L3 4102

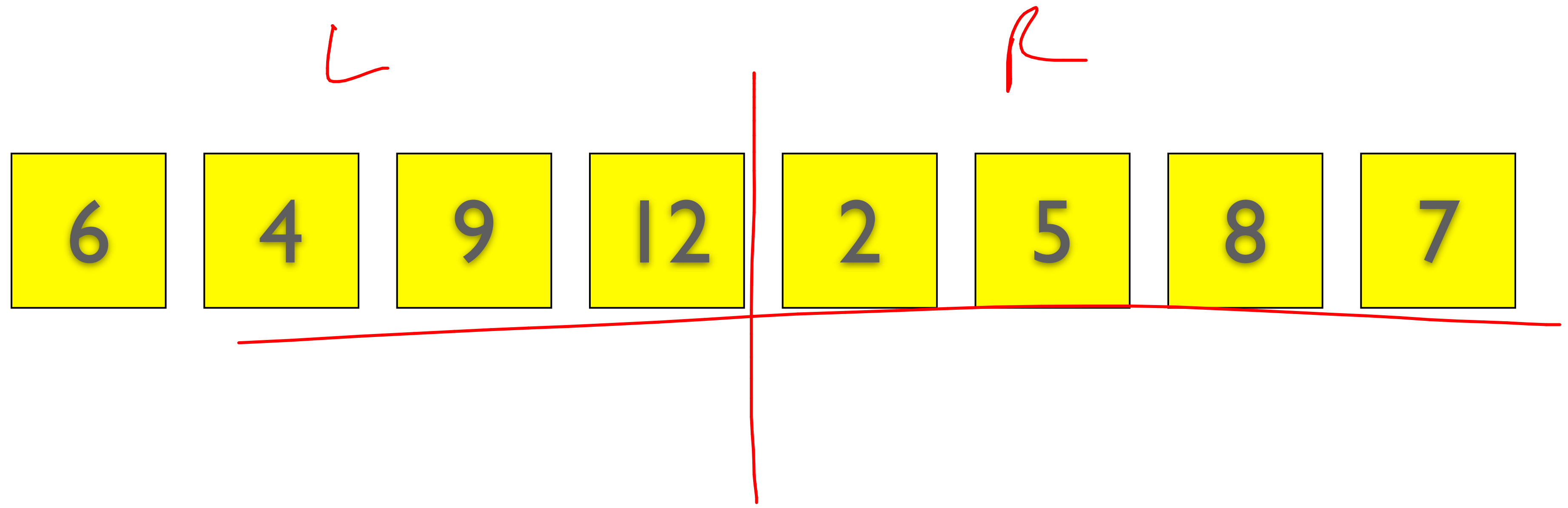
Sep 16 2016

shelat

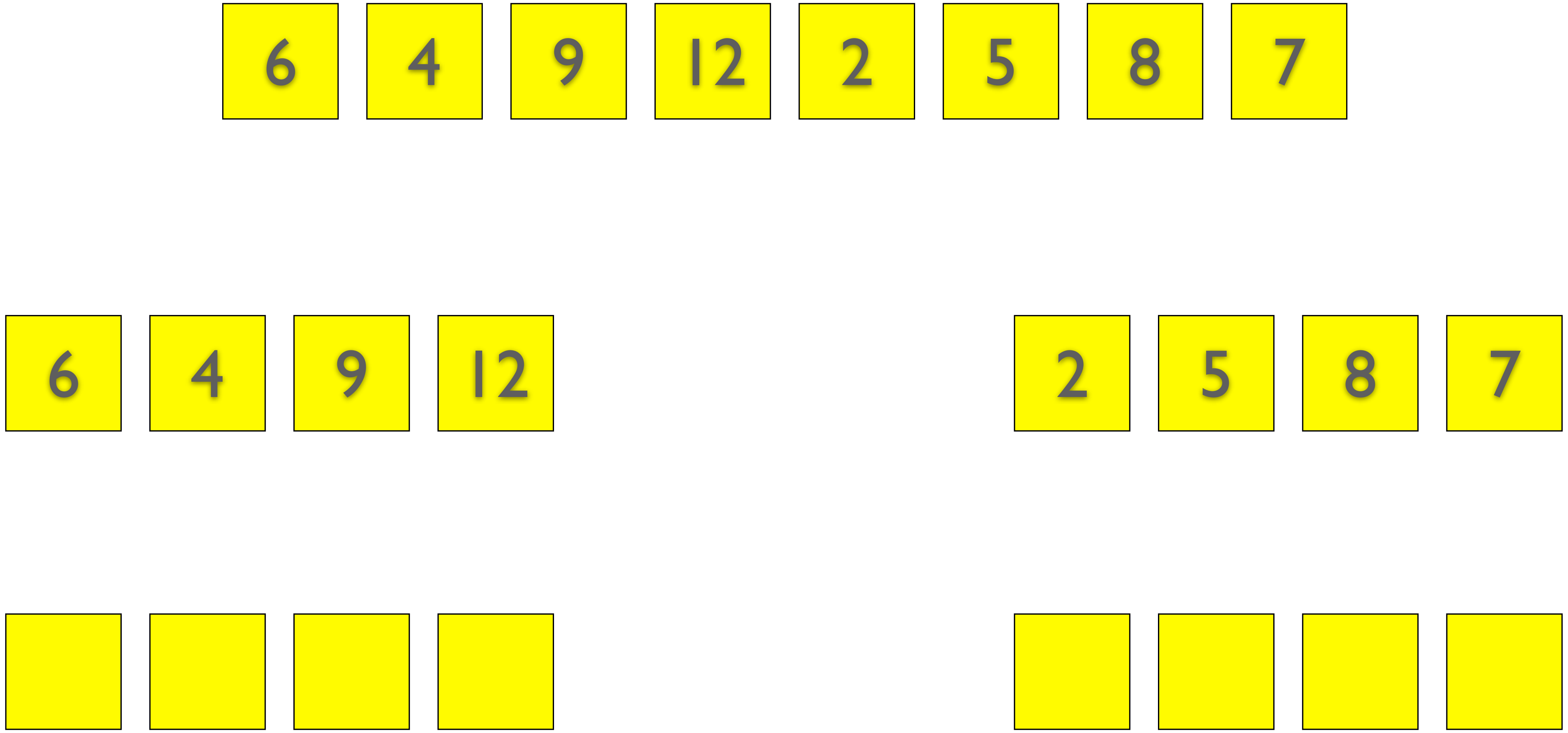
mergesort

goal:

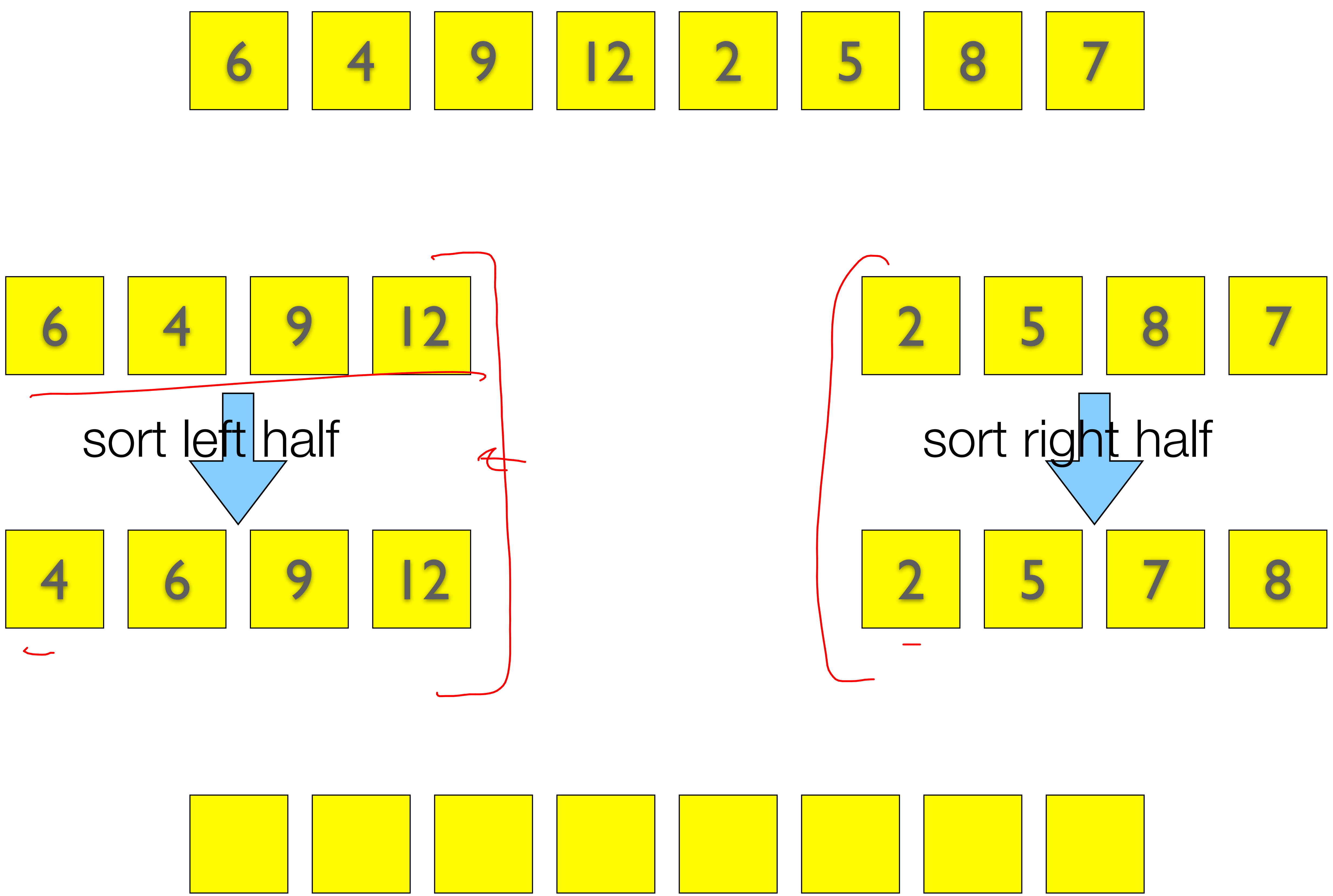
technique:



