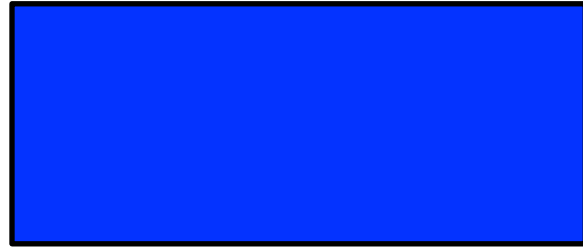


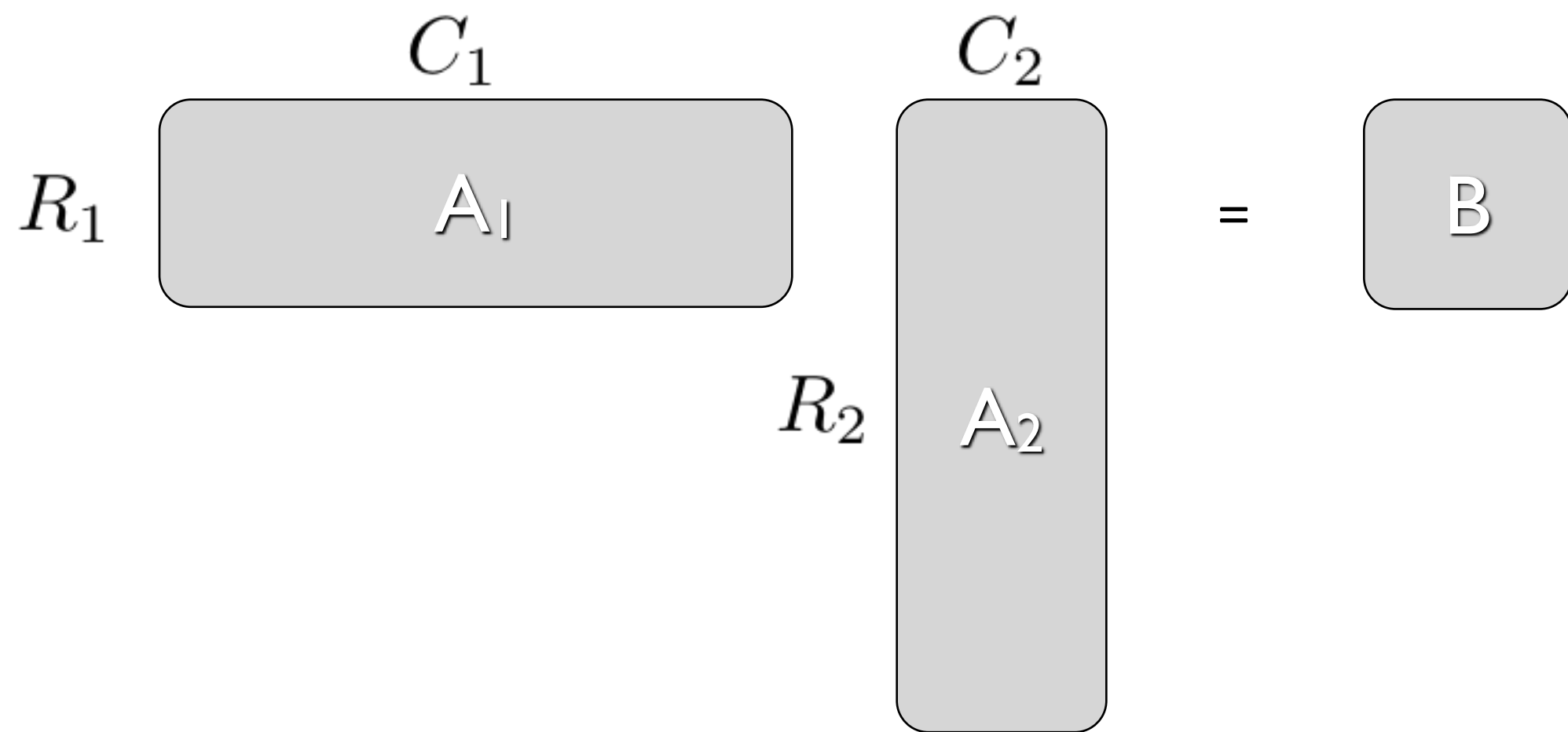
# La9

shelat  
16f-4800  
oct 7 2016

Dynamic Programming

# Matrix



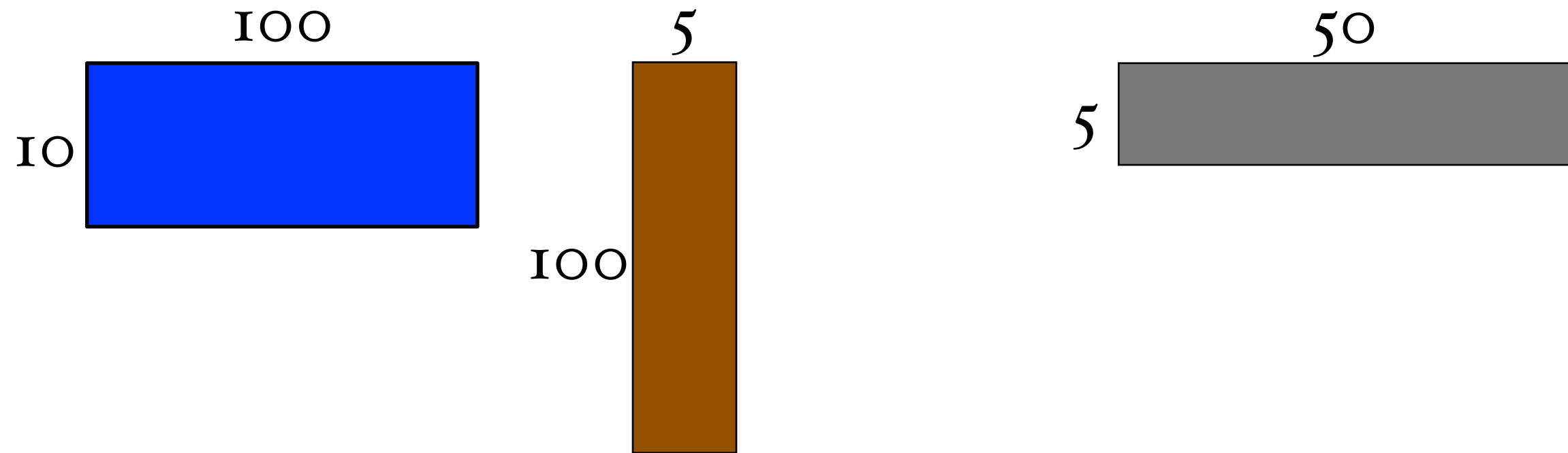


$$A_1 \cdot A_2 \cdot A_3$$

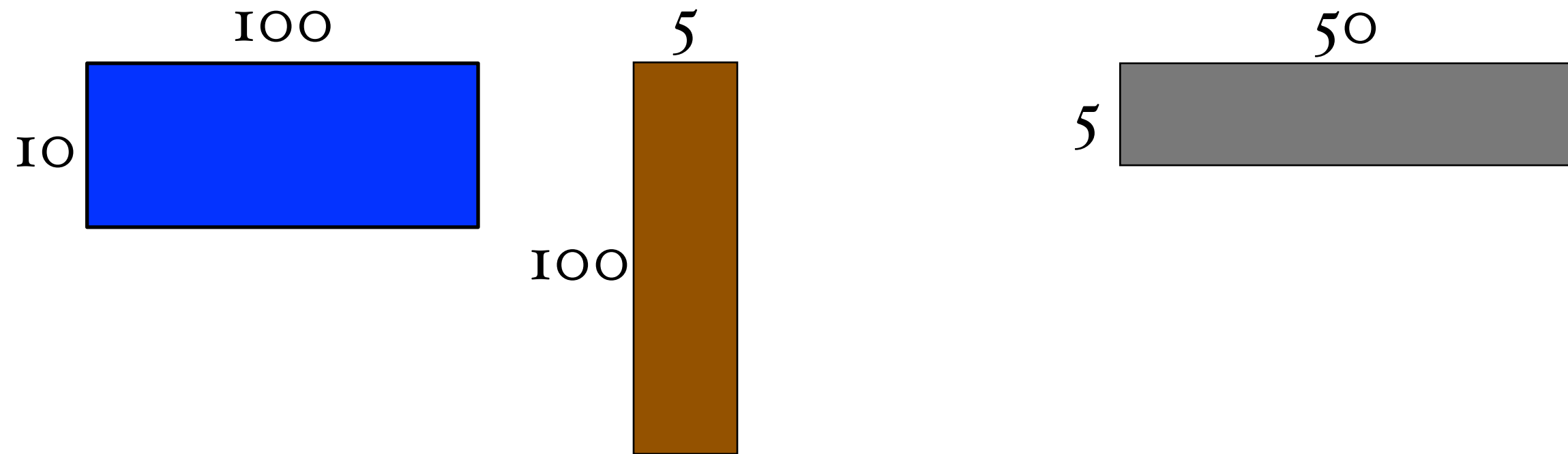
$$(A_1 \cdot A_2) \cdot A_3$$

$$A_1 \cdot (A_2 \cdot A_3)$$

$$(A_1 \cdot A_2) \cdot A_3$$



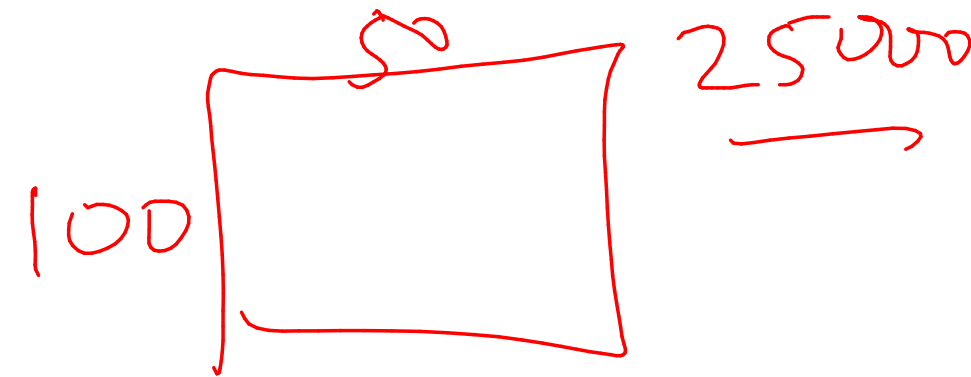
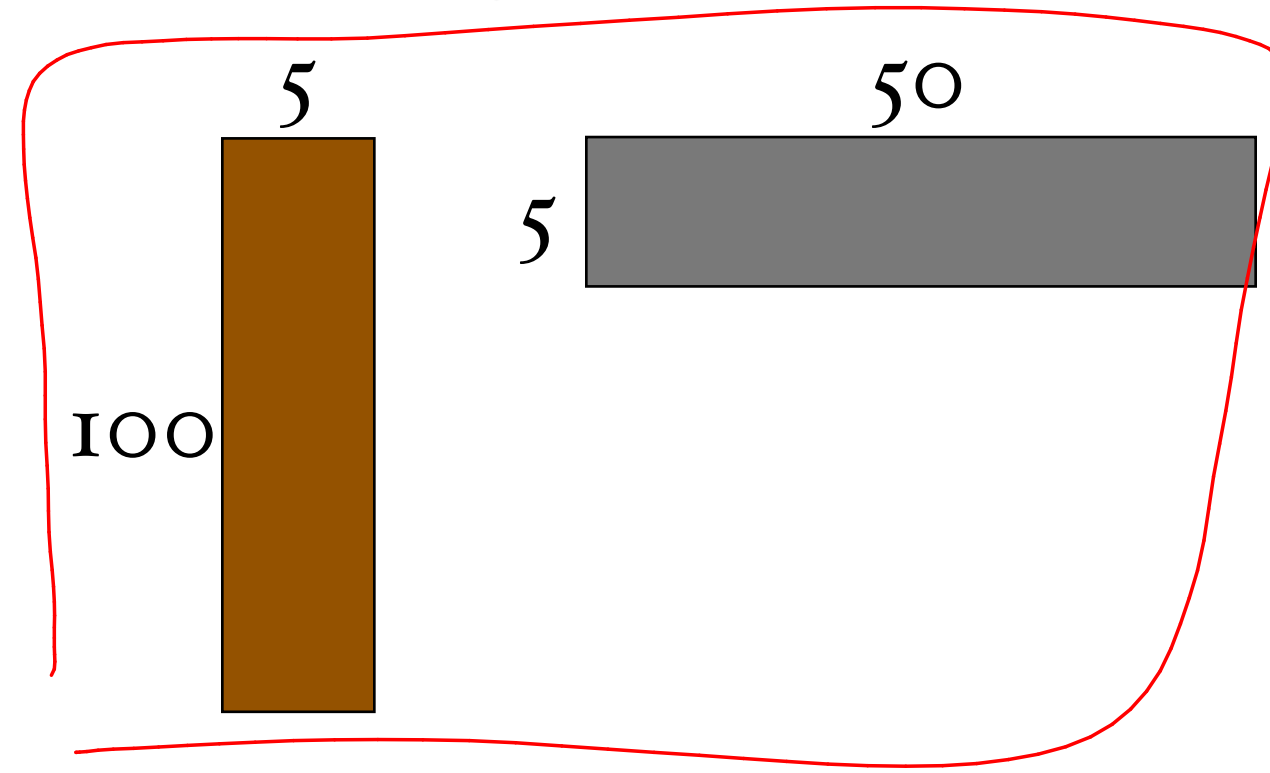
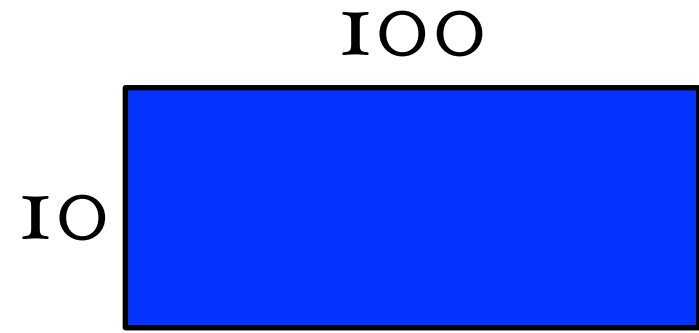
$$(A_1 \cdot A_2) \cdot A_3$$



