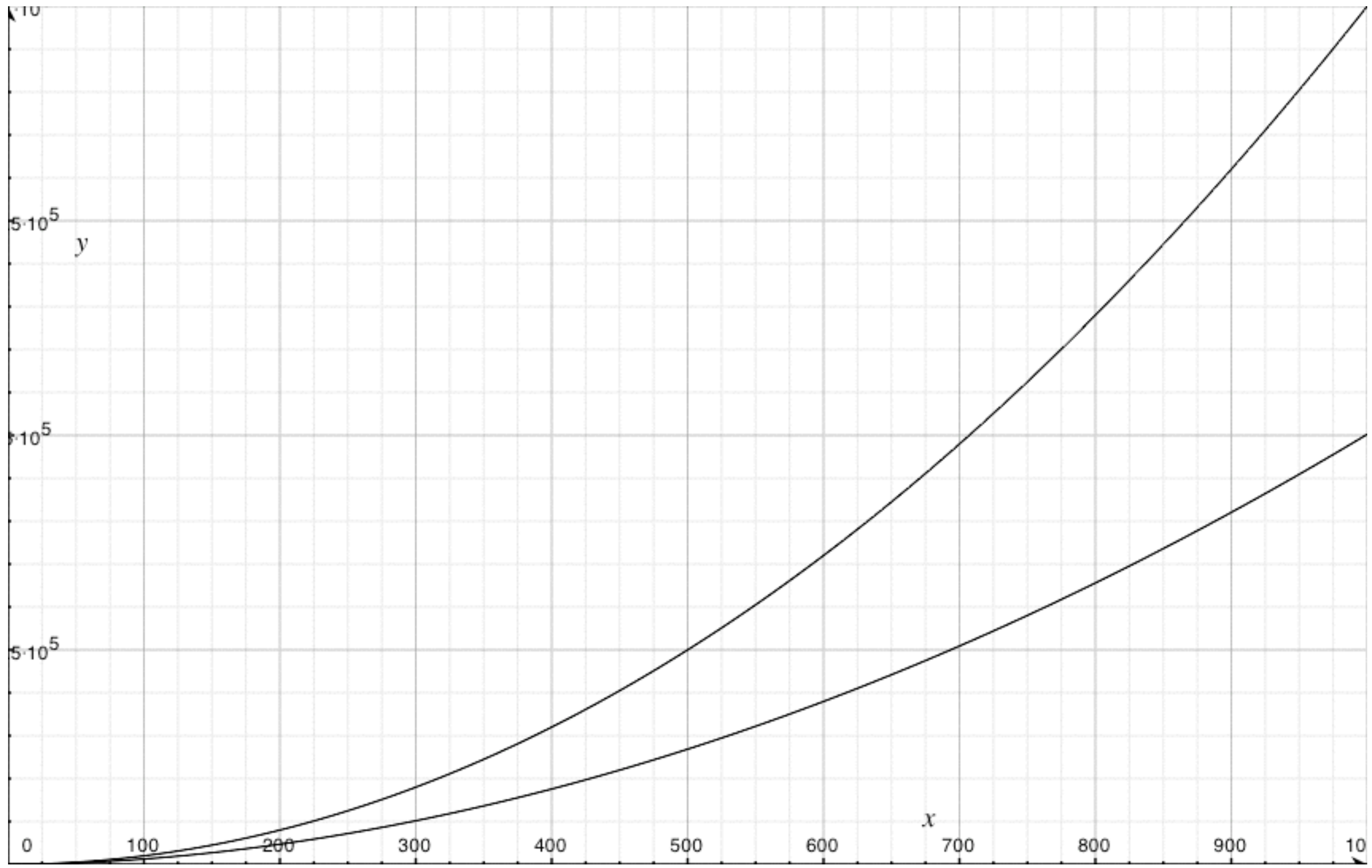
$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$



$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1: $f(n) = O(n^{\log_b a - \epsilon})$



y
x
0
100
200
300
400
500
600
700
800
900

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1: $f(n) = \Theta(n^{\log_b a - \epsilon})$

example: $T(n) = 4T(n/2) + n$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1 (cont):