mergesort

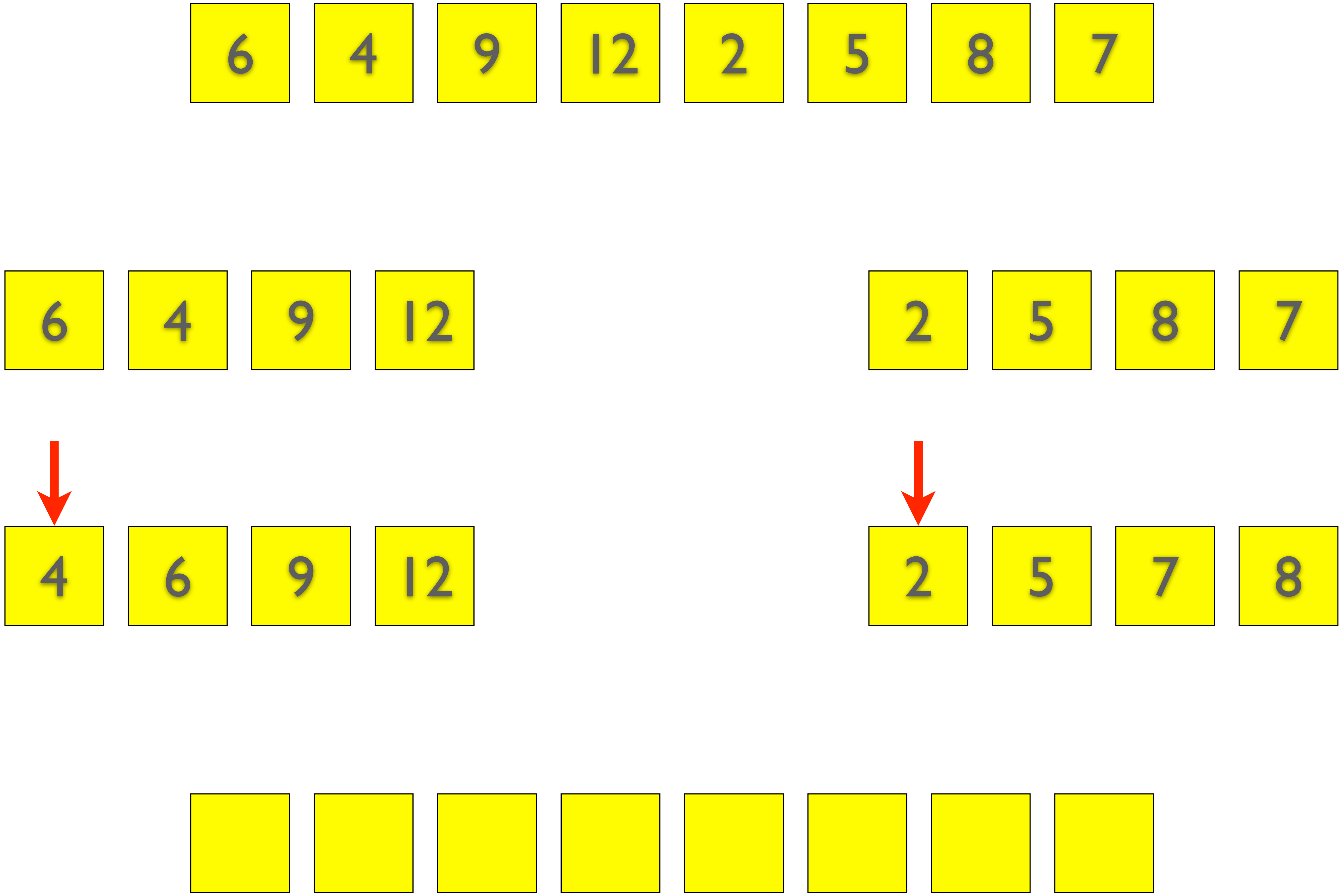


mergesort

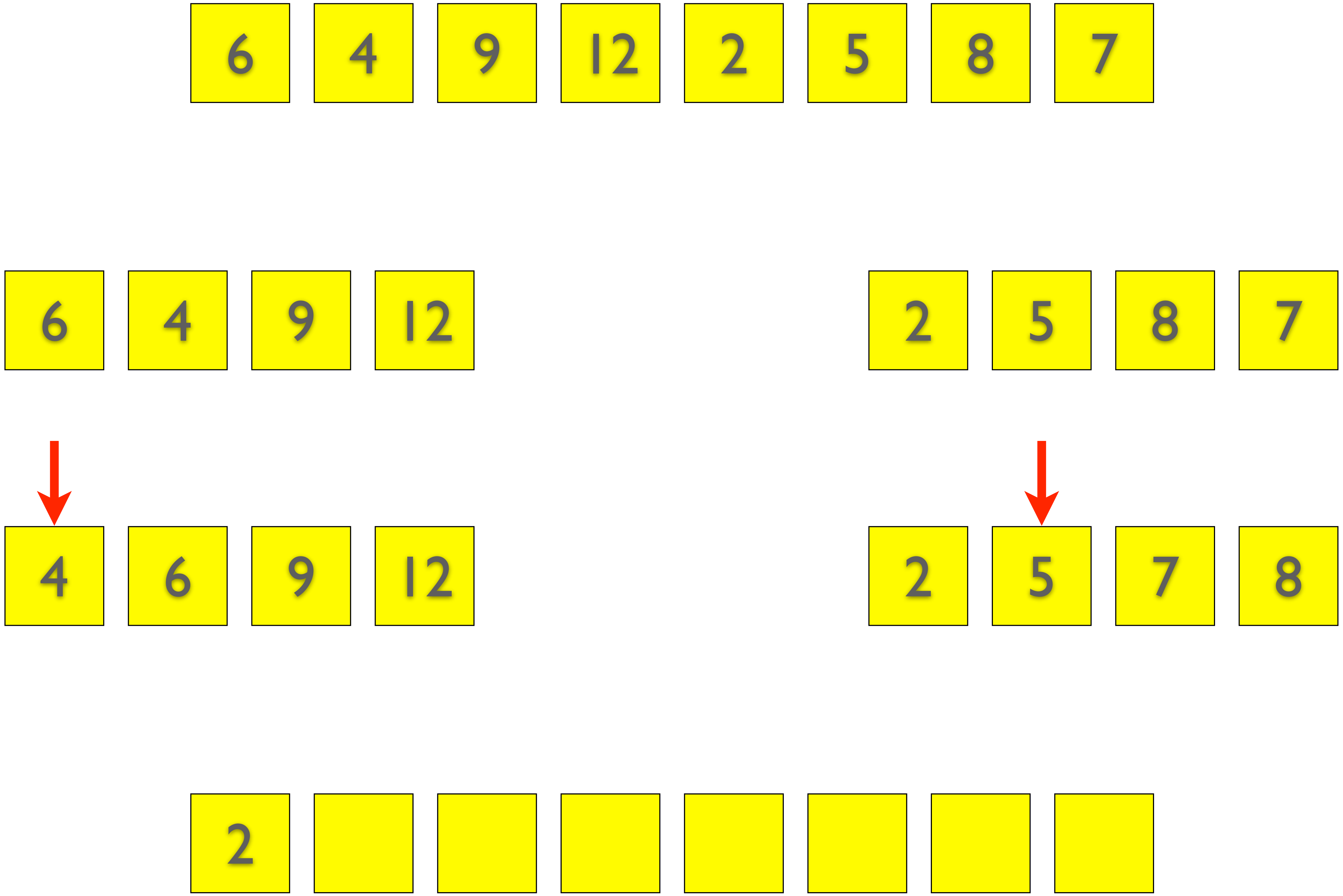


6 4 9 12 2 5 8 7
6 4 9 12
2 5 8 7
sort left half
sort right half
4 6 9 12
2 5 7 8