$$10 \cdot 100 \cdot 5 + 10 \cdot 5 \cdot 50$$

operations

$$A_1 \cdot A_2 \cdot A_3$$

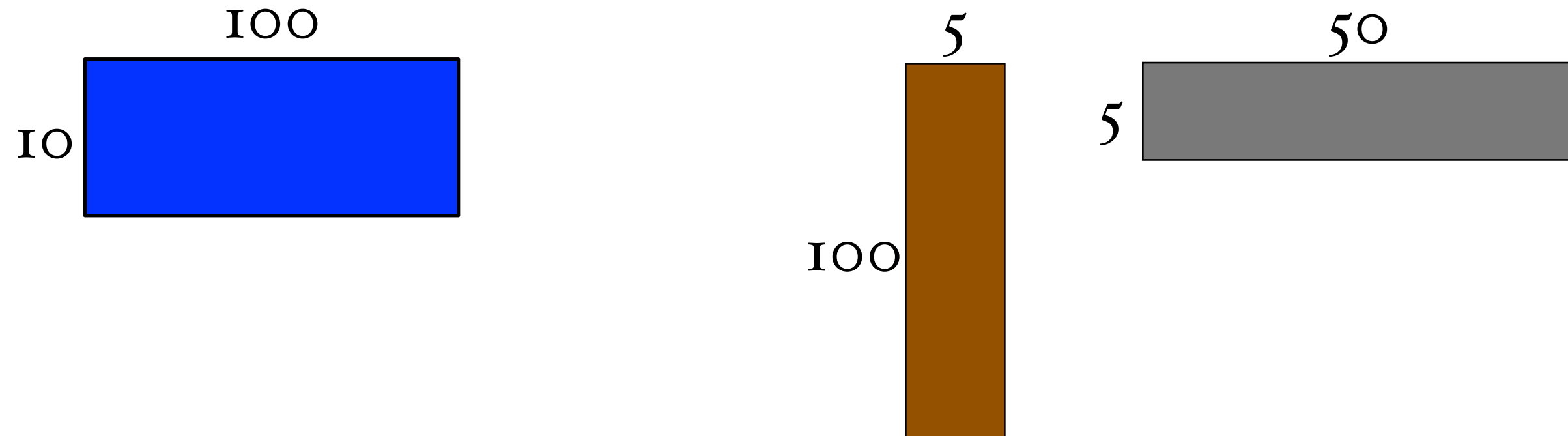


$$10 \cdot 100 \cdot 50$$

$$5000$$

$$= \underline{\underline{75,000}}$$

$$A_1 \cdot A_2 \cdot A_3$$



$$100 \cdot 5 \cdot 50 + 10 \cdot 100 \cdot 50$$

operations



# Order matters

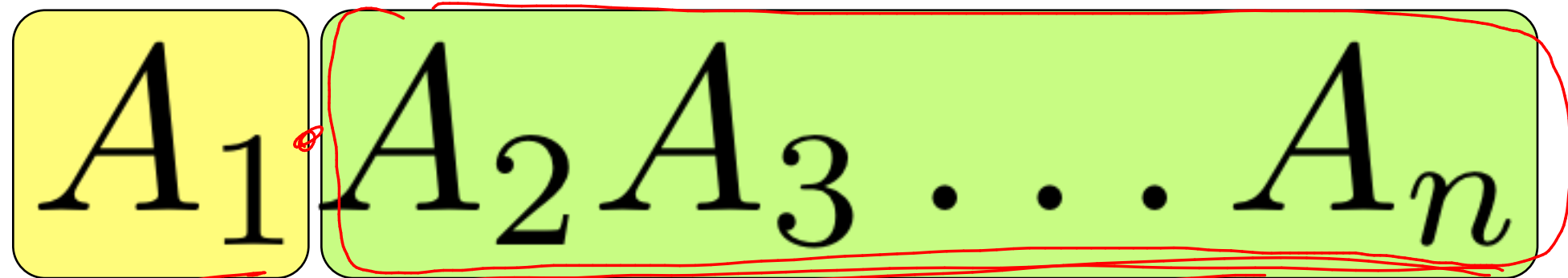
(for efficiency)

How many ways to compute?

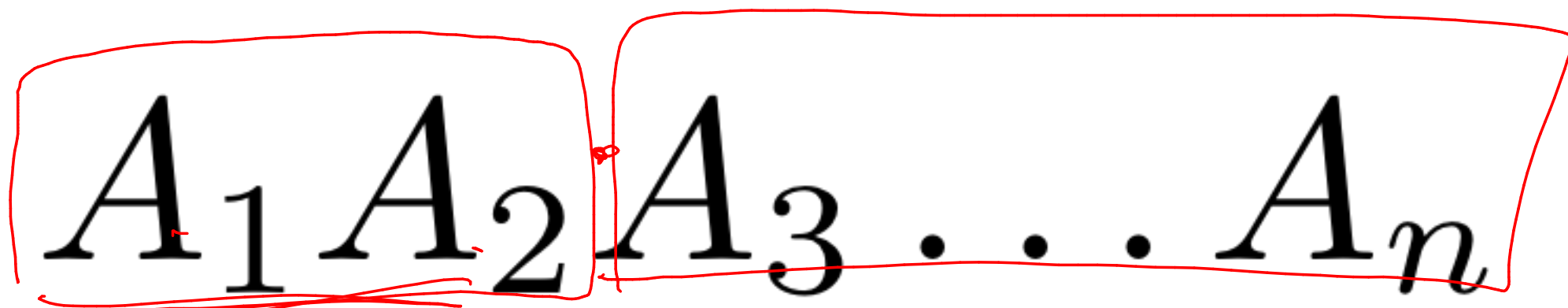
$A_1 A_2 A_3 \dots A_n$

$P(n) = \#$  of <sup>different</sup> ways to mult the  $n$  matrices.

# How many ways to compute?



$$P(1) \cdot P(n-1)$$



$$P(2) \cdot P(n-2)$$

$$P(3) \cdot P(n-3)$$

$$P(n) = P(1)P(n-1) + P(2)P(n-2) + \dots + P(n-1)P(1)$$

$$\approx 4^n$$

How many ways to compute?

$$A_1 A_2 A_3 \dots A_n$$

$$A_1 A_2 A_3 \dots A_n$$

$$A_1 A_2 A_3 \dots A_n$$

How many ways to compute?

$A_1$   $A_2 A_3 \dots A_n$

$A_1 A_2$   $A_3 \dots A_n$

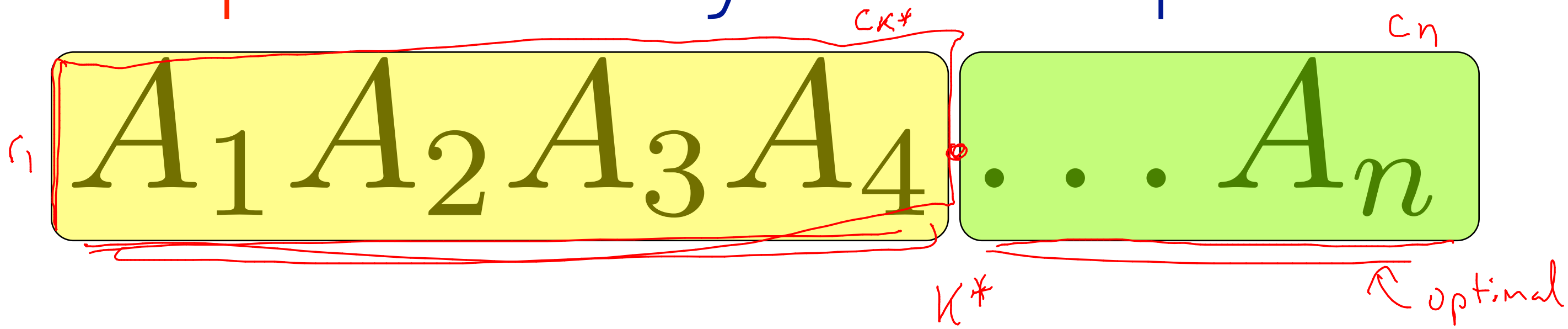
$A_1 A_2 A_3$   $\dots A_n$

# How do we solve it?

- ( identify smaller instances of the problem
- ( devise method to combine solutions
- ( small # of different subproblems  
solved them in the right order

# Optimal way to compute

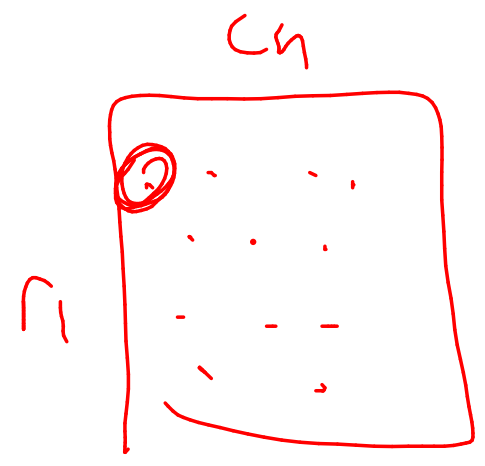
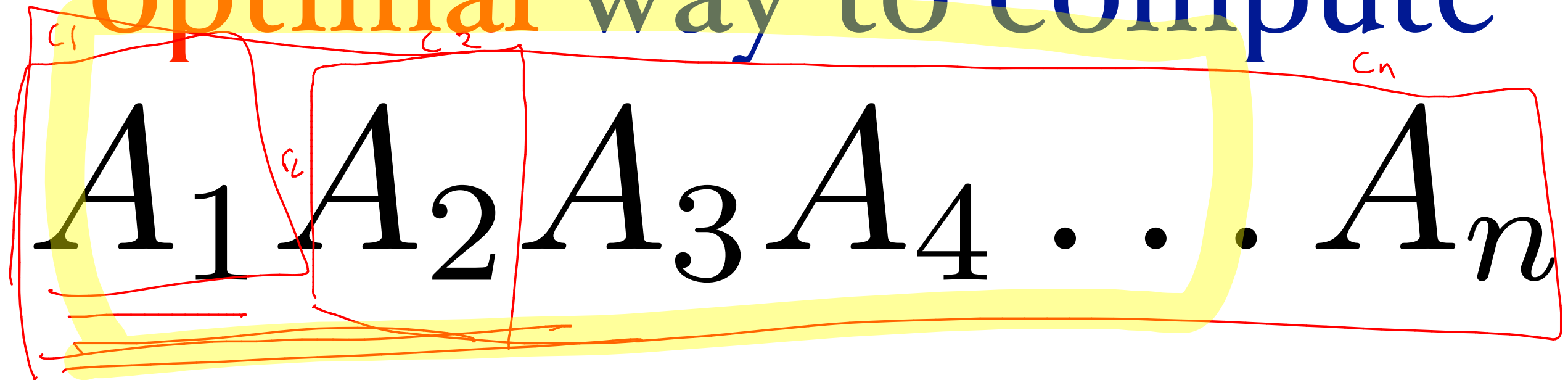
$B(1, n)$  = best way  
(least # of ops)  
to multiply  
 $A_1 \dots A_n$ .



$$B(1, n) = B(1, k) + B(k+1, n) + \boxed{\underline{\underline{r_1 \cdot c_{k^*} \cdot c_n}}}$$

how many choices are there for  $k^*$ ??