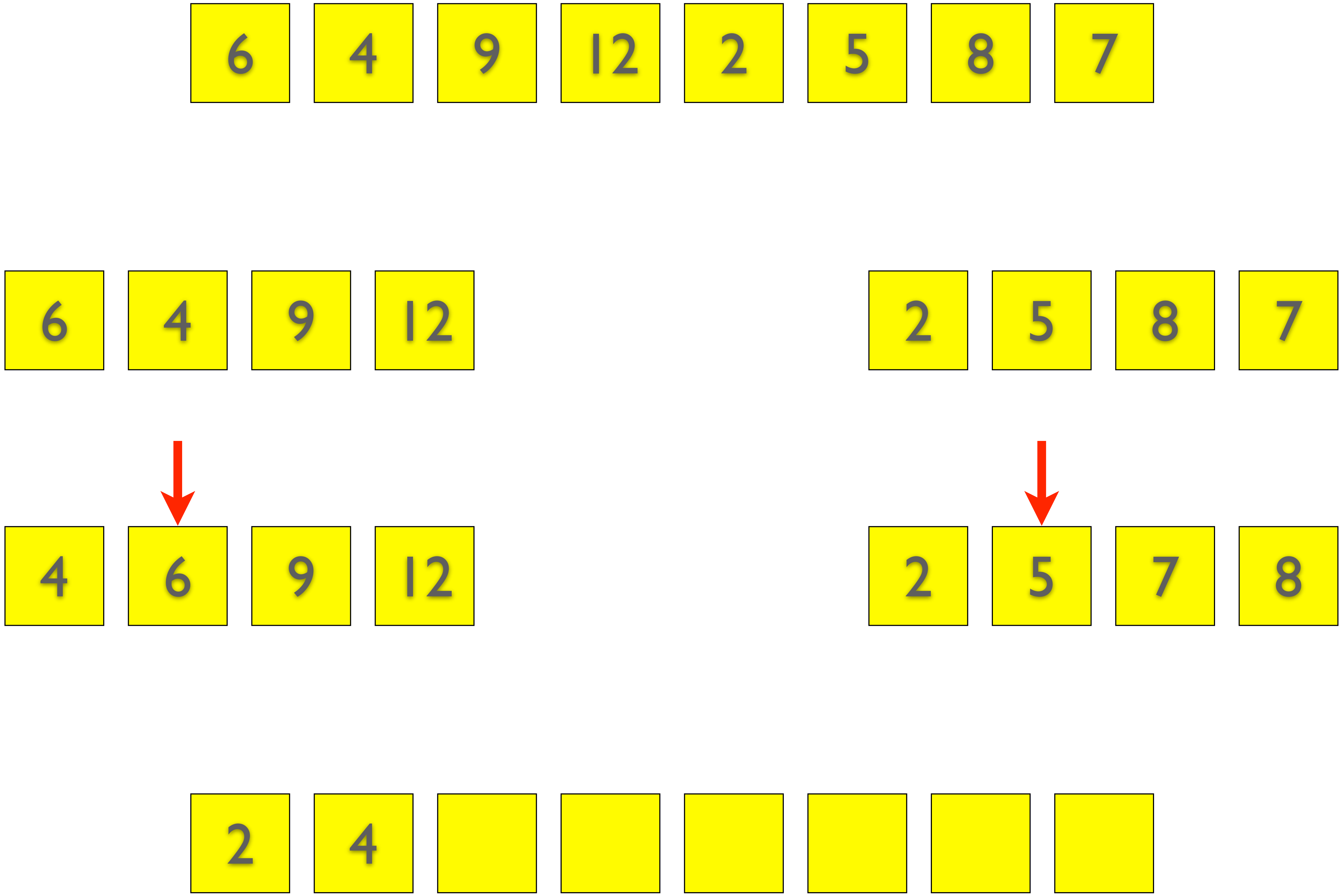
mergesort



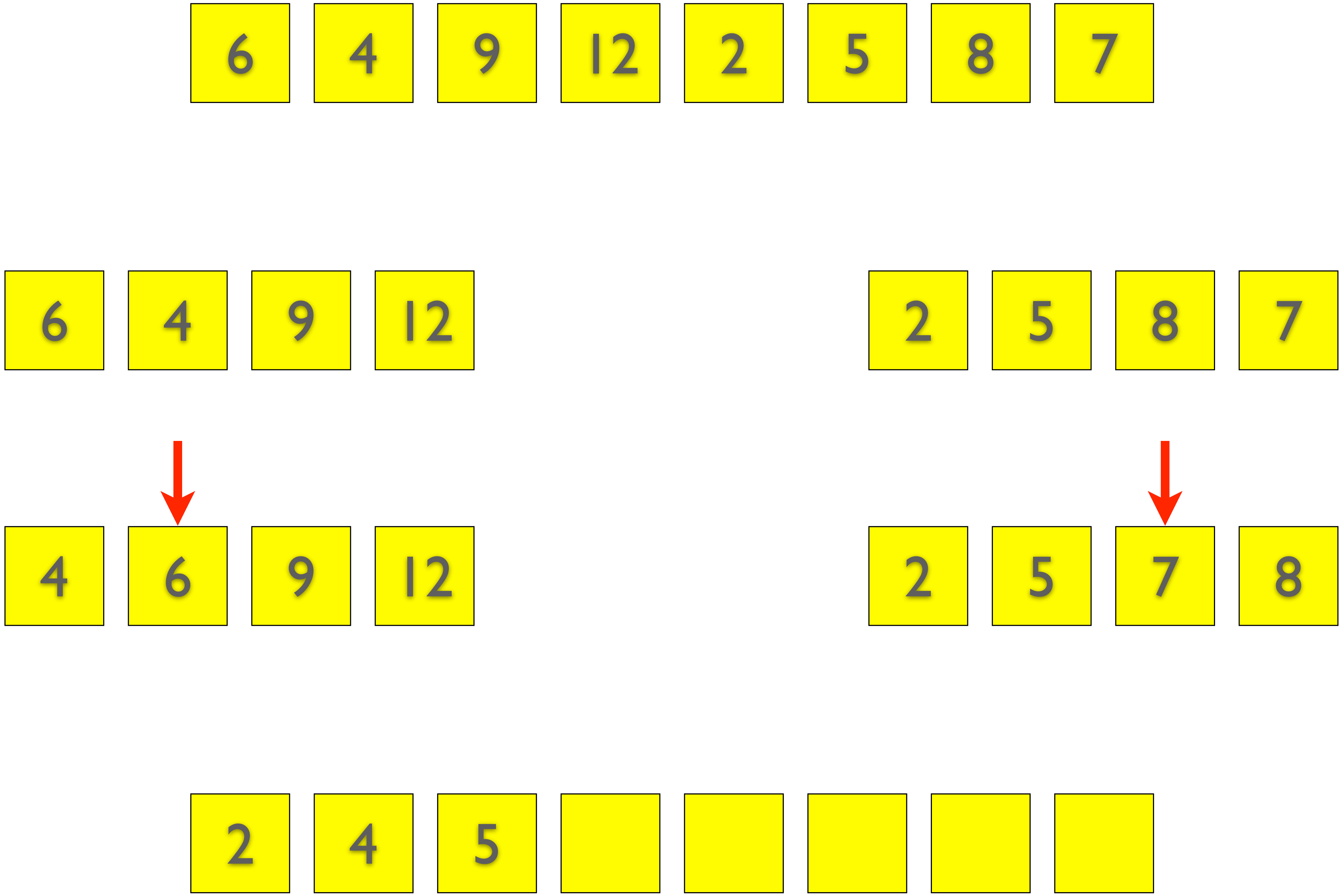
mergesort



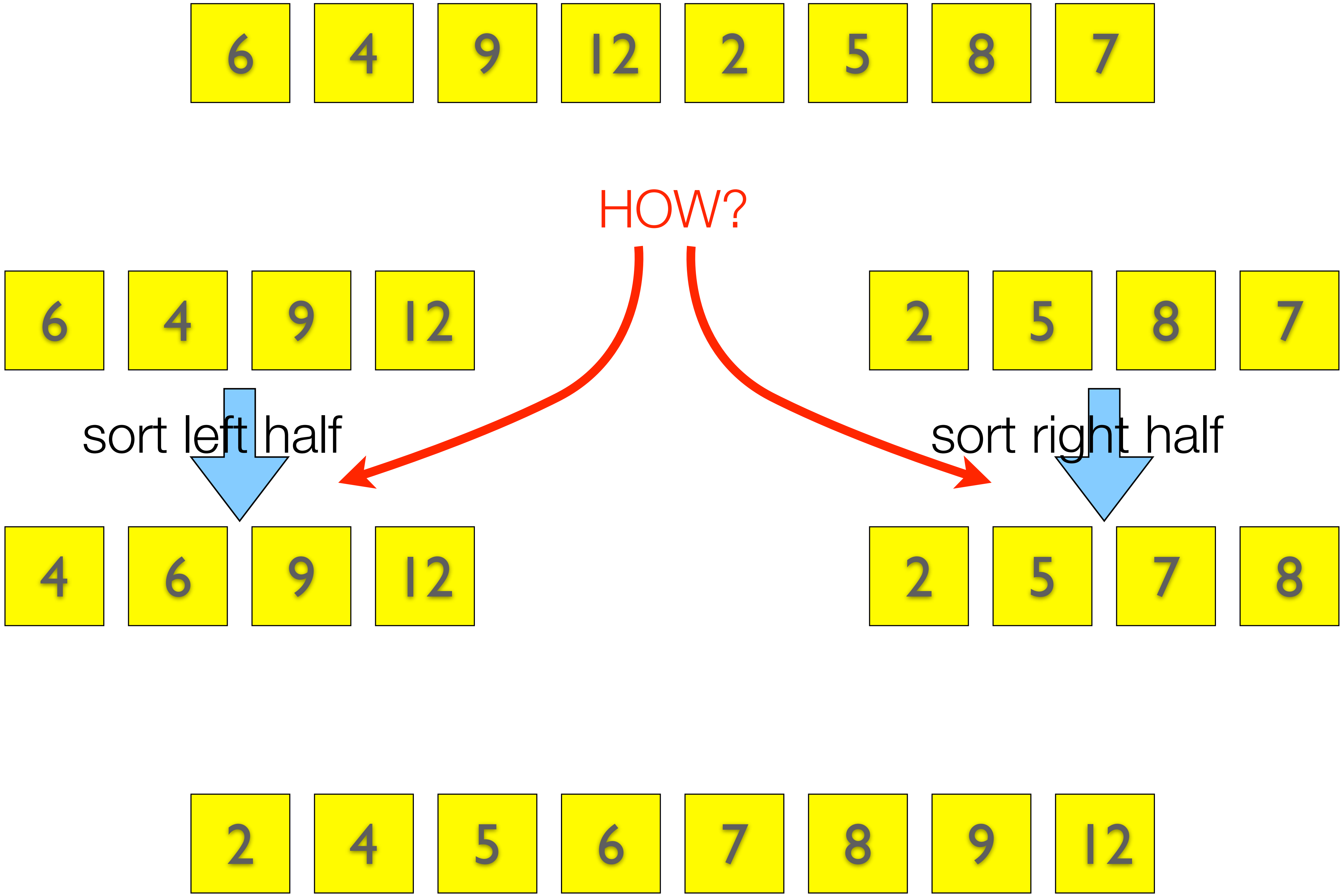
mergesort



mergesort



mergesort


6
4
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HOW?
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sort left half
sort right half
4
6
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12
2
5
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7
8
9
12

mergesort(`A,start,end`)

❶

❷

❸

❹

❺

mergesort(A,start,end)

```
1  if start < end

2      q ← ⌊(start + end)/2⌋

3      mergesort(A,start,q)
       mergesort(A,q+1,end)

4      merge(A,start,q,end)

5  else ... base case
```

mergesort(`A,start,end`)

❶ `if start < end`

❷ $q \leftarrow \lfloor (\text{start} + \text{end})/2 \rfloor$

❸ `mergesort(A,start,q)`
`mergesort(A,q+1,end)`

❹ `merge(A,start,q,end)`

❺ `else ...`

```
MERGE(A[1..n], m):
  i ← 1; j ← m + 1
  for k ← 1 to n
      if j > n
          B[k] ← A[i]; i ← i + 1
      else if i > m
          B[k] ← A[j]; j ← j + 1
      else if A[i] < A[j]
          B[k] ← A[i]; i ← i + 1
      else
          B[k] ← A[j]; j ← j + 1
  for k ← 1 to n
      A[k] ← B[k]
```

jeff erickson

mergesort(A,start,end)

running time?

1. `if start < end`
2. $q \leftarrow \lfloor (\text{start} + \text{end})/2 \rfloor$ → $O(1)$
3. `mergesort(A,start,q)` → $T(n/2)$
   `mergesort(A,q+1,end)` → $T(n/2)$
4. `merge(A,start,q,end)` → $\Theta(n)$
5. `else ...`

$$\begin{cases} T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n) \\ T(1) = 2 \end{cases}$$

$$T(n) = 2T(n/2) + n$$

goal is to show $T(n) = \Theta(n \log n)$

show: $T(n) \le n \log n$

Proof: Base case holds for $n \le 5$. Assume that the hypothesis holds for all $k \le n$. Consider

$$T(n+1) = 2T\left(\frac{n+1}{2}\right) + (n+1)$$

$\frac{n+1}{2} < n, \Rightarrow T\left(\frac{n+1}{2}\right) \le \left(\frac{n+1}{2}\right) \log\left(\frac{n+1}{2}\right)$ by ind hypothesis

$$\le 2\left(\frac{n+1}{2}\right) \log\left(\frac{n+1}{2}\right) + n+1$$

$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$

$$= (n+1)\left[\log(n+1) - 1\right] + n+1$$

$$= (n+1)\log(n+1) - (n+1) + n+1$$

$$= (n+1)\log(n+1)$$

$$T(n) = 2T(n/2) + n$$

prove:

hypothesis:

base case:

inductive step:

$$T(n) = 2T(n/2) + n$$

prove: $T(n) = O(n \log n)$

property: $T(n) < cn \log n$ for c>1

base case:

inductive step:

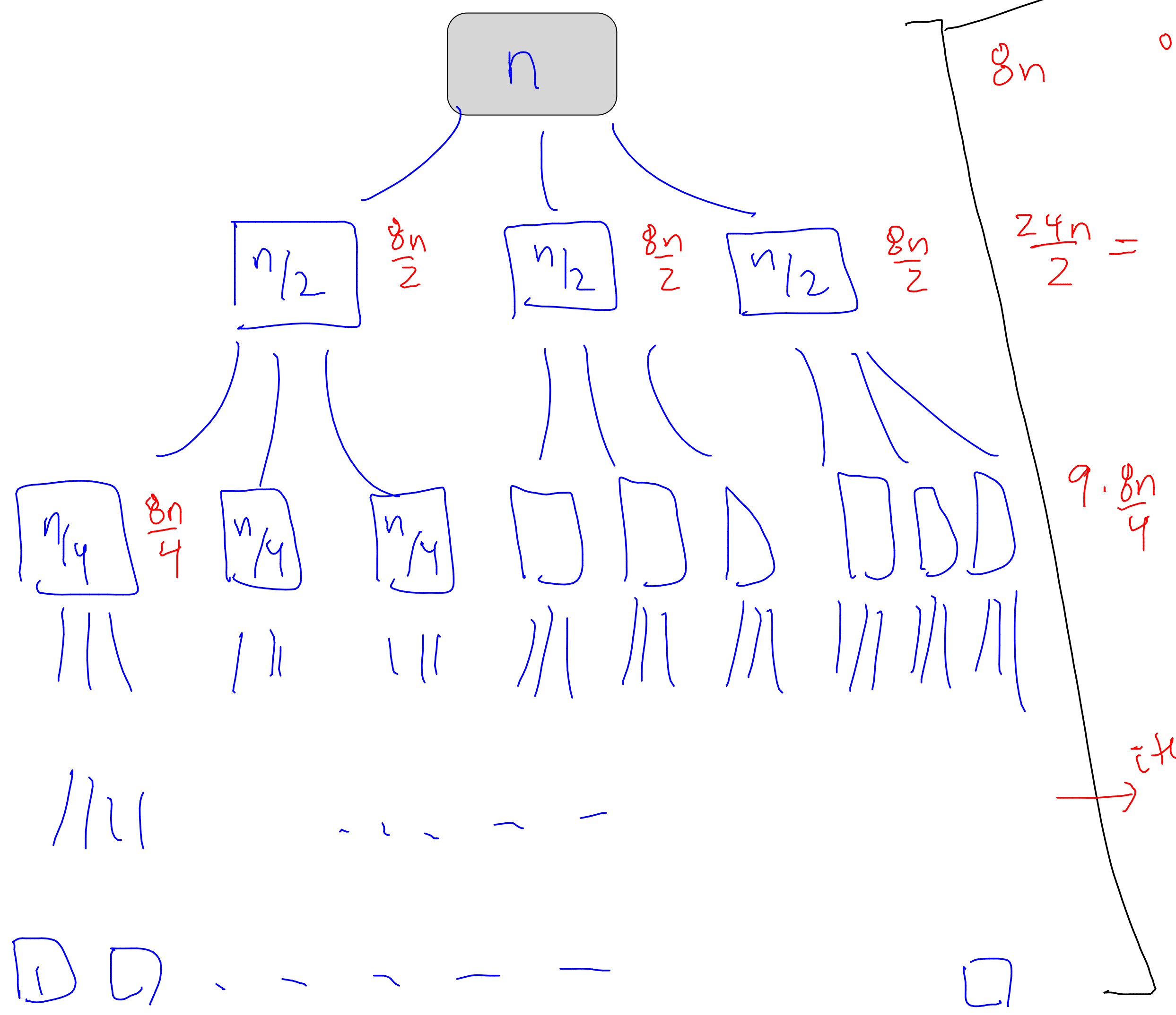
$$T(n) = 3T(n/2) + 10n$$

Karatsuba

$O(n^{1.589})$ 1.6

$O(n^{\log_2(3)})$ tree

$\lceil \log_2 n \rceil$



$8n$ 0

$\frac{24n}{2} = \left(\frac{3}{2}\right) 8n$ 1

$9 \cdot \frac{8n}{4} = \left(\frac{9}{4}\right) \cdot 8n$ 2

i^{th} level amount of work $= \left(\frac{3}{2}\right)^i \cdot 8n$

$$T(n) = 3T(n/2) + 10n$$ (guess +chk)

Prove that $T(n) \leq n^{\log_2 3} - 20n$. This implies $T(n) = O(n^{\log_2 3})$

① Base case holds for small n. Assume that the statement holds for all $k \leq n$.

Consider $T(n+1) =$

$$T(n) = 3T(n/2) + 10n$$ (guess +chk)

Lets prove that $T(n) \leq n^{\log_2 3} - 20n$

$$T(n) = 3T(n/2) + 10n$$ (guess +chk)

Lets prove that $T(n) \leq n^{\log_2 3} - 20n$

By inspection, indeed, $T(n) \leq n^{\log_2 3} - 20n$ when n<1024.

$$T(n) = 3T(n/2) + 10n$$ (guess +chk)

Lets prove that $T(n) \le n^{\log_2 3} - 20n$

By inspection, indeed, $T(n) \le n^{\log_2 3} - 20n$ when n<1024.

A1: Lets assume that $T(n) \le n^{\log_2 3} - 20n$ when $n < n_0$

$T(n_0+1) = 3T\left(\frac{n_0+1}{2}\right) + 10(n_0+1)$

$\le 3\left[\left(\frac{n_0+1}{2}\right)^{\log_2 3} - 20\left(\frac{n_0+1}{2}\right)\right] + 10(n_0+1)$ by A1 b/c $\frac{n_0+1}{2} \le n_0$

-200

$= 3\left(\frac{(n_0+1)^{\log_2 3}}{2^{\log_2 3}}\right) - 30(n_0+1) + 10(n_0+1)$

$2^{\log_2 3} = 3$

$= (n_0+1)^{\log_2 3} - 20(n_0+1)$

-240

$$T(n) = 3T(n/2) + 10n$$ (guess +chk)

Lets prove that $T(n) \le n^{\log_2 3} - 20n$

By inspection, indeed, $T(n) \le n^{\log_2 3} - 20n$ when n<1024.

A1: Lets assume that $T(n) \le n^{\log_2 3} - 20n$ when $n < n_0$

Consider the case of $T(n_0 + 1)$

$$T(n_0 + 1) = 3T((n_0 + 1)/2) + 10(n_0 + 1)$$ By definition

But since $(n_0 + 1)/2 < n_0$ and A1, it follows that

$$T(n_0 + 1) = 3\left[\left(\frac{n_0 + 1}{2}\right)^{\log_2 3} - 20\left(\frac{n_0 + 1}{2}\right)\right] + 10(n_0 + 1)$$

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$$T(n_0+1) = 3\left[\left(\frac{n_0+1}{2}\right)^{\log_2 3} - 20\left(\frac{n_0+1}{2}\right)\right] + 10(n_0+1)$$

$$< (n_0+1)^{\log_2 3} - 30(n_0+1) + 10(n_0+1)$$

$$< (n_0+1)^{\log_2 3} - 20(n_0+1)$$

$$T(n_0+1) = 3\left[\left(\frac{n_0+1}{2}\right)^{\log_2 3} - 20\left(\frac{n_0+1}{2}\right)\right] + 10(n_0+1)$$

$$< (n_0+1)^{\log_2 3} - 30(n_0+1) + 10(n_0+1)$$

$$< (n_0+1)^{\log_2 3} - 20(n_0+1)$$

This expression matches our Assumption A1.

A1: Lets assume that $T(n) < n^{\log_2 3} - 20n_0$ when $n < n_0$

Thus, we can conclude the proof via induction.

This establishes that $T(n) = O(n^{\log_2 3})$

Induction summary

1 $T(n) < n^{\log_2 3} - 20n_0$ IS TRUE for one case.

2 $T(n) < n^{\log_2 3} - 20n_0$ Suppose TRUE for $n < n_0$

3 Showed that 1,2 imply that

$$T(n_0 + 1) < (n+1)^{\log_2 3} - 20(n+1)$$

4 (Induction)

What happens if
we skip the -20n?

$$T(n) = 3T(n/2) + 10n$$ (guess +chk)

Lets prove that $T(n) \leq n^{\log_2 3} - 20n$

By inspection, indeed, $T(n) \leq n^{\log_2 3} - 20n$ when n<1024.

A1: Lets assume that $T(n) \leq n^{\log_2 3} - 20n$ when $n < n_0$

Consider the case of $T(n_0 + 1)$

$$T(n_0+1) = 3T((n_0+1)/2) + 10(n_0+1)$$ By definition

But since $(n_0+1)/2 < n_0$ and A1, it follows that

$$T(n_0+1) = 3\left[\left(\frac{n_0+1}{2}\right)^{\log_2 3} - 20\left(\frac{n_0+1}{2}\right)\right] + 10(n_0+1)$$

$$T(n_0+1) = 3\left[\left(\frac{n_0+1}{2}\right)^{\log_2 3} - 20\left(\frac{n_0+1}{2}\right)\right] + 10(n_0+1)$$

$$< (n_0+1)^{\log_2 3} + 10n_0$$

This expression DOES NOT matches our Assumption A1.
So the induction STOPS!

$$T(n) = 8T(n/2) + \Theta(n^2) \text{ (guess +chk)}$$

$$= 8T(n/2) + dn^2 \qquad d > 0$$

Prove: $T(n) \le n^3 - cn^2$

- Spse base case holds. Assume true for $k < n$.

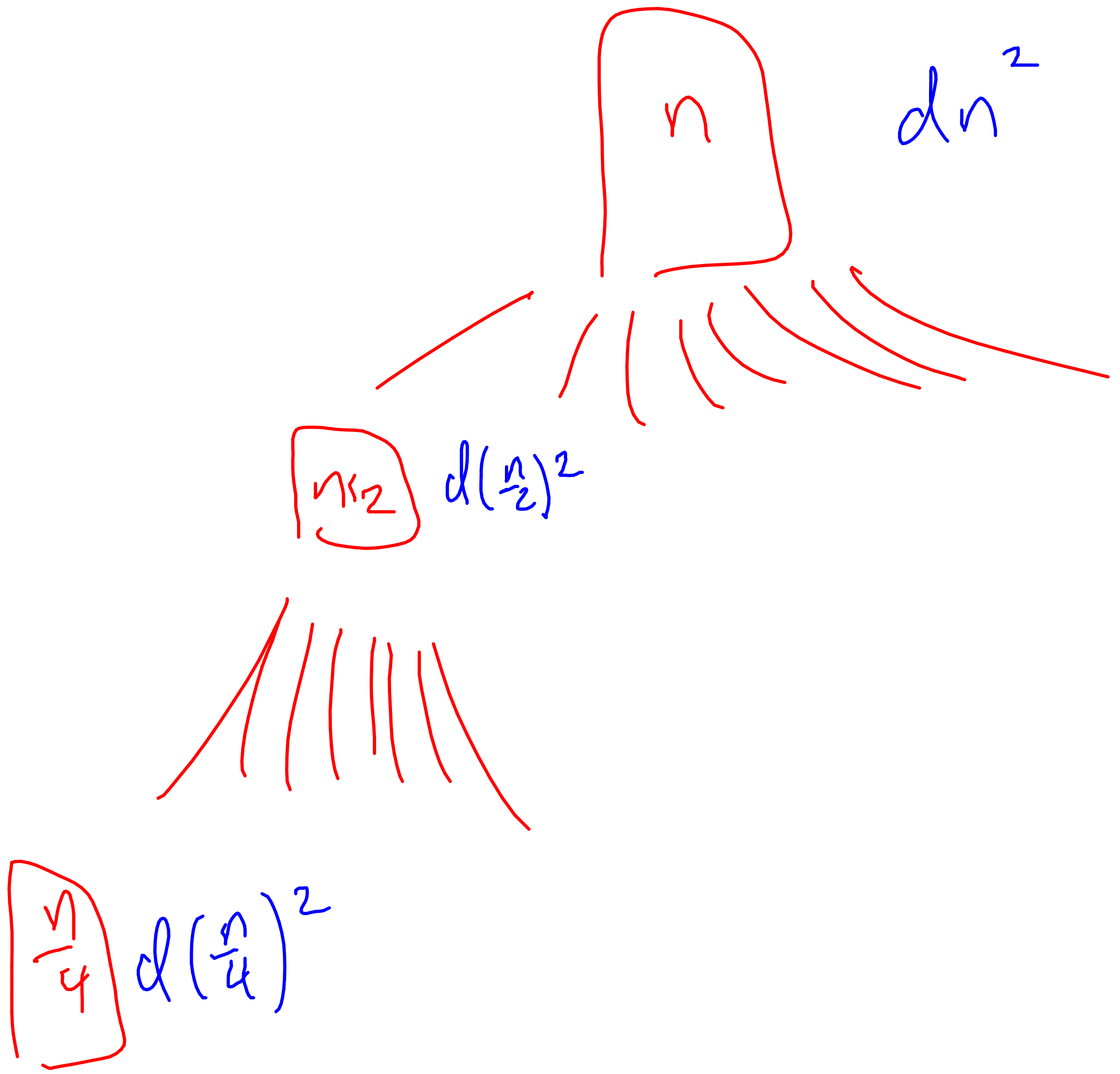
Consider $T(n+1) = 8T\left(\frac{n+1}{2}\right) + d(n+1)^2$

$$\le 8\left(\left(\frac{n+1}{2}\right)^3 - c\left(\frac{n+1}{2}\right)^2\right) + d(n+1)^2$$

$$= (n+1)^3 - 2c(n+1)^2 + d(n+1)^2$$

Works when $c \ge d$.

$$T(n) = 8T(n/2) + dn$$



$$dn^2 = dn^2 = 1dn^2$$

$$8 \cdot d\left(\frac{n}{2}\right)^2 = \frac{8}{2^2} dn^2 = 2dn^2$$

$$64 \cdot d\left(\frac{n}{4}\right)^2 = \frac{8^2}{4^2} dn^2 = 4dn^2$$

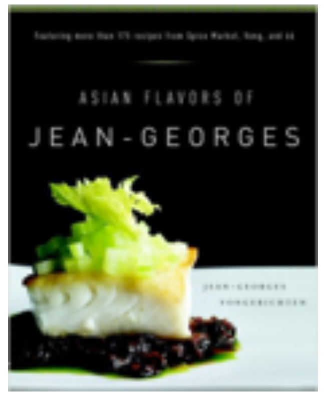
$$8^3 \cdot d\left(\frac{n}{8}\right)^2 = \frac{8^3}{8^2} dn^2 = 8dn^2$$