# optimal way to compute



$B[1, n]$

$B(1, 1) + B(2, n) +$

$r_1 \cdot c_1 \cdot c_n$

$B(1, 2) + B(3, n)$

$+ r_1 \cdot c_2 \cdot c_n$

$B(1, n-1) + B(n, n) +$

$r_1 \cdot c_{n-1} \cdot c_n$



optimal way to compute

$A_1 A_2 A_3 A_4 \dots A_n$

$B[1,n]$

$B[1,1]$

$B[2,n]$

$R_1 C_1 C_n$

optimal way to compute

$A_1 A_2 A_3 A_4 \dots A_n$

$B[1,n]$

$B[1,1]$	$B[1,2]$	...	$B[1,n-2]$	$B[1,n-1]$
$B[2,n]$	$B[3,n]$	...	$B[n-1,n]$	$B[n,n]$

$R_1 C_1 C_n$	$R_1 C_2 C_n$		$R_1 C_{n-2} C_n$	$R_1 C_{n-1} C_n$
---------------	---------------	--	-------------------	-------------------

Min

$$B(i, i) = 1$$

$$B(1, n) = \min$$



②

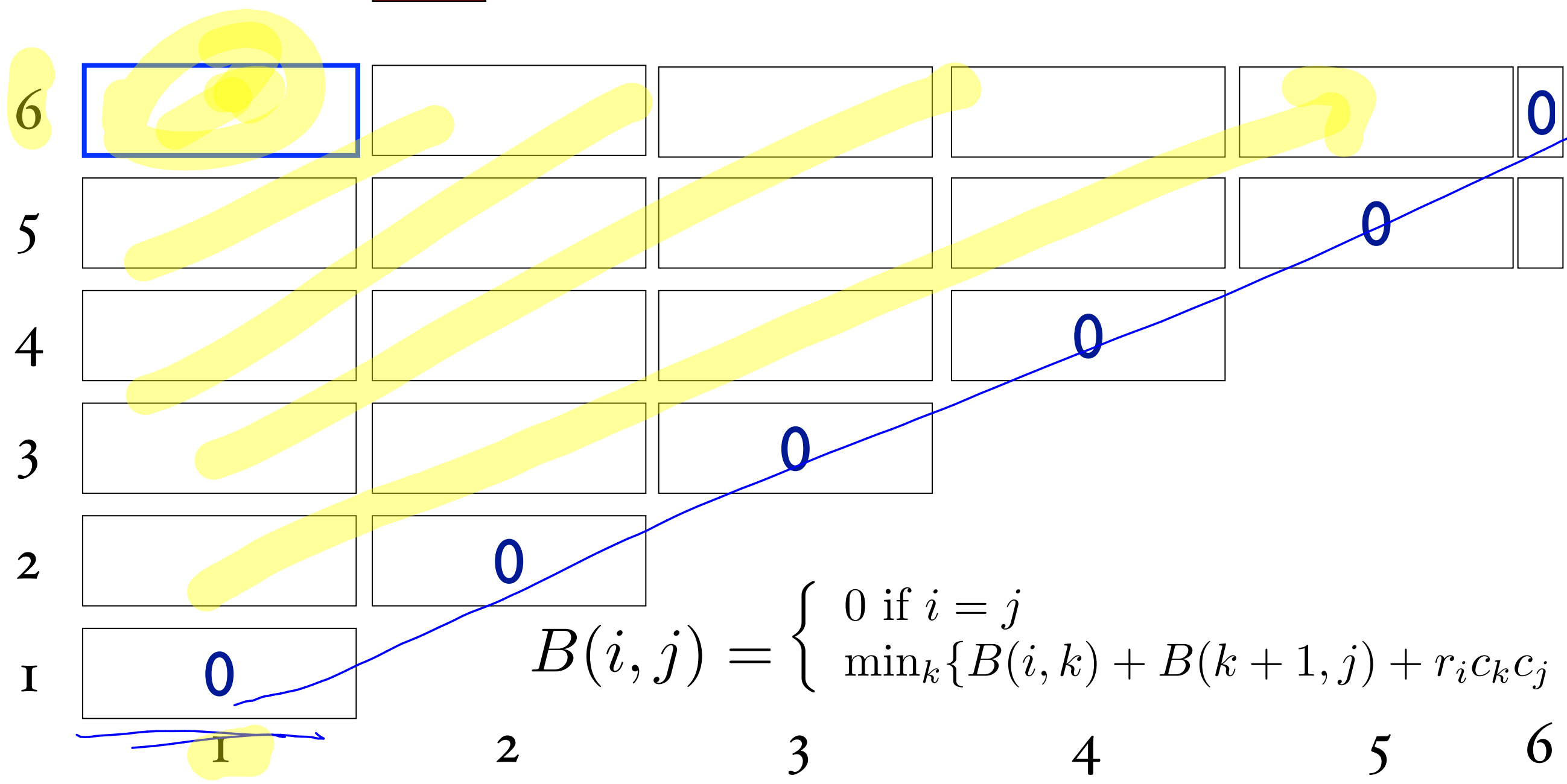
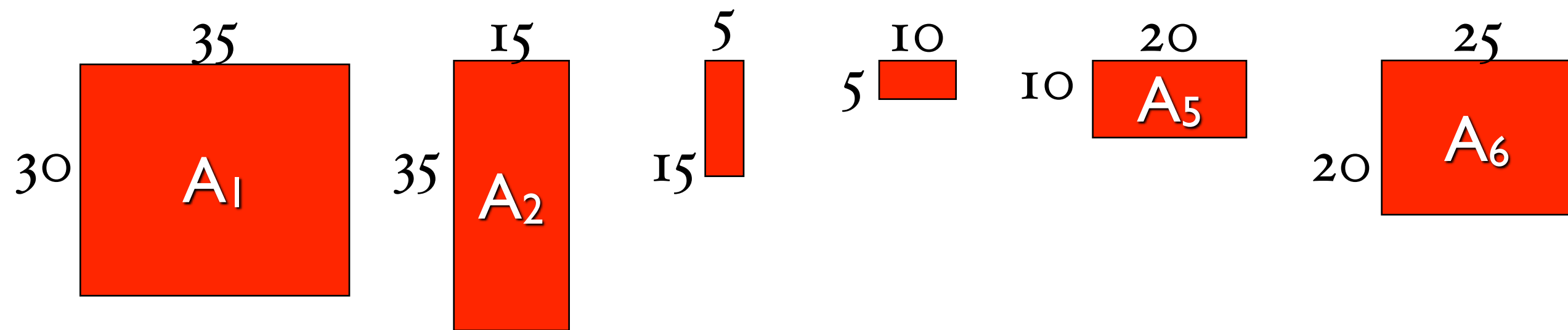
$$\underline{B(i, i) = 0}$$

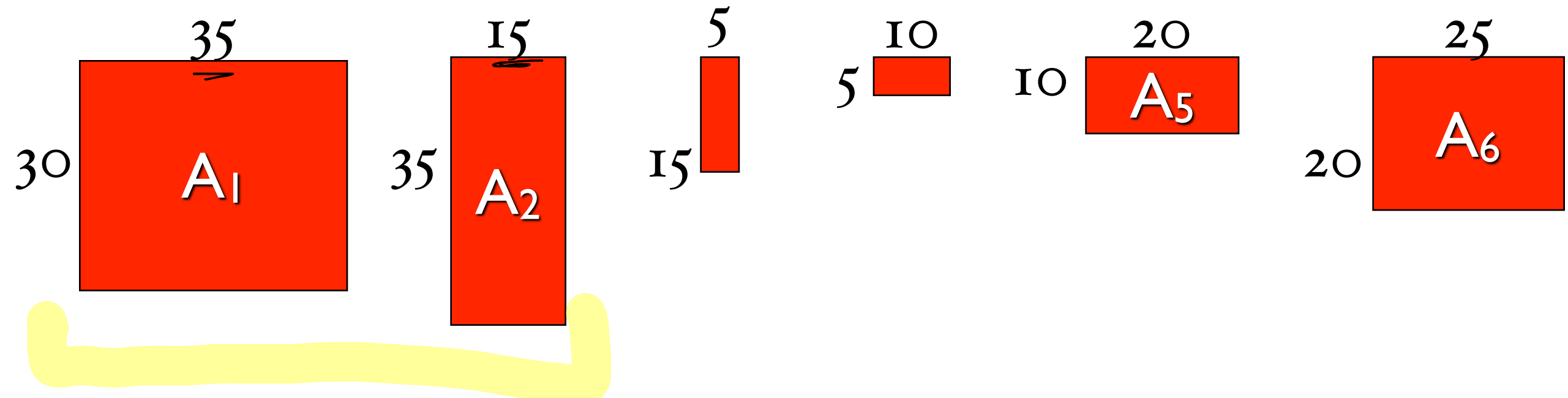
$$B(1, n) = \min \begin{cases} B(1, 1) + B(2, n) + r_1 c_1 c_n \\ B(1, 2) + B(3, n) + r_1 c_2 c_n \\ \vdots \\ B(1, n-1) + B(n, n) + r_1 c_{n-1} c_n \end{cases}$$

$$B(i, j) =$$

$$\begin{cases} 0 & \text{if } i = j \\ \min_k \{ B(i, k) + B(k + 1, j) + r_i c_k c_j \} & \end{cases}$$

which order to solve?

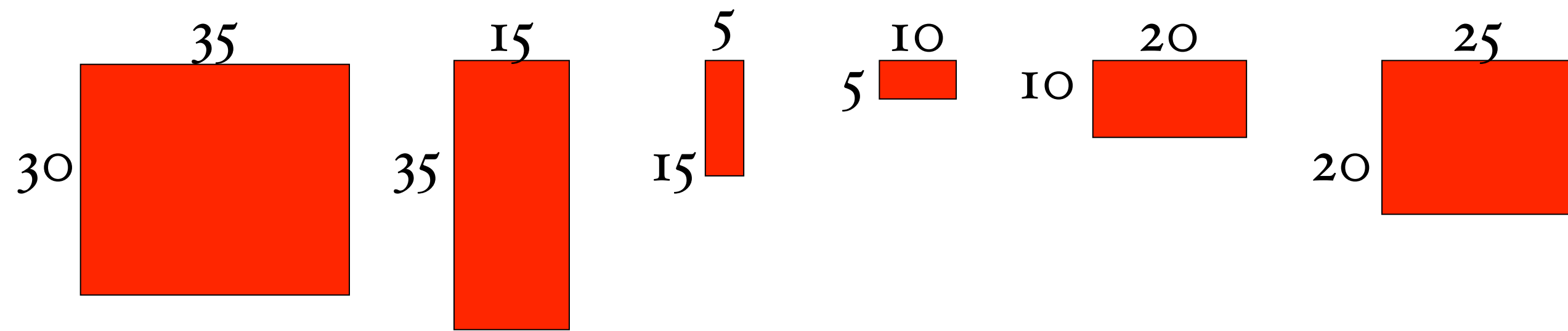




$$B(1,2) = \min \left\{ \underbrace{B(1,1)}_0 + \underbrace{B(2,2)}_0 + \underbrace{r_1 \cdot c_1 \cdot c_2}_{30 \cdot 35 \cdot 15} \right\}$$

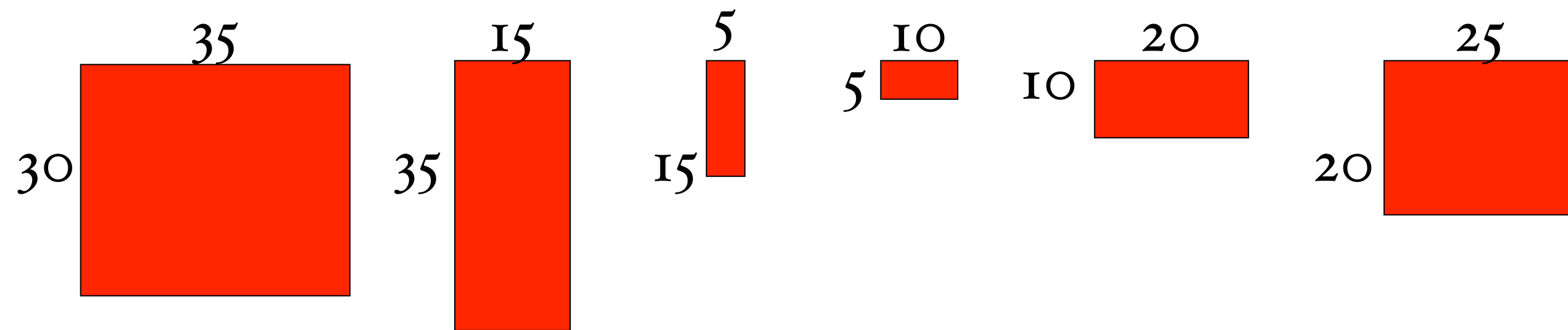
$450 \cdot 35 =$   
 $15000$   
 $1500 =$   
 $2000$   
500





6					$10 \cdot 20 \cdot 25 = 5000$	0
5				$5 \cdot 10 \cdot 20 = 1000$		0
4			$15 \cdot 5 \cdot 10 = 750$			0
3		$35 \cdot 15 \cdot 5 = 2625$				0
2	$30 \cdot 35 \cdot 15 = 15750$					0
1						0
	1	2	3	4	5	6

$$B(i, j) = \begin{cases} 0 & \text{if } i = j \\ \min_k \{ B(i, k) + B(k + 1, j) + r_i c_k c_j \} & \text{otherwise} \end{cases}$$



$B(1,3) = \min \left\{ \begin{array}{l} \underbrace{B(1,1)} + \underbrace{B(2,3)} + r_1 \cdot c_1 \cdot c_3 = 0 + 2625 + 30 \cdot 35 \cdot 5 = 7875 \\ \underbrace{B(1,2)} + \underbrace{B(3,3)} + r_1 \cdot c_2 \cdot c_3 = 15750 + 0 + 30 \cdot 15 \cdot 5 = 5250 \end{array} \right.$