$$\vdots$$

$$8^{\log_2 n} d\left(\frac{n}{n}\right)^2 = (2^3)^{\log_2 n} \cdot d$$

$$n \cdot dn^2$$

?-✓



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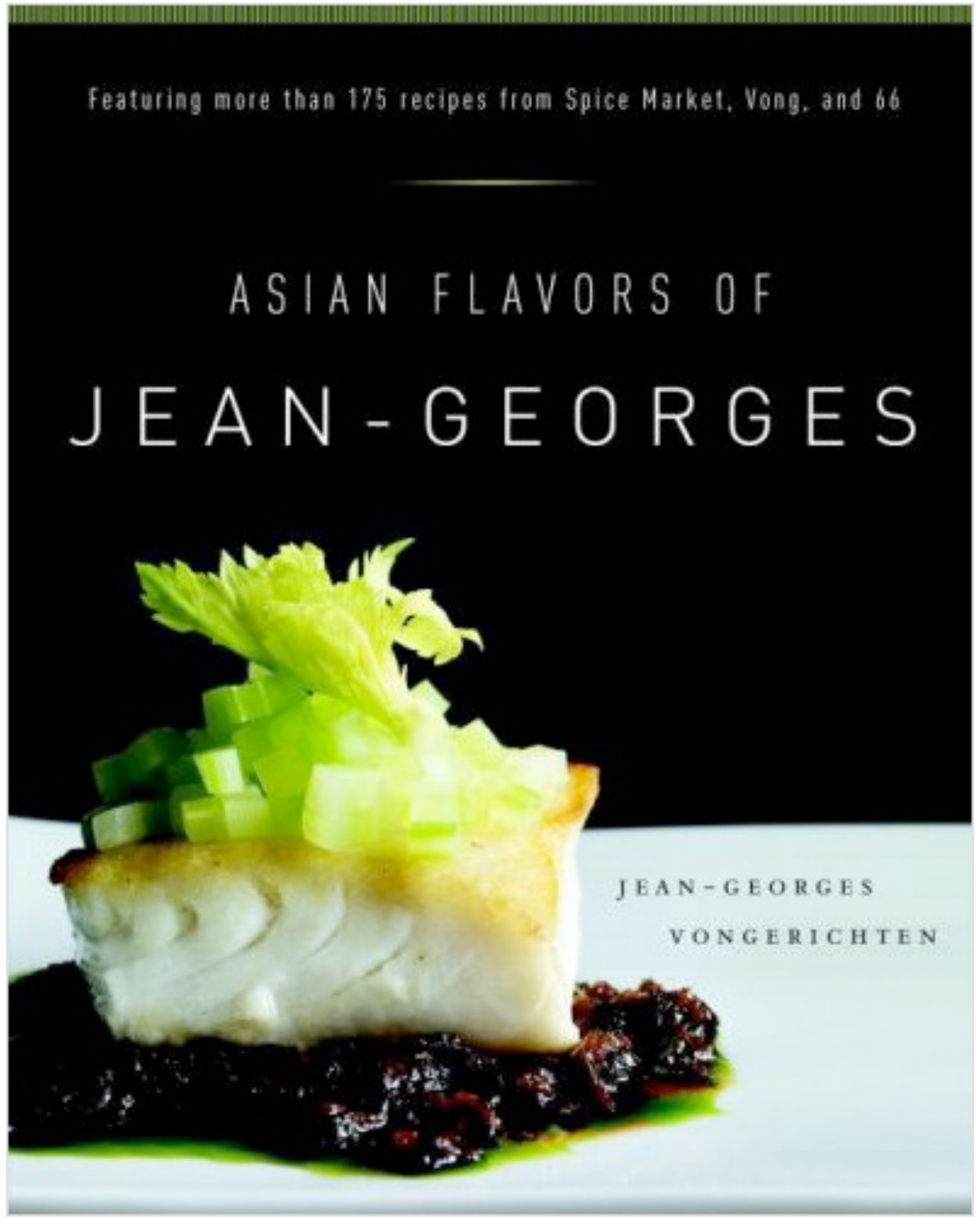
cookbook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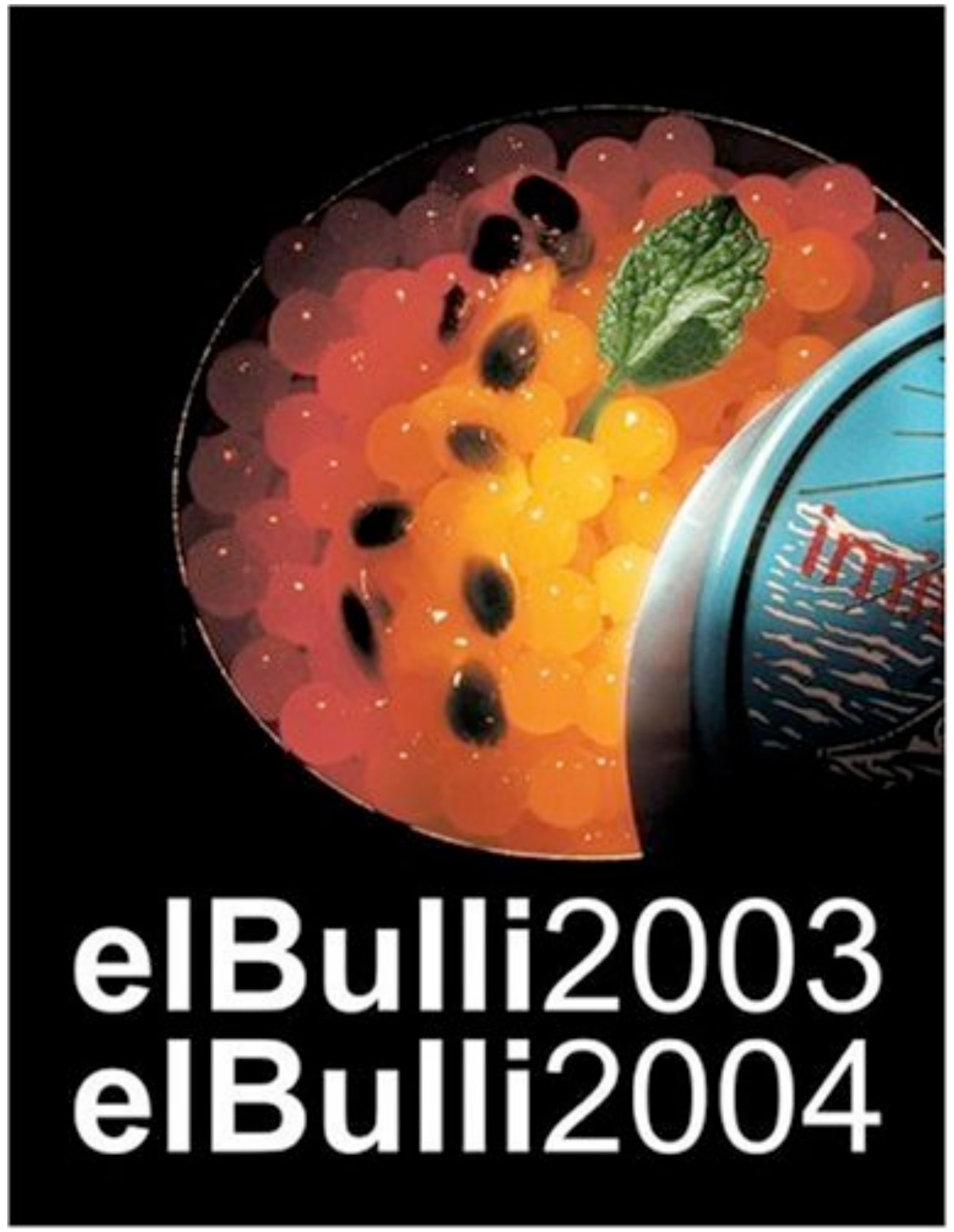
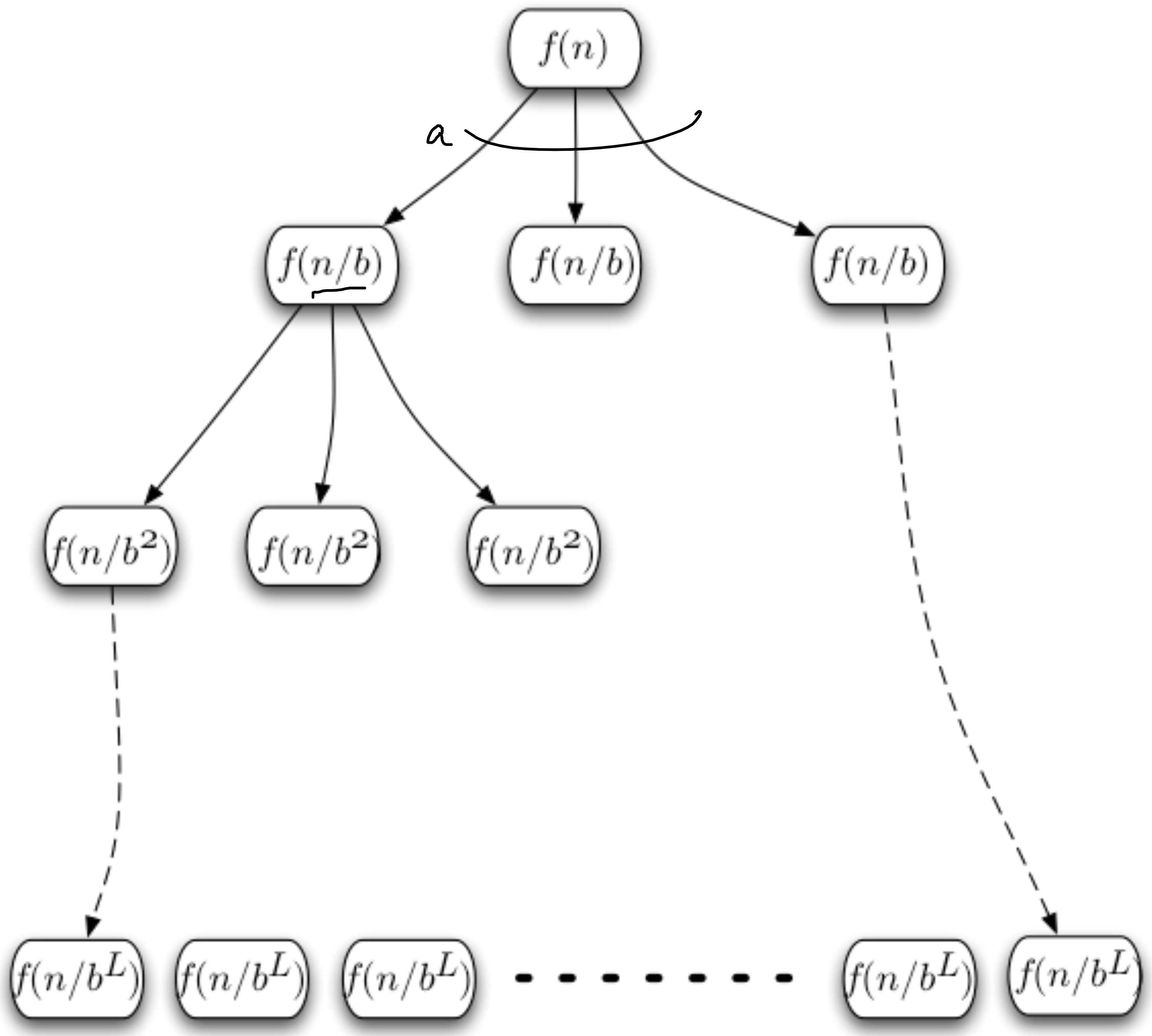

elBulli2003
elBulli2004

$$T(n) = aT(n/b) + f(n)$$

"divide problem into a smaller problems of size $\frac{n}{b}$ and use $f(n)$ time to combine solutions"