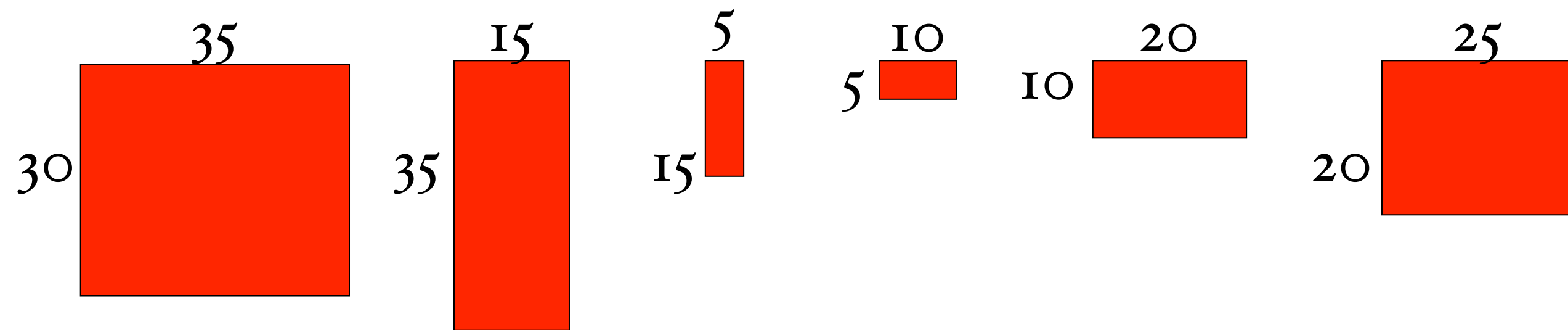
3	7875	$35 \cdot 15 \cdot 5 = 2625$	0
---	------	------------------------------	---

2	$30 \cdot 35 \cdot 15 = 15750$	0
---	--------------------------------	---

1	0
---	---

$$B(i, j) = \begin{cases} 0 & \text{if } i = j \\ \min_k \{ B(i, k) + B(k+1, j) + r_i c_k c_j \} & \text{otherwise} \end{cases}$$

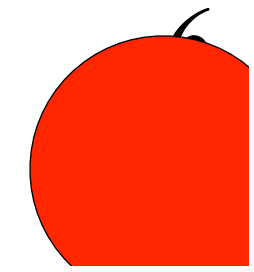
1                    2                    3                    4                    5                    6

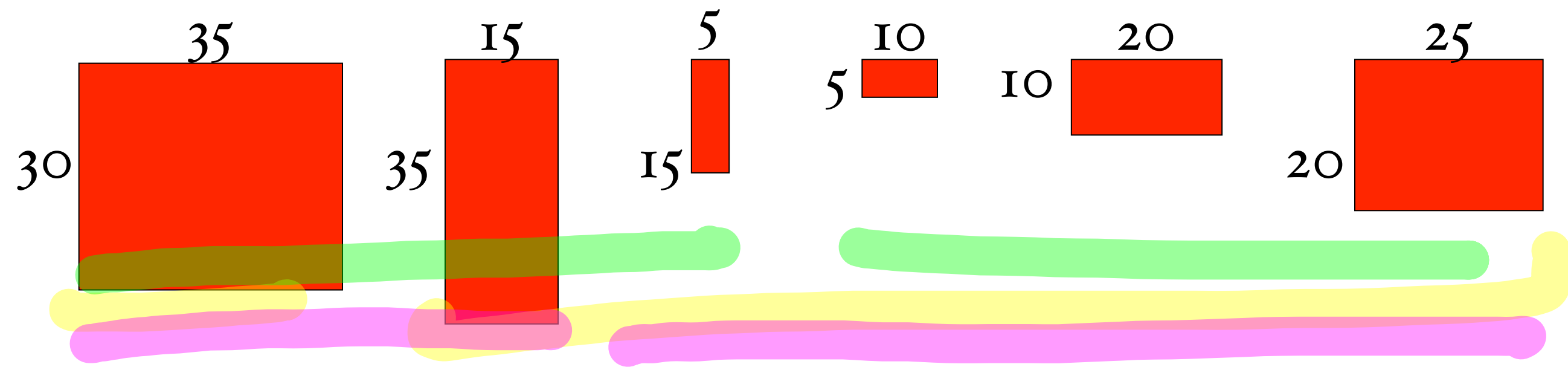


6		10500	5375	3500	$10 \cdot 20 \cdot 25 = 5000$	0
5	11875	7125	2500	$5 \cdot 10 \cdot 20 = 1000$	0	
4	9375	4375	$15 \cdot 5 \cdot 10 = 750$	0		
3	7875	$35 \cdot 15 \cdot 5 = 2625$	0			
2	$30 \cdot 35 \cdot 15 = 15750$	0				
1	0					

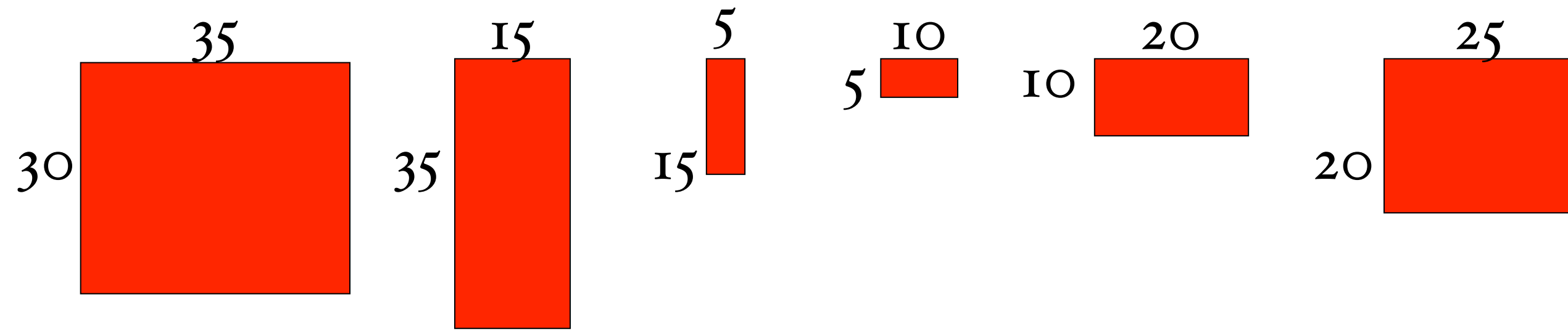
$$B(i, j) = \begin{cases} 0 & \text{if } i = j \\ \min_k \{ B(i, k) + B(k + 1, j) + r_i c_k c_j \} & \text{otherwise} \end{cases}$$

1                    2                    3                    4                    5                    6

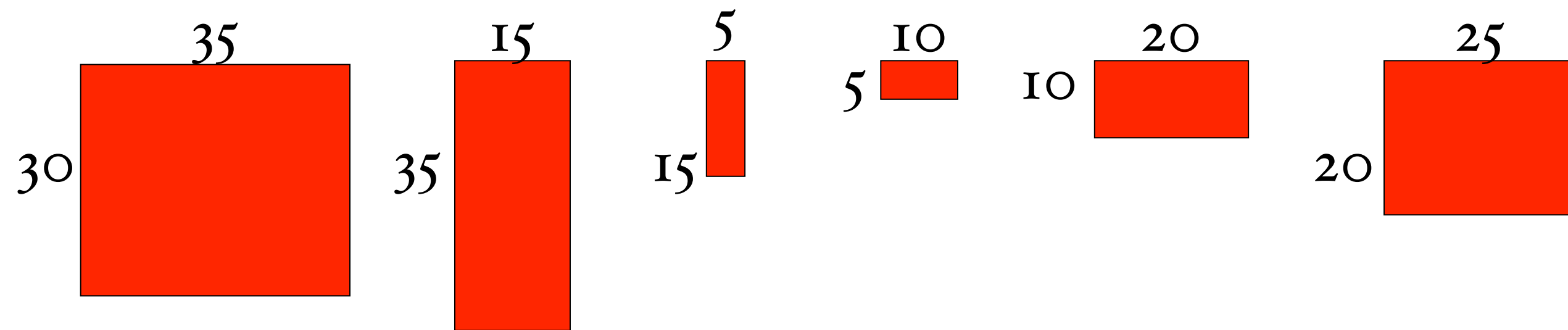





$$\begin{array}{l}
 6 \quad \boxed{\phantom{0000}} \\
 C(1, 6) = \min \left\{ \begin{array}{l}
 k = 1 \quad C(1, 1) + C(2, 6) + r_1 c_1 c_6 \\
 k = 2 \quad C(1, 2) + C(3, 6) + r_1 c_2 c_6 \\
 k = 3 \quad C(1, 3) + C(4, 6) + r_1 c_3 c_6 \\
 k = 4 \quad C(1, 4) + C(5, 6) + r_1 c_4 c_6 \\
 k = 5 \quad C(1, 5) + C(6, 6) + r_1 c_5 c_6
 \end{array} \right.
 \end{array}$$

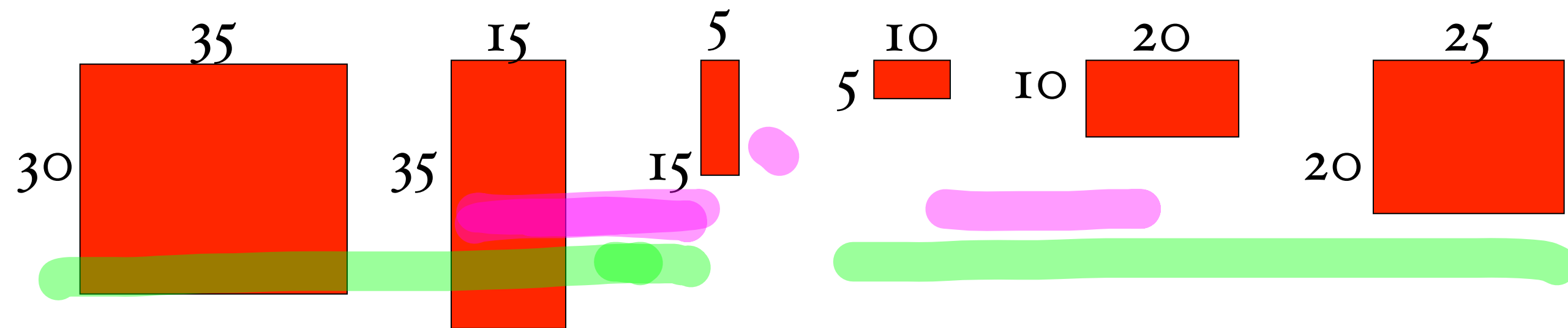


$$\begin{array}{l}
 6 \quad \boxed{\phantom{000000}} \\
 C(1, 6) = \min \left\{ \begin{array}{l}
 k = 1 \quad 0 + 10500 + 30 \cdot 35 \cdot 25 \\
 k = 2 \quad 15750 + 5375 + 30 \cdot 15 \cdot 25 \\
 k = 3 \quad 7875 + 3500 + 30 \cdot 5 \cdot 25 \\
 k = 4 \quad 9375 + 5000 + 30 \cdot 10 \cdot 25 \\
 k = 5 \quad 11875 + 0 + 30 \cdot 20 \cdot 25
 \end{array} \right.
 \end{array}$$

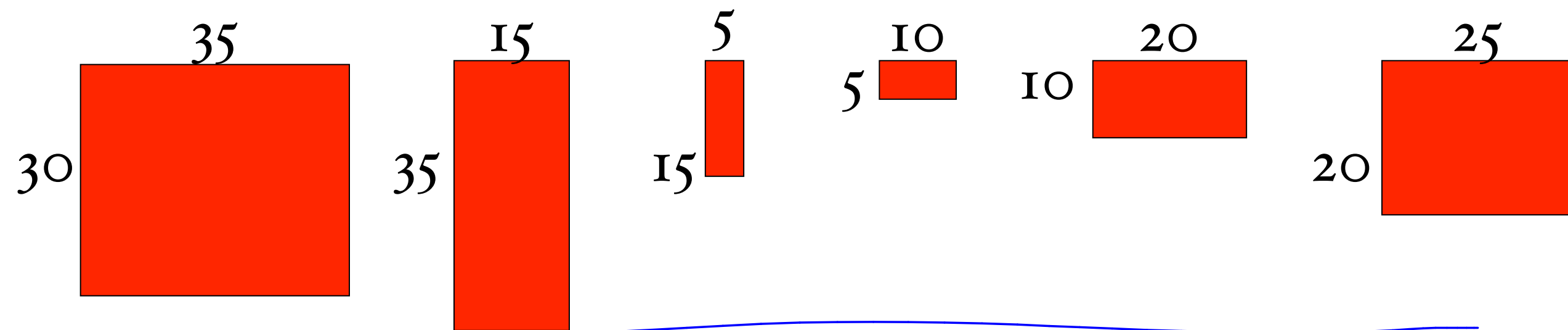


6 

$$C(1, 6) = \min \left\{ \begin{array}{l} k=1 \quad 0 + 10500 + 26250 \\ k=2 \quad 15750 + 5375 + 11250 \\ k=3 \quad 7875 + 3500 + 3750 \\ k=4 \quad 9375 + 5000 + 7500 \\ k=5 \quad 11875 + 0 + 15000 \end{array} \right.$$



6	15125 <small>3</small>	10500	5375	3500 <small>★</small>	$10 \cdot 20 \cdot 25 = 5000$	0
5	11875	7125	2500	$5 \cdot 10 \cdot 20 = 1000$	0	
4	9375	4375	$15 \cdot 5 \cdot 10 = 750$	0		
3	7875 <small>★</small>	$35 \cdot 15 \cdot 5 = 2625$	0			
2	$30 \cdot 35 \cdot 15 = 15750$	0				
1	0					
	1	2	3	4	5	6