$$T(n) = aT(n/b) + f(n)$$



$f(n)$

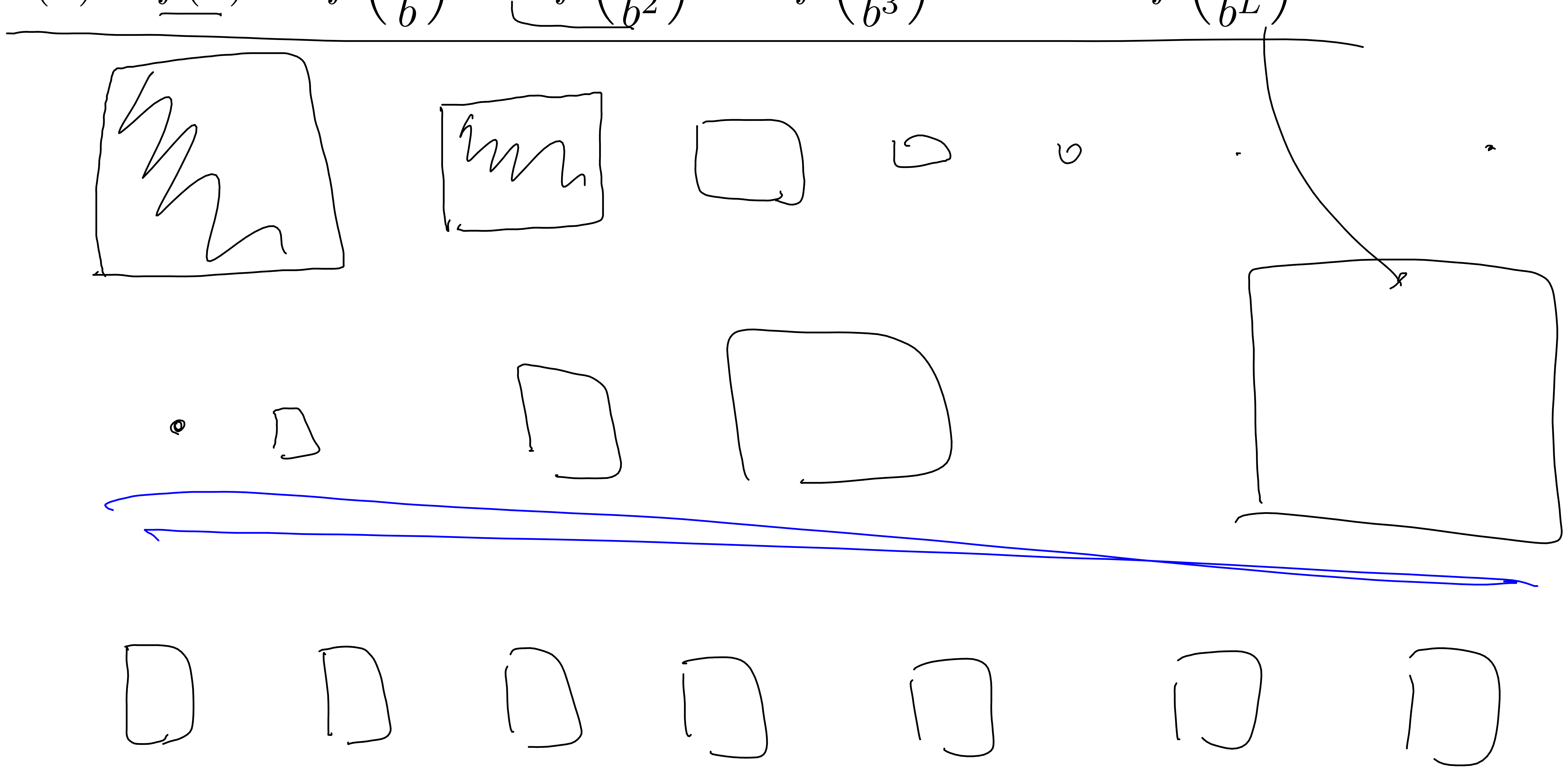
$a \cdot f\left(\frac{n}{b}\right)$

$a^2 \cdot f\left(\frac{n}{b^2}\right)$

$\vdots$

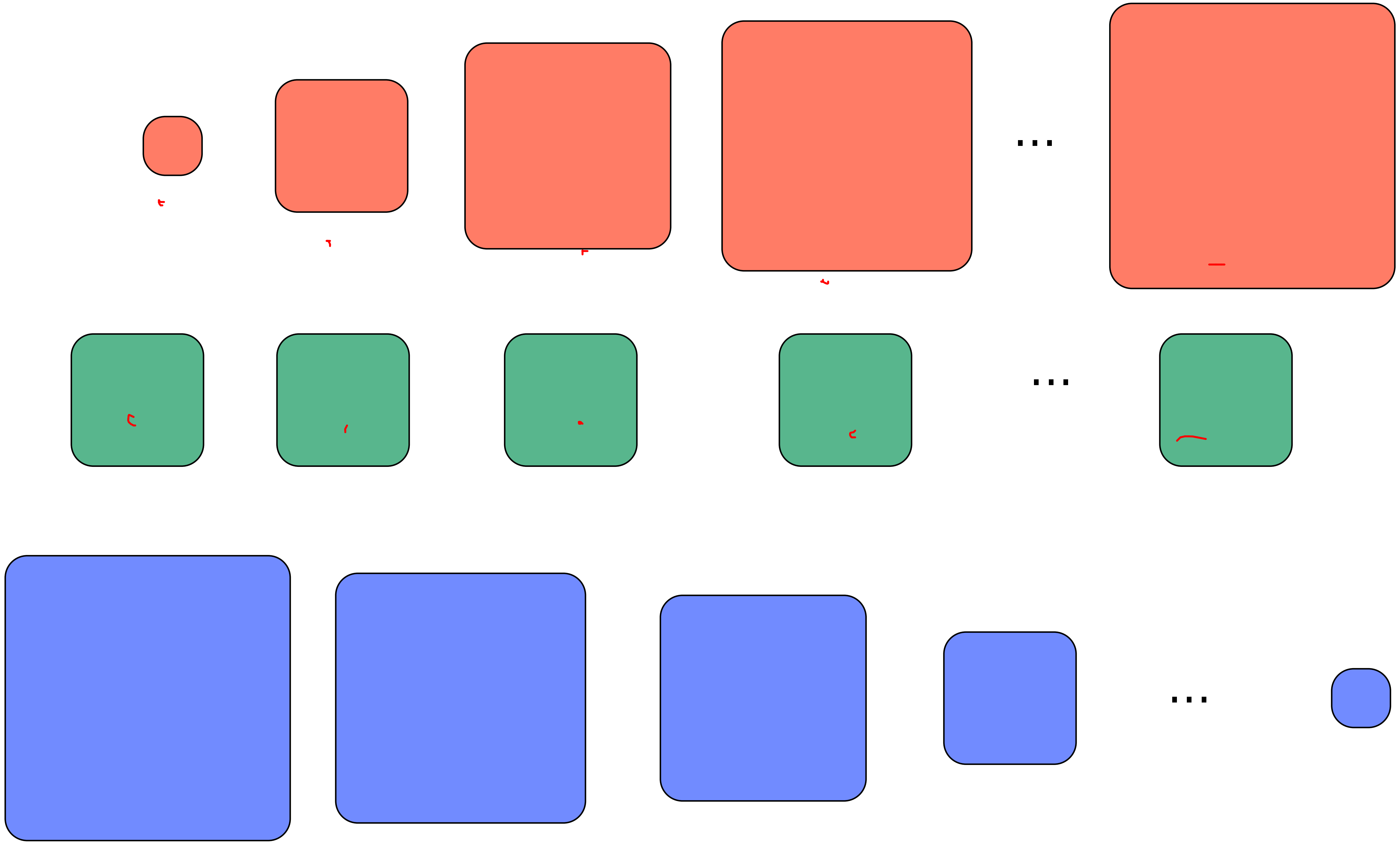
$a^L \cdot f\left(\frac{n}{b^L}\right)$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$



$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

Masters thm



case 1

2

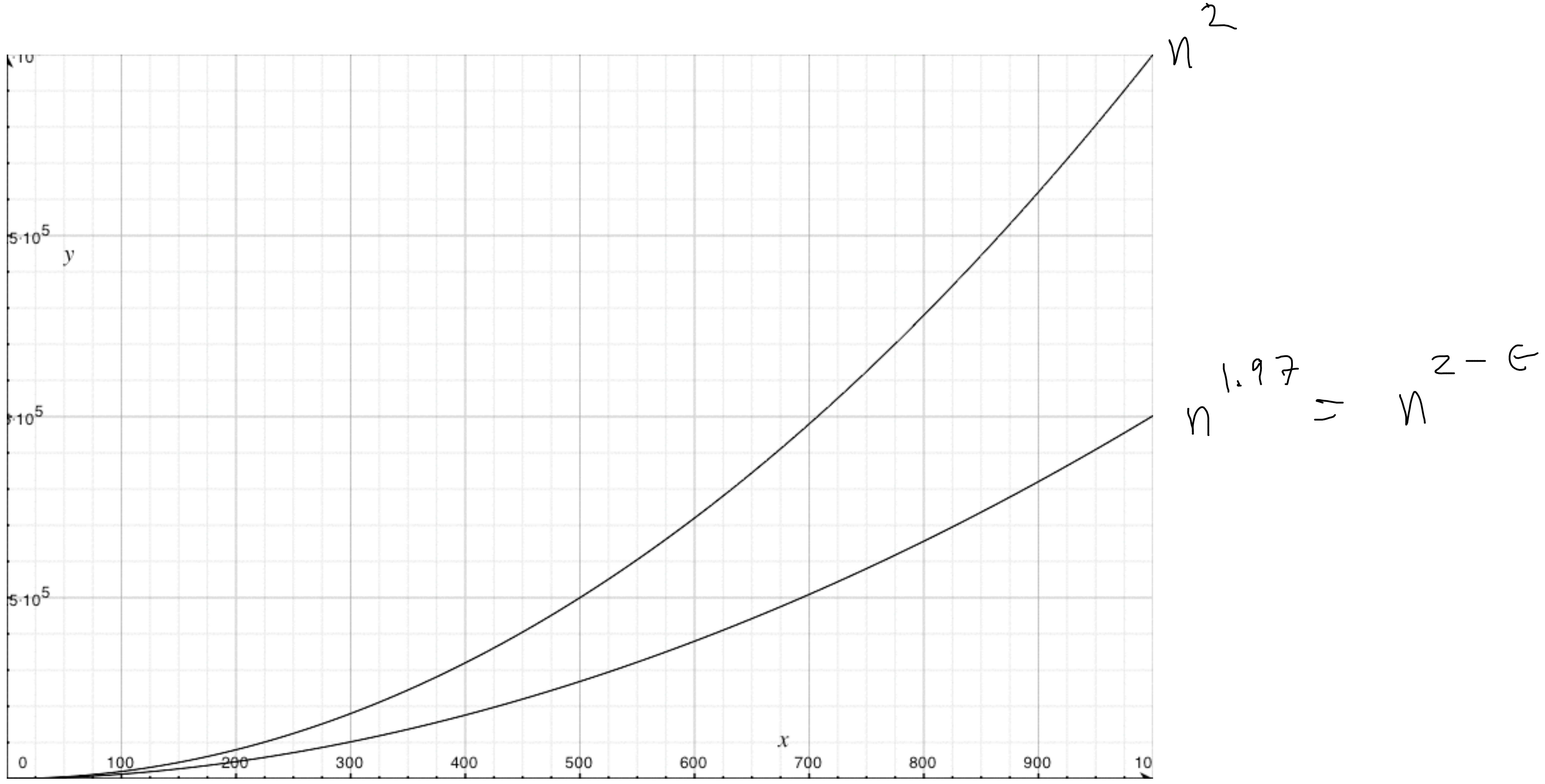
3

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1: $$f(n) = O(n^{\log_b a - \epsilon})$$

$\epsilon > 0$

then $T(n) = \Theta(n^{\log_b a})$



n^2
$n^{1.97} = n^{2-\epsilon}$
y
x
0
100
200
300
400
500
600
700
800
900

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1: $f(n) = O(n^{\log_b a - \epsilon})$

example: $T(n) = 4T(n/2) + n$

(a = 4, b = 2; $n^{1.999}$)

$f(n) = n \Rightarrow T(n) = \Theta(n^2)$

GOAL:

is $f(n) \overset{??}{=} O(n^{\log_2 4 - \epsilon})$ for some $\epsilon > 0$ ($\log_2 4 = 2$)

Yes $n = O(n^{2-\epsilon})$ for $\epsilon = 0.1$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1: $f(n) = O(n^{\log_b a - \epsilon})$

$$T(n) \le c \cdot n^{\log_b a - \epsilon} + c \cdot a\left(\frac{n}{b}\right)^{\log_b a - \epsilon} + c \cdot a^2\left(\frac{n}{b^2}\right)^{\log_b a - \epsilon} + \cdots + c \cdot a^{L-1}\left(\frac{n}{b^{L-1}}\right)^{\log_b a - \epsilon} + a^L f(1)$$

$$= c \cdot n^{\log_b a - \epsilon}\left[1 + \frac{a}{b^{\log_b a - \epsilon}} + \frac{a^2}{(b^2)^{\log_b a - \epsilon}} + \cdots + \frac{a^{L-1}}{(b^{L-1})^{\log_b a - \epsilon}}\right] + a^L \cdot d$$

$$= c n^{\log_b a - \epsilon}\left[1 + \frac{a}{\frac{a}{b^{\epsilon}}} + \frac{a^2}{\frac{a^2}{b^{2\epsilon}}} + \cdots + \frac{a^{L-1}}{\frac{a^{L-1}}{b^{(L-1)\epsilon}}}\right] + a^L$$

$$= c n^{\log_b a - \epsilon}\left[1 + b^{\epsilon} + b^{\epsilon 2} + \cdots b^{\epsilon \cdot L-1}\right] + a^L$$

$$1 + b^{\epsilon} + b^{2\epsilon} + \cdots + b^{(L-1)\epsilon} = \frac{b^{L\cdot\epsilon} - 1}{b^{\epsilon} - 1} = \frac{n^{\epsilon} - 1}{b^{\epsilon} - 1}$$

$$a^{L} = a^{\log_b n} = \left(b^{\log_b a}\right)^{\log_b n} = n^{\log_b a}$$

$$b^{L} = b^{\log_b n} = n$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1 (cont):

$$T(n) \le cn^{\log_b a - \epsilon}\left[1 + b^\epsilon + b^{2\epsilon} + \cdots + b^{\epsilon(L-1)}\right] + n^{\log_b a}$$

$$= c \cdot n^{\log_b a - \epsilon}\left[n^\epsilon - 1 / b^\epsilon - 1 \right] + n^{\log_b a}$$

$$= c \cdot n^{\log_b a} \cdot (b^\epsilon - 1) + n^{\log_b a} = O(n^{\log_b a})$$

upper

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1: Lower bound

We have:

$$T(n) \geq a^L f\left(\frac{n}{b^L}\right)$$

$> n^{\log_b a}$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1: Lower bound