6	15125 <span style="color: red;">3</span>	10500	5375	3500 <span style="color: orange;">★</span>	$10 \cdot 20 \cdot 25 = 5000$	0
5	11875	7125	2500	$5 \cdot 10 \cdot 20 = 1000$ <span style="color: orange;">★</span>	0	
4	9375	4375	$15 \cdot 5 \cdot 10 = 750$	0		
3	7875 <span style="color: orange;">★</span>	$35 \cdot 15 \cdot 5 = 2625$ <span style="color: orange;">★</span>	0			
2	$30 \cdot 35 \cdot 15 = 15750$	0				
1	0					
	1	2	3	4	5	6

$\Theta(n^2)$



# Matrix-chain-mult(p)

initialize array  $m[x,y]$  to zero

# Matrix-chain-mult(p)

initialize array  $m[x,y]$  to zero

starting at diagonal, working towards upper-left

compute  $m[i,j]$  according to

$$\begin{cases} 0 & \text{if } i = j \\ \min_k \{ B(i, k) + B(k + 1, j) + r_i c_k c_j \} \end{cases}$$

how many boxes??

$$\Theta(n^2)$$

$$\underline{\Theta(n^3)} \rightarrow \underline{\Theta(n^2)} \rightarrow \underline{\Theta(n \log n)}$$

# running time?

initialize array  $m[x,y]$  to zero

starting at diagonal, working towards upper-left

compute  $m[i,j]$  according to

$$\begin{cases} 0 & \text{if } i = j \\ \min_k \{ B(i, k) + B(k + 1, j) + r_i c_k c_j \} & \end{cases}$$

# Typesetting

It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair, we had everything before us, we had nothing before us, we were all going direct to heaven, we were all going direct the other way - in short, the period was so far like the present period, that some of its noisiest authorities insisted on its being received, for good or for evil, in the superlative degree of comparison only.

slack



~~was~~ overfit



It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair, we had everything before us, we had nothing before us, we were all going direct to heaven, we were all going direct the other way - in short, the period was so far like the present period, that some of its noisiest authorities insisted on its being received, for good or for evil, in the superlative degree of comparison only.

# First rule of typesetting

never print in the margin!

↔ are simply not allowed

It was the best of times, it was the worst  
of times, it was the age of wisdom, it was  
the age of foolishness, it was the epoch  
of belief, it was the epoch of <sup>1,2</sup>  
incredulity, it was the season of Light,  
it was the season of Darkness, it was the  
spring of hope, it was the winter of  
despair, we had everything before us, we  
had nothing before us, we were all going  
direct to heaven, we were all going direct  
the other way - in short, the period was  
so far like the present period, that some of its  
noisiest authorities insisted on its being  
received, for good or for evil, in the superlative  
degree of comparison only.

g reedy . bad.

\_\_\_\_\_ is....

It was the best of times, it was the worst  
of times, it was the age of wisdom, it was  
the age of foolishness, it was the epoch  
of belief, it was the epoch of  
incredulity, it was the season of Light,  
it was the season of Darkness, it was the  
spring of hope, it was the winter of  
despair, we had everything before us, we  
had nothing before us, we were all going  
direct to heaven, we were all going direct  
the other way - in short, the period was  
so far like the present period, that some of its  
noisiest authorities insisted on its being  
received, for good or for evil, in the superlative  
degree of comparison only.

0	0
0	0
2	4
12	144
2	4
1	1
6	36
2	4
2	4
0	0
	197



It was the best of times, it was the             
worst of times, it was the age of wisdom,     
it was the age of foolishness, it was the     
epoch of belief, it was the epoch of             
incredulity, it was the season of Light,         
it was the season of Darkness, it was the         
spring of hope, it was the winter of             
despair, we had everything before us, we         
had nothing before us, we were all going         
direct to heaven, we were all going direct  
the other way - in short, the period was  
so far like the present period, that some  
of its noisiest authorities insisted on  
its being received, for good or for evil,  
in the superlative degree of comparison  
only.

6  
1  
1  
6  
2  
1  
6  
2  
2  
0

36  
1  
1  
36  
4  
1  
36  
4  
4  
0

123



# Typesetting problem

input:  $M$ ,  $w_1 w_2 w_3 \dots$   $w_n$  ← lengths of the words —

output:  $(w_1 \dots w_{l_1})$ ,  $(w_{l_1+1} \dots w_{l_2})$ ,  $\dots$ ,  $\dots (w_{l_{k-1}+1} \dots w_n)$

such that

# Typesetting problem

input:  $W = \{w_1, w_2, w_3, \dots, w_n\}$   $M$

output:  $L = (w_1, \dots, w_{\ell_1}), (w_{\ell_1+1}, \dots, w_{\ell_2}), \dots, (w_{\ell_x+1}, \dots, w_n)$

such that

# Typesetting problem

input:  $W = \{w_1, w_2, w_3, \dots, w_n\}$   $M$

$l_0 = 0$

output:  $L = (\underbrace{w_1, \dots, w_{l_1}}_{\uparrow}, \underbrace{w_{l_1+1}, \dots, w_{l_2}}_{\uparrow}, \dots, (w_{l_{x+1}}, \dots, w_n))$

$c_0 = \sum_{j=l_0+1}^{l_1}$

$$c_i = \left( \sum_{j=l_i+1}^{l_{i+1}} |w_j| \right) + \underbrace{(l_{i+1} - l_i - 1)}$$

spaces between words

such that

$$\underline{c_i} \leq M \quad \forall i$$

$$\min \sum (M - c_i)^2$$

slack

# how to solve

define the right variable:

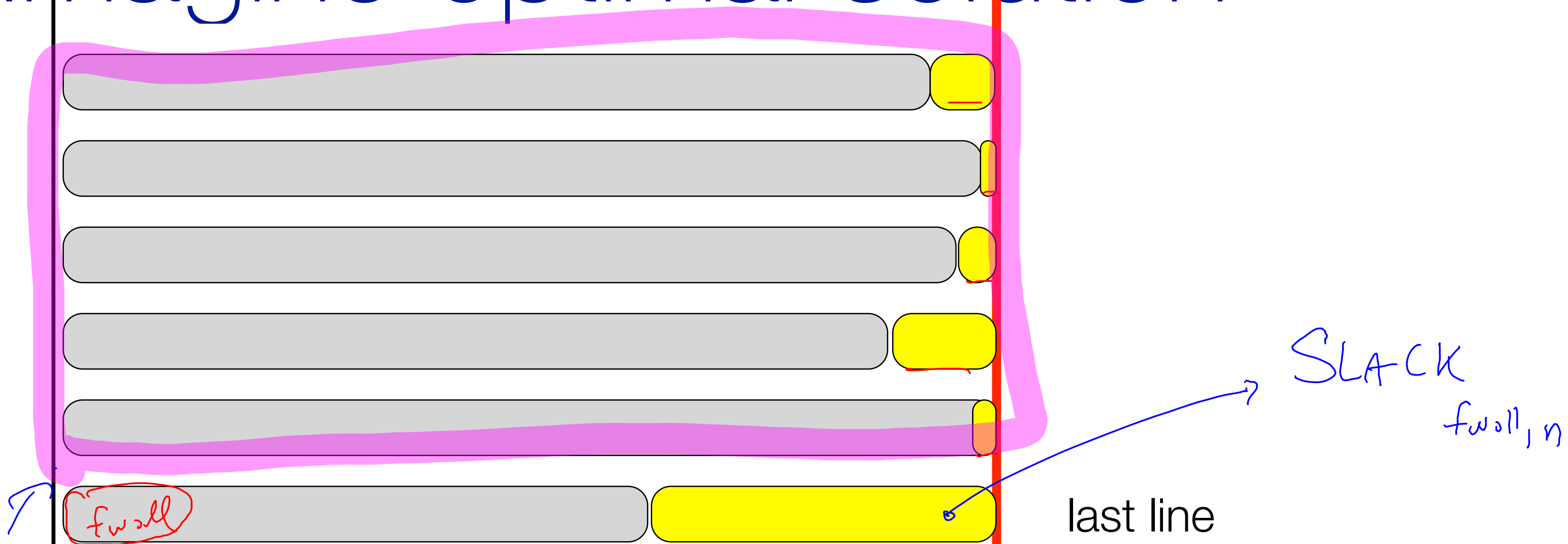
$Best_n$ : smallest  $(penalty)^2$  for typesetting <sup>the first</sup>  $n$  words

# Imagine optimal solution



↑  
first word of the last line

# Imagine optimal solution



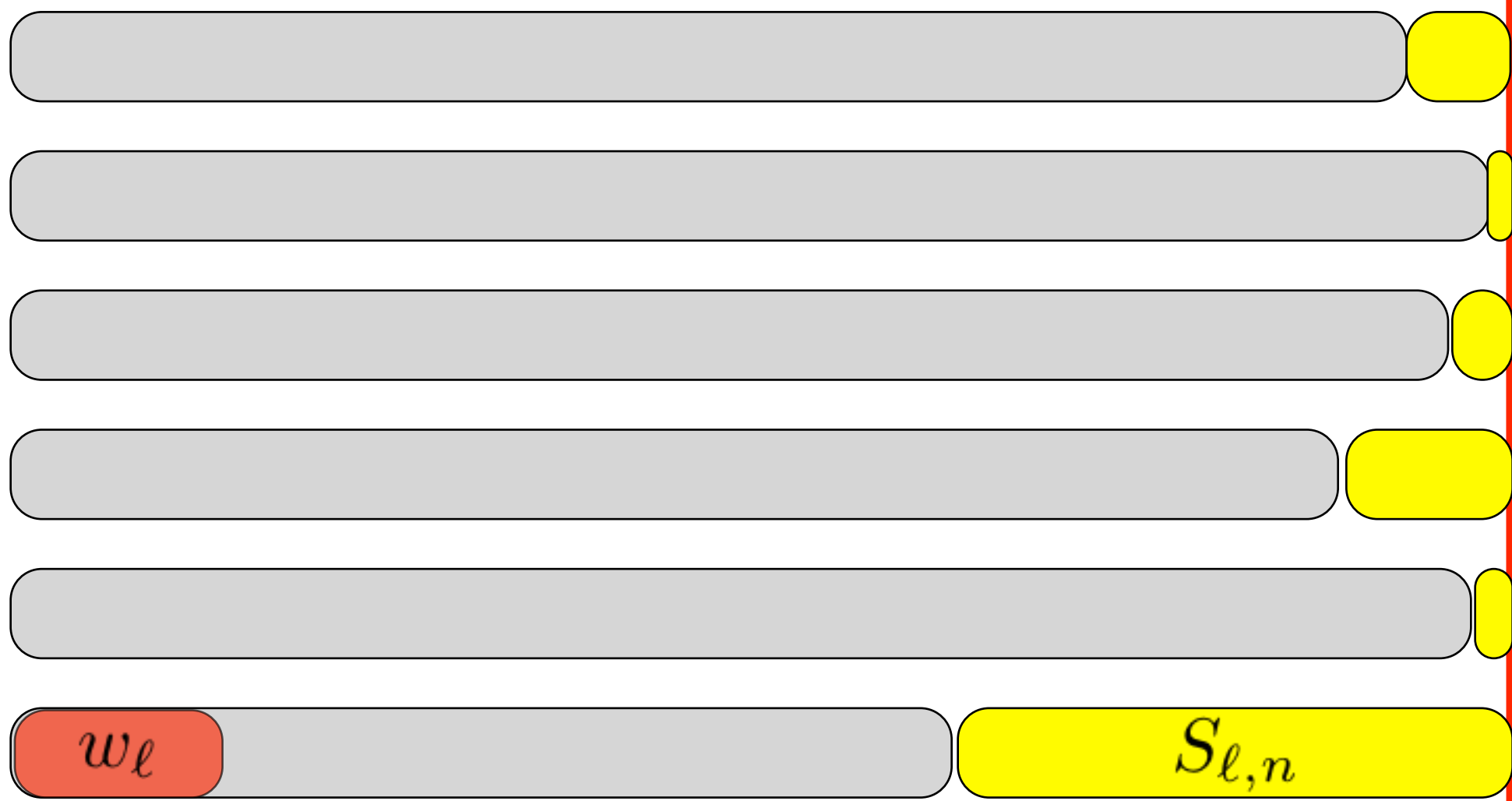
$$Best_n = Best_{f_{wall, n-1}} + (S_{f_{wall, n}})^2$$

$w_n$

Some word has to  
be the first-word-of-  
last-line  
(fwoll)



# Imagine optimal solution

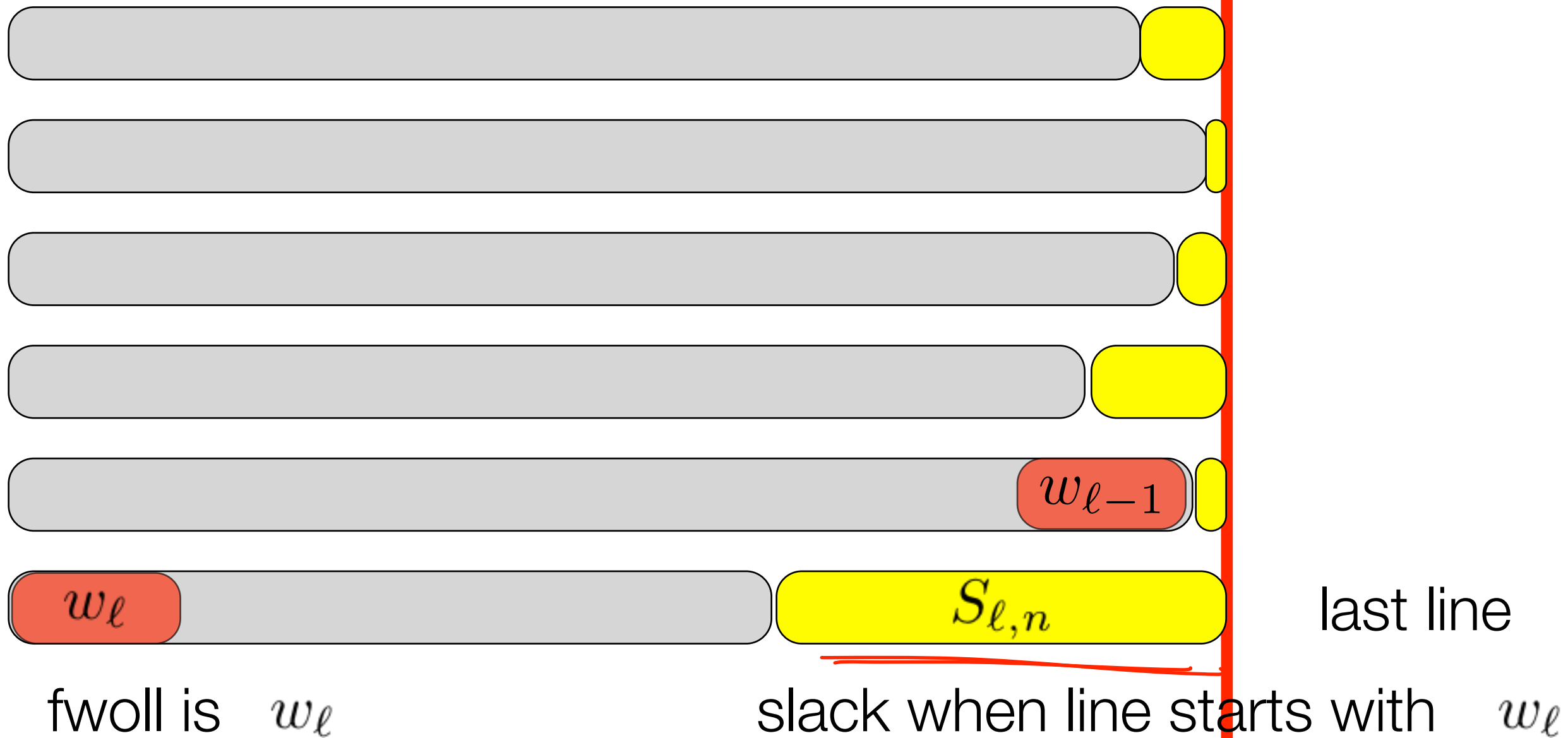


last line

fwoll is  $w_l$

slack when line starts with  $w_l$

# Imagine optimal solution



$$\text{BEST}_n = \text{BEST}_{l-1} + S_{l,n}^2$$

How many candidates  
are there for the fwooll?

n

# Is $w_I$ fwoll?

$w_1$

there is no slack (no solution even)  
because words go beyond edge!

define  $S_{1,n}$  =  $\infty$  if this happens

$$\text{Best}_0 + (S_{1,n})^2$$

# Is $w_2$ fwoll?



$$S_{2,n} = \infty$$

$$\text{Best}_1 + (S_{2,n})^2$$

$w_1$     $w_2$

$w_3$

$$\text{Best}_2 + (S_{3,1})^2$$

# Is $w_i$ fwoll?

$w_1$

$w_j$

$s_{j,n}$

↑  
f word of the  
last line

# Which word is fwoll?

$$\text{BEST}_n = \min \left\{ \begin{array}{l} \text{Best, + S} \\ \end{array} \right.$$

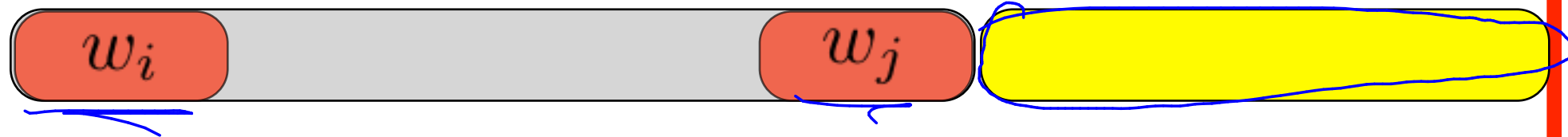


# Which word is fwoll?

$$\underline{\text{BEST}}_n = \min \left\{ \begin{array}{l} \underline{\text{BEST}}_0 + \underline{S_{1,n}^2} \\ \underline{\text{BEST}}_1 + \underline{S_{2,n}^2} \\ \text{BEST}_2 + S_{3,n}^2 \\ \dots \\ \underline{\text{BEST}}_{\ell-1} + \underline{S_{\ell,n}^2} \\ \dots \\ \underline{\text{BEST}}_{n-1} + \underline{S_{n,n}^2} \end{array} \right.$$

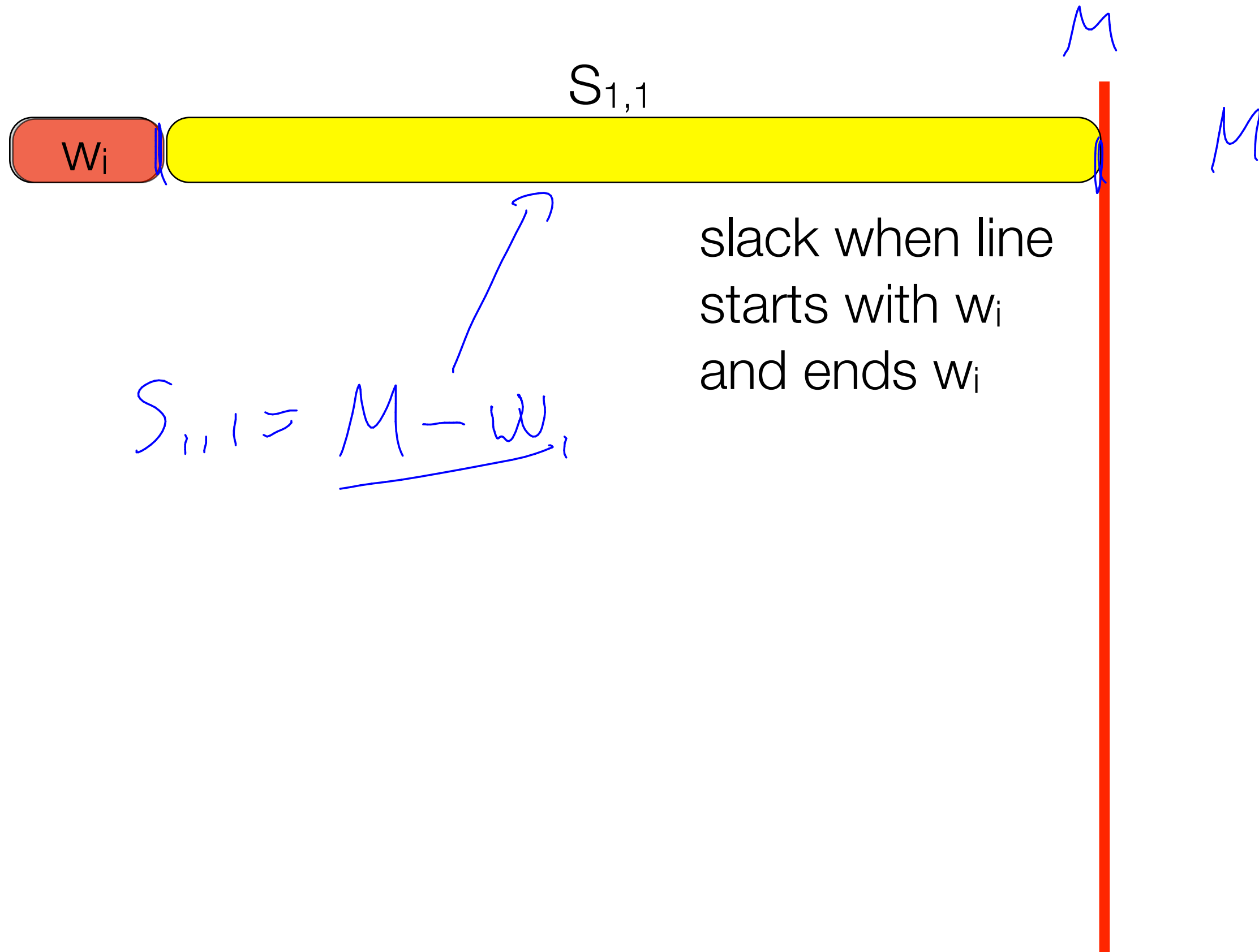
# How to compute $S_{i,j}$

$$S_{i,j}$$

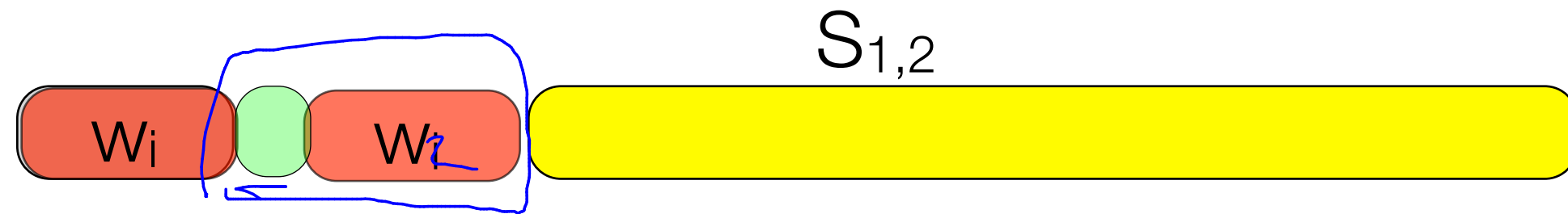


slack when line  
starts with  $w_i$   
and ends  $w_j$

# Simplest case



# Simplest case



slack when line  
starts with  $w_i$   
and ends  $w_2$

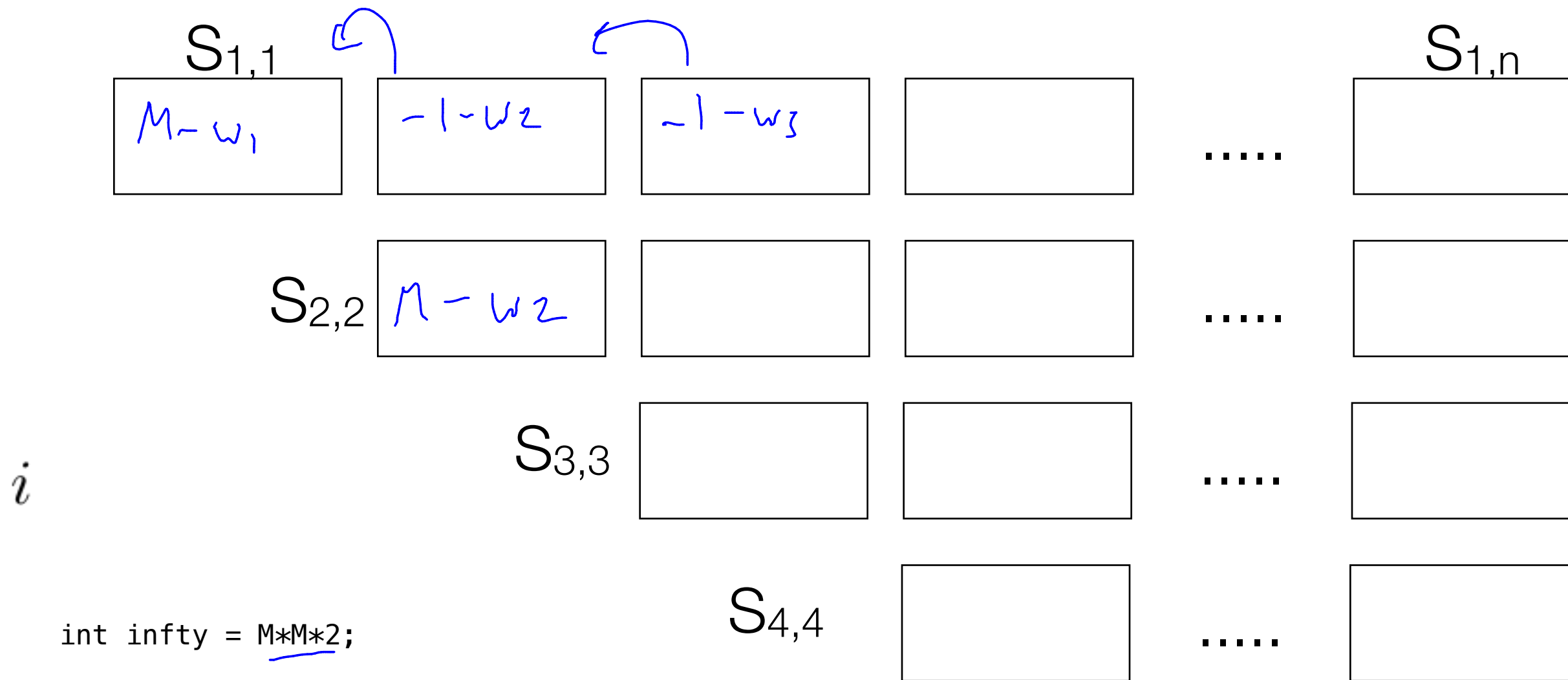
$$S_{1,2} = S_{1,1} - 1 - w_2$$

# how to compute $S_{i,j}$

$$S_{i,j}$$



slack when line  
starts with  $w_i$   
and ends  $w_j$



```

int infty = M*M*2;

// compute S_ij
→ int S[][] = new int[n+1][n+1];
for(int i=1; i<=n; i++) {
    S[i][i] = M - lens[i];
    for(int j=i+1; j<=n; j++) {
        S[i][j] = S[i][j-1] - lens[j] - 1;
        if (S[i][j] < 0) {
            while(j<=n) { S[i][j++] = infty; }
        }
    }
}