We have:

$$T(n) \geq a^L f\left(\frac{n}{b^L}\right)$$

$$\geq a^{\log_b(n)} = (b^{\log_b a})^{\log_b(n)}$$

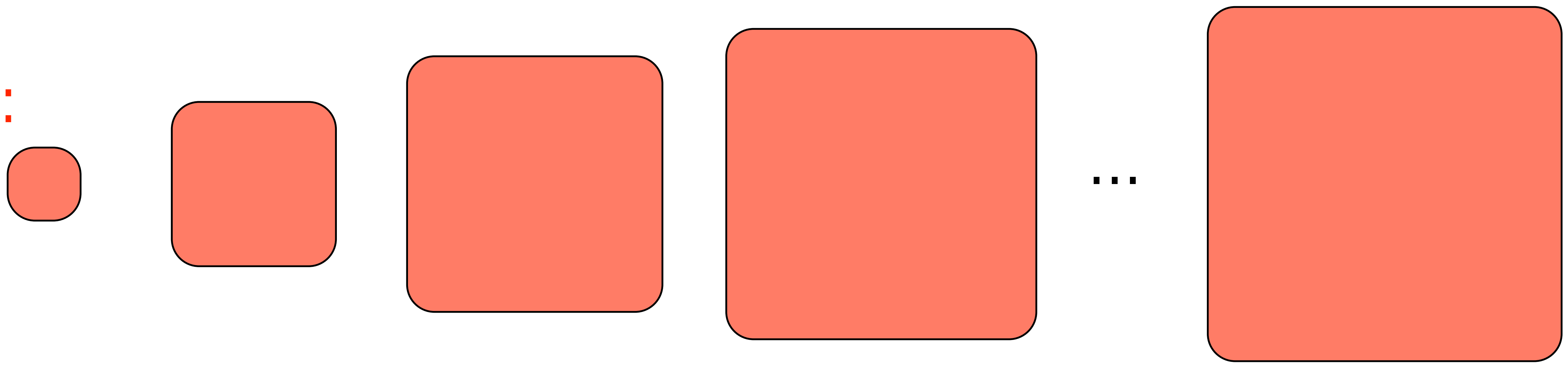
$$= n^{\log_b(a)}$$

$$= \Omega(n^{\log_b(a)})$$

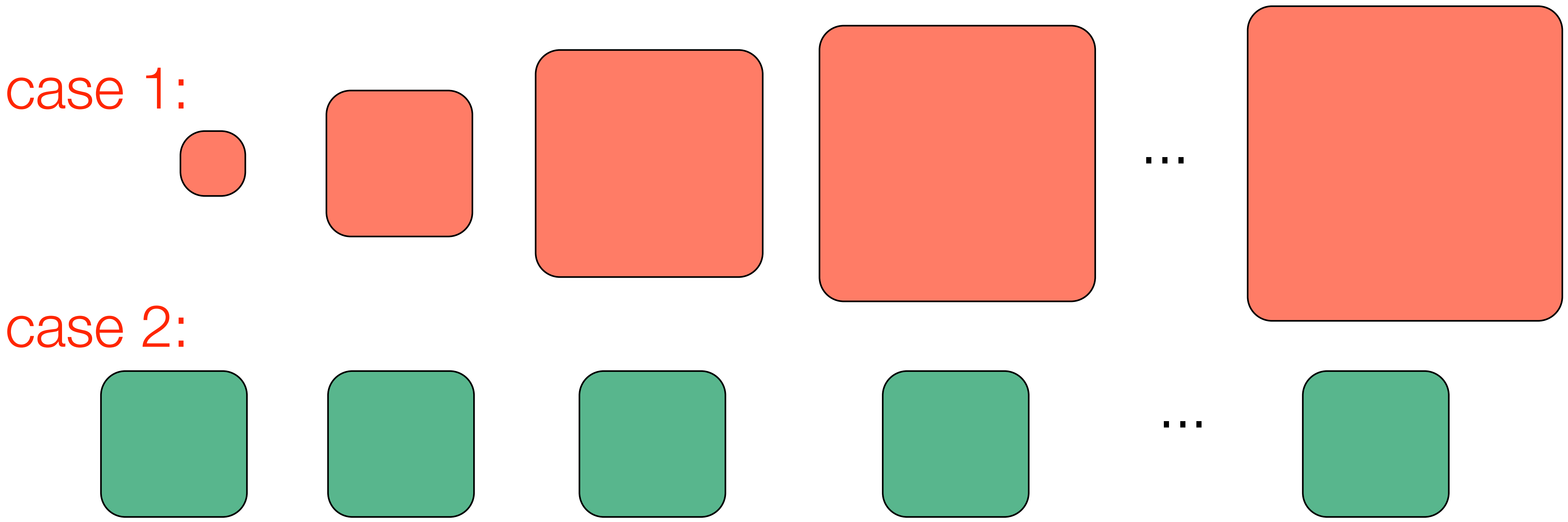
$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1:



$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$



$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

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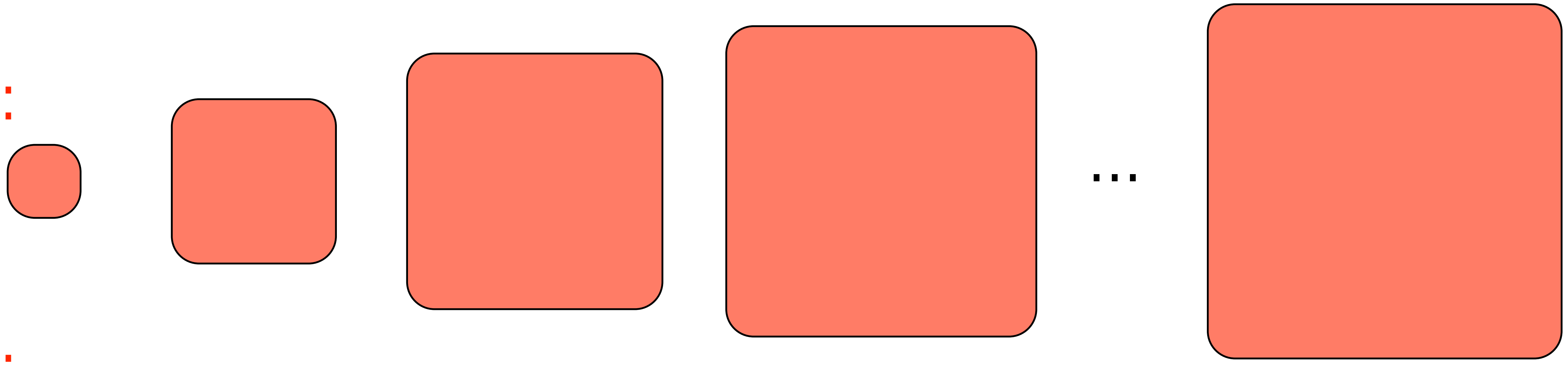
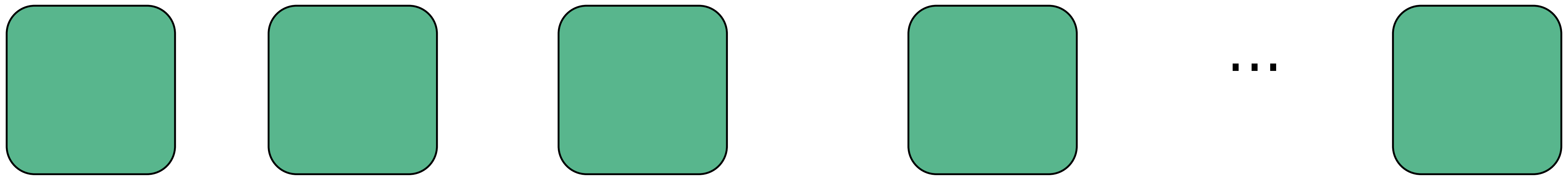
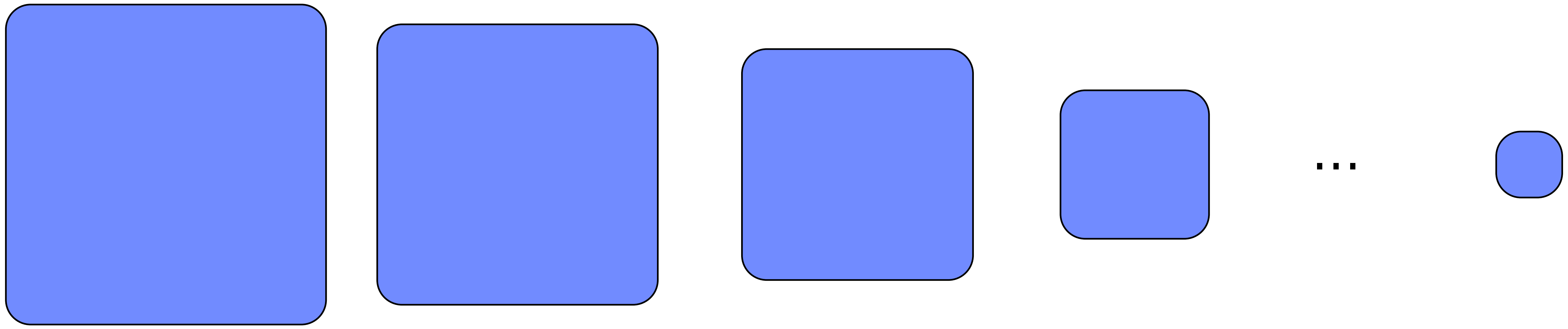


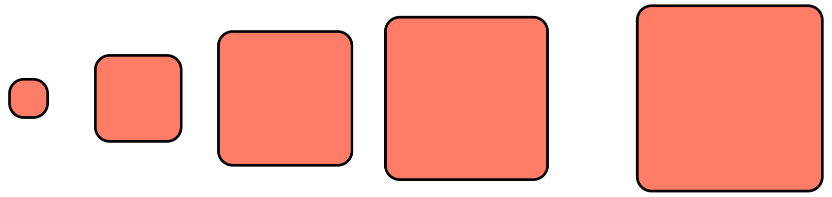
case 2:



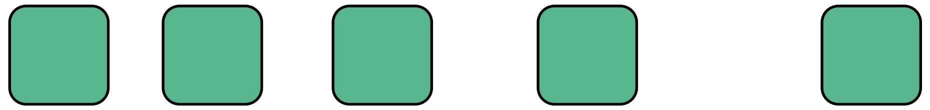
case 3:



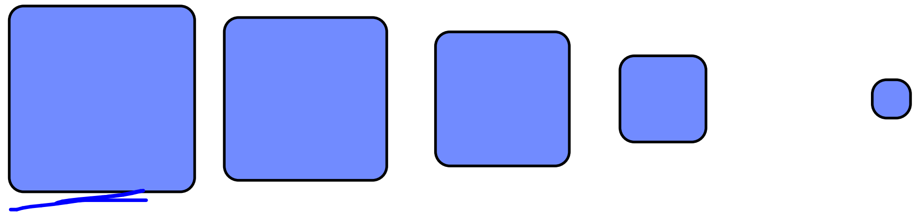
$$T(n) = aT(n/b) + f(n)$$

case 1: 

$$f(n) = O(n^{\log_b a - \epsilon}), \epsilon > 0$$

case 2: 

$$f(n) = \Theta(n^{\log_b a})$$

case 3: 

$$f(n) = \Omega(n^{\log_b a + \epsilon}), \epsilon > 0$$

and c<1 s.t

$$a \cdot f(\tfrac{n}{b}) \le c \cdot f(n)$$

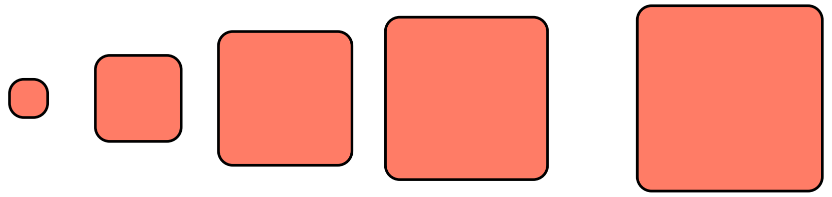
Then:

$$T(n) = \Theta(n^{\log_b a})$$

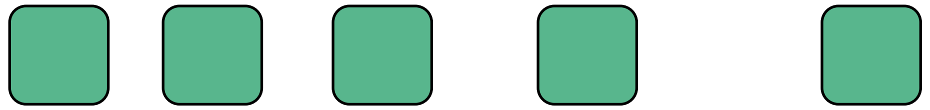
$$T(n) = \Theta(n^{\log_b a} \cdot \log n)$$

$$T(n) = \Theta(f(n))$$

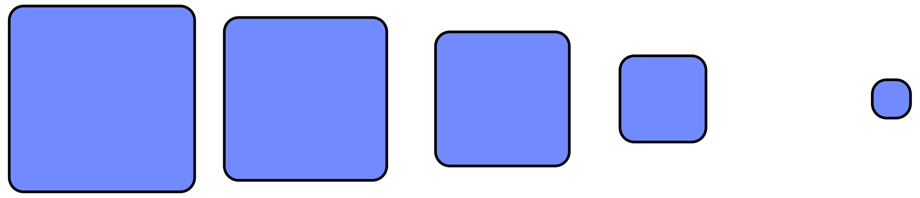
$$T(n) = aT(n/b) + f(n)$$

case 1: 

$$f(n) = O(n^{\log_b a - \epsilon}), \epsilon > 0$$

case 2: 

$$f(n) = \Theta(n^{\log_b a})$$

case 3: 

$$f(n) = \Omega(n^{\log_b a + \epsilon}), \epsilon > 0$$

and c<1 s.t $af(n/b) < cf(n)$

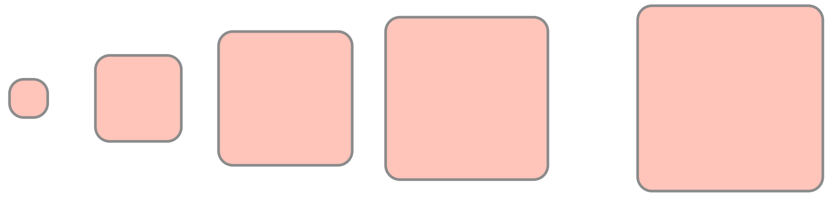
Then:

$$T(n) = \Theta(n^{\log_b a})$$

$$T(n) = \Theta(n^{\log_b a} \log n)$$

$$T(n) = \Theta(f(n))$$

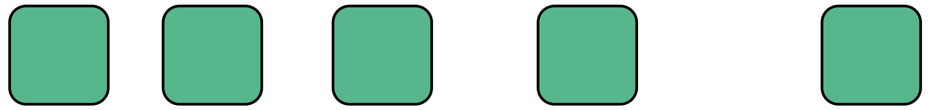
$T(n) = aT(n/b) + f(n)$

case 1: 

Then:

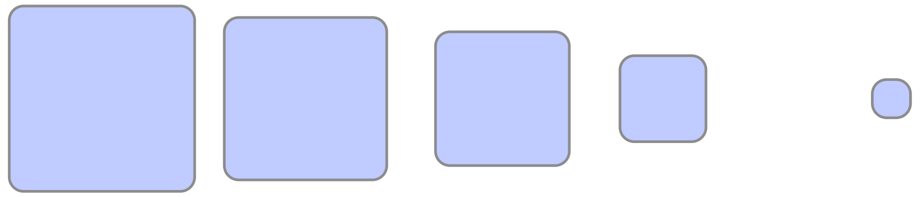
$f(n) = O(n^{\log_b a - \epsilon}), \epsilon > 0$

$T(n) = \Theta(n^{\log_b a})$

case 2: 

$f(n) = \Theta(n^{\log_b a})$

$T(n) = \Theta(n^{\log_b a} \log n)$

case 3: 

$f(n) = \Omega(n^{\log_b a + \epsilon}), \epsilon > 0$

$T(n) = \Theta(f(n))$

and c<1 s.t $af(n/b) < cf(n)$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 2 : When $f(n) < cn^{\log_b a}$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 2 : When $f(n) < cn^{\log_b a}$

$$T(n) \leq cn^{\log_b(a)}\left[1 + \frac{a}{b^{\log_b(a)}} + \frac{a^2}{(b^2)^{\log_b(a)}} + \cdots + \frac{a^{L-1}}{(b^{L-1})^{\log_b(a)}}\right] + cn^{\log_b(a)}$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

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$$T(n) \leq cn^{\log_b(a)}\left[1 + 1 + \cdots + 1\right] + cn^{\log_b(a)}$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 2 : When $f(n) < cn^{\log_b a}$

$$T(n) \leq cn^{\log_b(a)} \left[1 + \frac{a}{b^{\log_b(a)}} + \frac{a^2}{(b^2)^{\log_b(a)}} + \cdots + \frac{a^{L-1}}{(b^{L-1})^{\log_b(a)}}\right] + cn^{\log_b(a)}$$

$$T(n) \leq cn^{\log_b(a)} [1 + 1 + \cdots + 1] + cn^{\log_b(a)}$$

$$\leq cn^{\log_b(a)} [\log_b(n)]$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 2 : When $f(n) < cn^{\log_b a}$

$$T(n) \leq cn^{\log_b(a)} \left[1 + \frac{a}{b^{\log_b(a)}} + \frac{a^2}{(b^2)^{\log_b(a)}} + \cdots + \frac{a^{L-1}}{(b^{L-1})^{\log_b(a)}}\right] + cn^{\log_b(a)}$$

$$T(n) \leq cn^{\log_b(a)} [1 + 1 + \cdots + 1] + cn^{\log_b(a)}$$

$$\leq cn^{\log_b(a)} [\log_b(n)]$$

$$= O(n^{\log_b a} \log n)$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 2 : When $f(n) > cn^{\log_b(a)}$

lower-bound

$$T(n) \geq cn^{\log_b(a)}\left[1 + \frac{a}{b^{\log_b(a)}} + \frac{a^2}{(b^2)^{\log_b(a)}} + \cdots + \frac{a^{L-1}}{(b^{L-1})^{\log_b(a)}}\right]$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 2 : When $f(n) > cn^{\log_b(a)}$

lower-bound

$$T(n) \geq cn^{\log_b(a)} \left[1 + \frac{a}{b^{\log_b(a)}} + \frac{a^2}{(b^2)^{\log_b(a)}} + \cdots + \frac{a^{L-1}}{(b^{L-1})^{\log_b(a)}}\right]$$

$$T(n) \geq cn^{\log_b(a)} [1 + 1 + \cdots + 1]$$

$$\geq cn^{\log_b(a)} \log_b(a)$$

$$\Omega(n^{\log_b(a)} \log_b(a))$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 3: $f(n) > n^{\log_b a+\epsilon}$ and c<1 s.t $af(n/b) < cf(n)$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 3: $f(n) > n^{\log_b a + \epsilon}$ and c<1 s.t $af(n/b) < cf(n)$

$$T(n) \leq f(n) + cf(n) + c^2 f(n) + \cdots + c^L f(n)$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 3: $f(n) > n^{\log_b a+\epsilon}$ and c<1 s.t $af(n/b) < cf(n)$

$$T(n) \leq f(n) + cf(n) + c^2 f(n) + \cdots + c^L f(n)$$

$$T(n) \leq f(n)[1 + c + c^2 + \cdots + c^L]$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 3: $f(n) > n^{\log_b a + \epsilon}$ and c<1 s.t $af(n/b) < cf(n)$

$$T(n) \leq f(n) + cf(n) + c^2 f(n) + \cdots + c^L f(n)$$

$$T(n) \leq f(n)[1 + c + c^2 + \cdots + c^L]$$

$$= O(f(n))$$

example 2: $T(n) = 8T(n/2) + \Theta(n^2)$

a b f

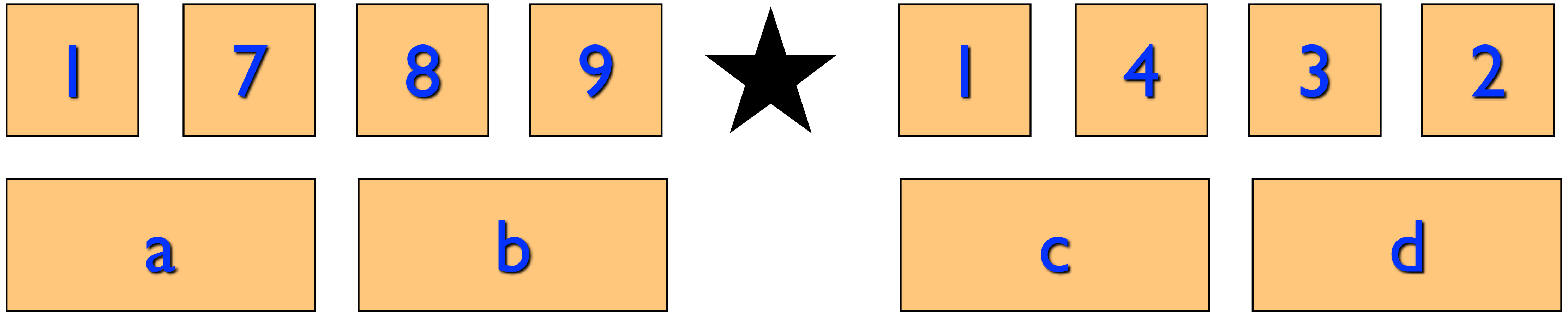
$f(n) = n^2$

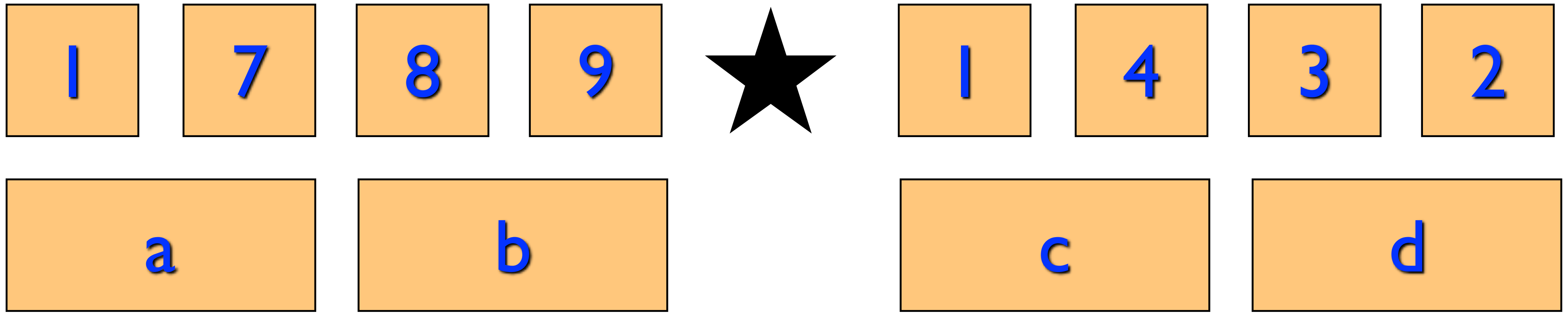
$O(n^{\log_2 8 - \epsilon})$ (3)

?? $\Rightarrow O(n^{3-\epsilon})$

Yes. Case I applies

$T(n) = \Theta(n^{\log_2 8}) = \Theta(n^3)$


1
7
8
9
1
4
3
2
a
b
c
d



$$T(n) = 4T(n/2) + 3O(n)$$

a b f

$$f = O(n) = O(n^{\log_2 4 - \epsilon})$$ yes

Case I. $T(n) = \Theta(n^2)$

$$T(n) = 2T(n/2) + n^3$$

a b f

$f(n) = n^3$ is $\Omega(n^{\log_2 2 + \epsilon}) = \Omega(n^{1+\epsilon})$

case 3 applies if regularity holds

$\exists\ c = \frac{1}{2}$

$a \cdot f(\frac{n}{b}) = 2 \cdot f(\frac{n}{2}) = 2 \cdot (\frac{n}{2})^3 = \frac{1}{4} n^3 \leq \frac{1}{2} n^3$ ✓

$c \cdot f(n)$

$$T(n) = \Theta(n^3)$$

$$T(n) = 7T(n/2) + O(n^2)$$

example:

$$T(n) = T\left(\frac{14}{17}n\right) + 24$$


8
13
FIFA

$$T(n) = 2T\left(\sqrt{n}\right) + \lg n$$