```

# Typesetting algorithm

make table for  $S_{i,j}$

# Typesetting algorithm

make table for  $S_{i,j}$

for i=1 to n

$$\underline{\underline{\text{best}[i] = \min\{ \text{best}[j] + s[j+1][i]^2 \}}}$$

*equation*

```
// compute best_0, ..., best_n
int best[] = new int[n+1];
int choice[] = new int[n+1];
best[0] = 0;
for(int i=1; i<=n; i++) {
    int min = infty;
    int ch = 0;
    for(int j=0; j<i; j++) {
        int t = best[j] + S[j+1][i]*S[j+1][i];
        if (t < min) { min = t; ch = j; }
    }
    best[i] = min;
    choice[i] = ch;
}
```



# Example

It was the best of times, it was the worst of times; it was the age of wisdom, it was the age of foolishness; it was the epoch of belief, it was the epoch of incredulity; it was the season of

2 3 3 4 2 6 2 3 3 5 2 6 2 3 3 3 2 7 2 3 3  
3 2 12 2 3 3 5 2 7 2 3 3 5 2 12 2 3 3 6 2

# first step: make $S_{i,j}$

	1	2	3	4	5	6	7	8	9	10	11	12	...
1	40	36	32	27	24								

$$S_{1,1} = 42 - 2$$

$\textcircled{2}$  3 3 4 2 6 2 3 3 5 2 6 2 3 3 3 2 7 2 3 3  
 3 2 12 2 3 3 5 2 7 2 3 3 5 2 12 2 3 3 6 2

$$M = 42$$

$$S_{i,i} = M - |w_i|$$

$$S_{i,j} = S_{i,j-1} - 1 - |w_j|$$

# First step: make $S_{i,j}$

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30									
3													

2 3 3 4 2 6 2 3 3 5 2 6 2 3 3 3 2 7 2 3 3  
 3 2 12 2 3 3 5 2 7 2 3 3 5 2 12 2 3 3 6 2

$$S_{i,i} = M - |w_i|$$

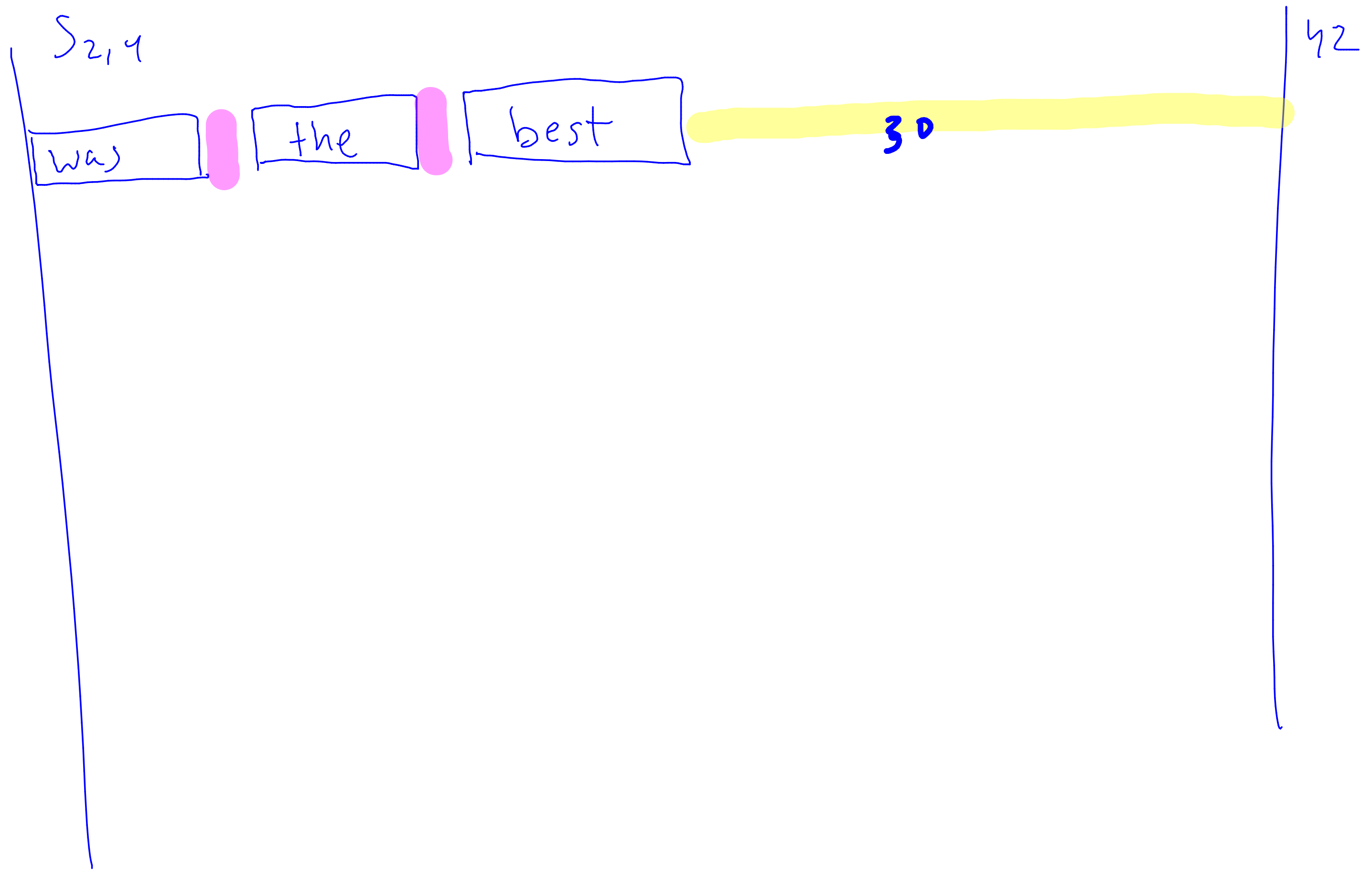
$$S_{i,j} = S_{i,j-1} - 1 - |w_j|$$

$$S_{2,2} = 42 - 3 = 39$$

$$S_{2,3} = S_{2,2} - 1 - 3 = 35$$

$$S_{2,4} = S_{2,3} - 1 - 4 = 30$$

$M = 42$



# First step: make $S_{i,j}$

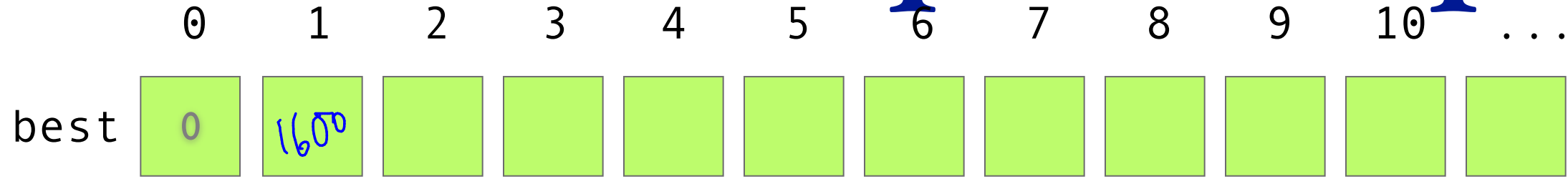
	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30	27	20	17	13	9	3	0	99	99
3													

2 3 3 4 2 6 2 3 3 5 2 6 2 3 3 3 2 7 2 3 3  
 3 2 12 2 3 3 5 2 7 2 3 3 5 2 12 2 3 3 6 2

$$S_{i,i} = M - |w_i|$$

$$S_{i,j} = S_{i,j-1} - 1 - |w_j|$$

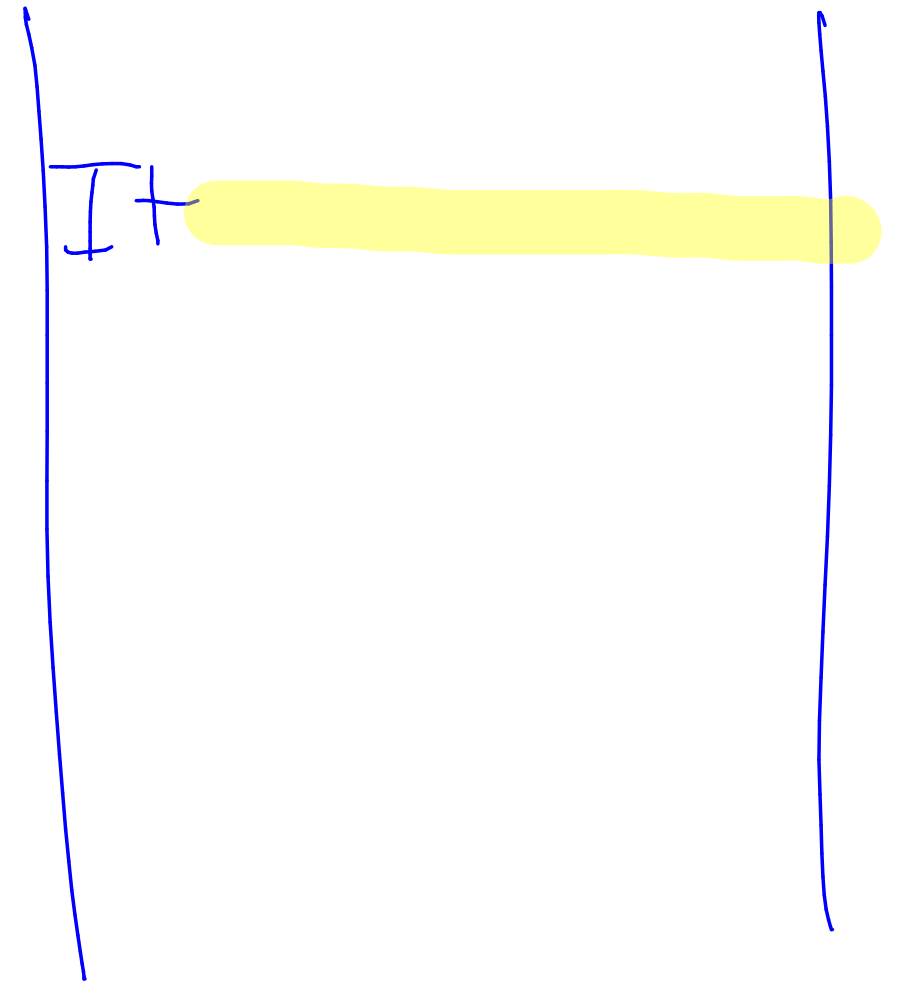
# second step: compute



$$Best_1 = Best_0 + (S_{1,1})^2 = 0 + 40^2 = 1600$$

$$BEST_i = \min_{j=0}^{i-1} \{ BEST_j + S_{j+1,i}^2 \}$$

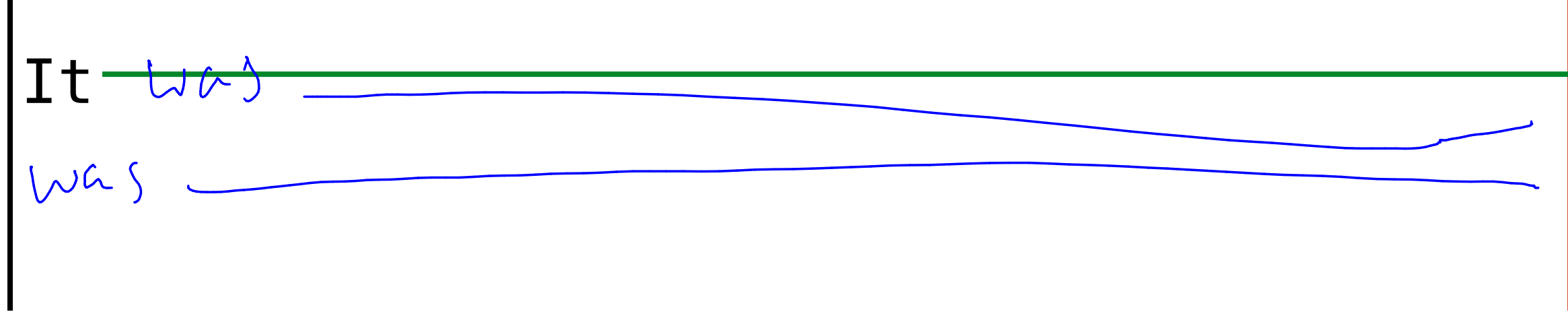
	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30	27	20	17	13	9	3	0	99	99



$$\text{BEST}_i = \min_{j=0}^{i-1} \{ \text{BEST}_j + S_{j+1,i}^2 \}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
best	0	1600												
choice	0	0	0	0	0	0	0	0	0	0	0			

aa



Best<sub>2</sub> = min {

$$\begin{matrix} 0 + 36^2 \\ \text{Bot} + (S_{1,2})^2 \\ \text{Bot} + (S_{2,2})^2 \end{matrix}$$

↓

1600 + 39<sup>2</sup>

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30	27	20	17	13	9	3	0	99	99

$$\text{BEST}_i = \min_{j=0}^{i-1} \{ \text{BEST}_j + S_{j+1,i}^2 \}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
best	0	1600	1296											
choice	0	0	0	0	0	0	0	0	0	0	0			

aa

It was \_\_\_\_\_

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30	27	20	17	13	9	3	0	99	99



$$\text{BEST}_i = \min_{j=0}^{i-1} \{ \text{BEST}_j + S_{j+1,i}^2 \}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
best	0	1600	1296	1024										
choice	0	0	0	0	0	0	0	0	0	0	0			

aa

It was the

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30	27	20	17	13	9	3	0	99	99

$$\text{BEST}_i = \min_{j=0}^{i-1} \{ \text{BEST}_j + S_{j+1,i}^2 \}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
best	0	1600	1296	1024	729	576	289	196	100	36	0			
choice	0	0	0	0	0	0	0	0	0	0	0			

aa

It was the best

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30	27	20	17	13	9	3	0	99	99

$$\text{BEST}_i = \min_{j=0}^{i-1} \{ \text{BEST}_j + S_{j+1,i}^2 \}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
best	0	1600	1296	1024	729	576	289	196	100	36	0			
choice	0	0	0	0	0	0	0	0	0	0	0			

aa

It was the best of

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30	27	20	17	13	9	3	0	99	99

$$\text{BEST}_i = \min_{j=0}^{i-1} \{ \text{BEST}_j + S_{j+1,i}^2 \}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
best	0	1600	1296	1024	729	576	289	196	100	36	0			
choice	0	0	0	0	0	0	0	0	0	0	0			

aa

It was the best of times, it was the worst of

was the →

$$\text{Best}_{11} = \min \left\{ \begin{array}{l} B_0 + (S_{1,11})^2 \\ B_1 + (S_{2,11})^2 \end{array} \right. \rightarrow \sigma^2$$

$$\text{BEST}_i = \min_{j=0}^{i-1} \{ \text{BEST}_j + S_{j+1,i}^2 \}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
best	0	1600	1296	1024	729	576	289	196	100	36	0			
choice	0	0	0	0	0	0	0	0	0	0	0			

aa

It was the best of times, it was the worst  
of \_\_\_\_\_

$$\text{Best}_{11} = \min \{$$

$$\text{BEST}_i = \min_{j=0}^{i-1} \{ \text{BEST}_j + S_{j+1,i}^2 \}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
best	0	1600	1296	1024	729	576	289	196	100	36	0			
choice	0	0	0	0	0	0	0	0	0	0	0			

aa

It was the best of times, it was the  
worst of \_\_\_\_\_  
 10

$$\text{BEST}_{11} = \min \left\{ \begin{array}{l} \text{BEST}_{10} + S_{11,11}^2 \\ \underline{\text{BEST}_9 + S_{10,11}^2} \\ \text{BEST}_8 + S_{9,11}^2 \\ \text{BEST}_7 + S_{8,11}^2 \\ \text{BEST}_6 + S_{7,11}^2 \\ \dots \end{array} \right.$$

$$\text{BEST}_i = \min_{j=0}^{i-1} \{ \text{BEST}_j + S_{j+1,i}^2 \}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
best	0	1600	1296	1024	729	576	289	196	100	36	0			
choice	0	0	0	0	0	0	0	0	0	0	0			

aa

It was the best of times, it was  
the worst of \_\_\_\_\_

$$\text{BEST}_{11} = \min \left\{ \begin{array}{l} \text{BEST}_{10} + S_{11,11}^2 \\ \text{BEST}_9 + S_{10,11}^2 \\ \text{BEST}_8 + S_{9,11}^2 \\ \text{BEST}_7 + S_{8,11}^2 \\ \text{BEST}_6 + S_{7,11}^2 \\ \dots \end{array} \right.$$

$$\text{BEST}_i = \min_{j=0}^{i-1} \{ \text{BEST}_j + S_{j+1,i}^2 \}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
best	0	1600	1296	1024	729	576	289	196	100	36	0	818		
choice	0	0	0	0	0	0	0	0	0	0	0			

aa

It was the best of times,  
 it was the worst of \_\_\_\_\_

$$\text{BEST}_{11} = \min \left\{ \begin{array}{l} \text{BEST}_{10} + S_{11,11}^2 \\ \text{BEST}_9 + S_{10,11}^2 \\ \text{BEST}_8 + S_{9,11}^2 \\ \text{BEST}_7 + S_{8,11}^2 \\ \text{BEST}_6 + S_{7,11}^2 \\ \dots \end{array} \right.$$



$$\text{BEST}_i = \min_{j=0}^{i-1} \{ \text{BEST}_j + S_{j+1,i}^2 \}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
best	0	1600	1296	1024	729	576	289	196	100	36	0	818	545	
choice	0	0	0	0	0	0	0	0	0	0	0	6	6	

aa

It was the best of times, ~~it~~

it was the worst of times, ~~it was~~

$$\text{BEST}_{13} = \min \left\{ \begin{array}{l} \text{BEST}_{12} + S_{13,13}^2 \\ \text{BEST}_{11} + S_{12,13}^2 \\ \dots \\ \text{BEST}_7 + S_{8,13}^2 \\ \text{BEST}_6 + S_{7,13}^2 \end{array} \right.$$

$$\text{BEST}_i = \min_{j=0}^{i-1} \{ \text{BEST}_j + S_{j+1,i}^2 \}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
best	0	1600	1296	1024	729	576	289	196	100	36	0	818	545	
choice	0	0	0	0	0	0	0	0	0	0	0	6	6	

aa

It was the best of times, it  
 was the worst of times, it

$$\text{BEST}_{13} = \min \left\{ \begin{array}{l} \text{BEST}_{12} + S_{13,13}^2 \\ \text{BEST}_{11} + S_{12,13}^2 \\ \dots \\ \text{BEST}_7 + S_{8,13}^2 \\ \text{BEST}_6 + S_{7,13}^2 \end{array} \right.$$

d-172-25-159-219-typeset abhi\$ java typeset charly 42

0 best: 0 ch 0  
1 best: 1600 ch 0  
2 best: 1296 ch 0  
3 best: 1024 ch 0  
4 best: 729 ch 0  
5 best: 576 ch 0  
6 best: 289 ch 0  
7 best: 196 ch 0  
8 best: 100 ch 0  
9 best: 36 ch 0  
10 best: 0 ch 0  
11 best: 818 ch 6  
12 best: 545 ch 6  
13 best: 452 ch 7  
14 best: 340 ch 7  
15 best: 244 ch 8  
16 best: 164 ch 8  
17 best: 117 ch 9  
18 best: 37 ch 9  
19 best: 16 ch 10  
20 best: 0 ch 10  
21 best: 509 ch 14  
22 best: 413 ch 15  
23 best: 344 ch 15  
24 best: 133 ch 17  
25 best: 118 ch 17  
26 best: 62 ch 18  
27 best: 32 ch 19  
28 best: 4 ch 20  
29 best: 444 ch 23  
30 best: 348 ch 23  
31 best: 277 ch 24  
32 best: 197 ch 24  
33 best: 149 ch 24  
34 best: 87 ch 26  
35 best: 66 ch 26  
36 best: 446 ch 31  
37 best: 377 ch 31  
38 best: 297 ch 32  
39 best: 233 ch 32

0 best: 0 ch 0	It
1 best: 1600 ch 0	It was
2 best: 1296 ch 0	It was the
3 best: 1024 ch 0	It was the best
4 best: 729 ch 0	It was the best of
5 best: 576 ch 0	It was the best of times,
6 best: 289 ch 0	It was the best of times, it
7 best: 196 ch 0	It was the best of times, it was
8 best: 100 ch 0	It was the best of times, it was the
9 best: 36 ch 0	It was the best of times, it was the worst
10 best: 0 ch 0	It was the best of times, \n it was the worst
11 best: 818 ch 6	It was the best of times, \n it was the worst of
12 best: 545 ch 6	It was the best of times, \n it was the worst of times,
13 best: 452 ch 7	It was the best of times, it \n was the worst of times, it
14 best: 340 ch 7	It was the best of times, it \n was the worst of times, it was
15 best: 244 ch 8	It was the best of times, it was \n the worst of times, it was the
16 best: 164 ch 8	It was the best of times, it was \n the worst of times, it was the age
17 best: 117 ch 9	It was the best of times, it was the \n worst of times, it was the age of
18 best: 37 ch 9	It was the best of times, it was the \n worst of times, it was the age of wisdom,
19 best: 16 ch 10	It was the best of times, it was the worst \n of times, it was the age of wisdom, it
20 best: 0 ch 10	It was the best of times, it was the worst \n of times, it was the age of wisdom, it was
21 best: 509 ch 14	It was the best of times, it \n was the worst of times, it was \n the age of wisdom, it was the
22 best: 413 ch 15	It was the best of times, it was \n the worst of times, it was the \n age of wisdom, it was the age
23 best: 344 ch 15	It was the best of times, it was \n the worst of times, it was the \n age of wisdom, it was the age
24 best: 133 ch 17	It was the best of times, it was the \n worst of times, it was the age of \n wisdom, it was the age
25 best: 118 ch 17	It was the best of times, it was the \n worst of times, it was the age of \n wisdom, it was the age
it	
26 best: 62 ch 18	It was the best of times, it was the \n worst of times, it was the age of wisdom, \n it was the age

```

// read input

try {
BufferedReader bin = new BufferedReader(new FileReader(args[0]));
String line = bin.readLine();
String words[] = line.split(" ");
int infty = M*M*2;
int n = words.length;
int M = Integer.parseInt(args[1]);
// compute S[i]
int lens[] = new int[n+1];
for(int i=1; i<=n; i++) {
for(int j=1; j<=i; j++) {
lens[i] = words[i-1].length();
if(lens[i]>M) {
for(int k=i+1; k<=n; k++) {
System.out.println("word too long");
System.exit(1) - lens[j] - 1;
if (S[i][j]<0) {
while(j<=n) { S[i][j++] = infty; }
}
}
}
}
}
}

```

```

    lens[i] = words[i-1].length();
    if (lens[i]>M) {
        System.out.println("word too long");
        System.exit(1);
    }
}

int infity = M*M*2;

// compute S_ij
int S[][] = new int[n+1][n+1];
for(int i=1;i<=n;i++) {
    S[i][i] = M - lens[i];
    for(int j=i+1; j<=n; j++) {
        S[i][j] = S[i][j-1] - lens[j] - 1;
        if (S[i][j]<0) {
            while(j<=n) { S[i][j++] = infity; }
        }
    }
}

// compute best_0,...,best_n
int best[] = new int[n+1];
int choice[] = new int[n+1];
best[0] = 0;
for(int i=1;i<=n;i++) {
    int min = infity;
    int ch = 0;
    for(int j=0;j<i;j++) {
        int t = best[j] + S[j+1][i]*S[j+1][i];
        if (t<min) { min = t; ch = j;}
    }
    best[i] = min;
    choice[i] = ch;
}

```

```
    lens[i] = words[i-1].length();
    if (lens[i]>M) {
        System.out.println("word too long");
        System.exit(1);
    }
}
int infty = M*M*2;

// compute S_ij
int S[][] = new int[n+1][n+1];
for(int i=1;i<=n;i++) {
    S[i][i] = M - lens[i];
    for(int j=i+1; j<=n; j++) {
        S[i][j] = S[i][j-1] - lens[j] - 1;
        if (S[i][j]<0) {
            while(j<=n) { S[i][j++] = infty; }
        }
    }
}
}
```

```

    lens[i] = words[i-1].length();
    if (lens[i]>M) {
        System.out.println("word too long");
        System.exit(1);
    }
}
int infty = M*M*2;

// compute S_ij
int S[][] = new int[n+1][n+1];
for(int i=1;i<=n;i++) {
    S[i][i] = M - lens[i];
    for(int j=i+1; j<=n; j++) {
        S[i][j] = S[i][j-1] - lens[j] - 1;
        if (S[i][j]<0) {
            while(j<=n) { S[i][j++] = infty; }
        }
    }
}

// compute best_0,...,best_n
int best[] = new int[n+1];
int choice[] = new int[n+1];
best[0] = 0;
for(int i=1;i<=n;i++) {
    int min = infty;
    int ch = 0;
    for(int j=0;j<i;j++) {
        int t = best[j] + S[j+1][i]*S[j+1][i];
        if (t<min) { min = t; ch = j;}
    }
    best[i] = min;
    choice[i] = ch;
}

```



```
// backtrack to output linebreaks
int end = n;
int start = choice[end]+1;
String lines[] = new String[n];
int cnt = 0;
while (end>0) {
    StringBuffer buf = new StringBuffer();
    for(int j=start; j<=end; j++) {
        buf.append(words[j-1] + " ");
    }
    lines[cnt++] = buf.toString();
    end = start-1;
    start = choice[end]+1;
}
```

Gerrymander

# Congressional District 5



**5** C  
Nelson C



0 50 100 Miles

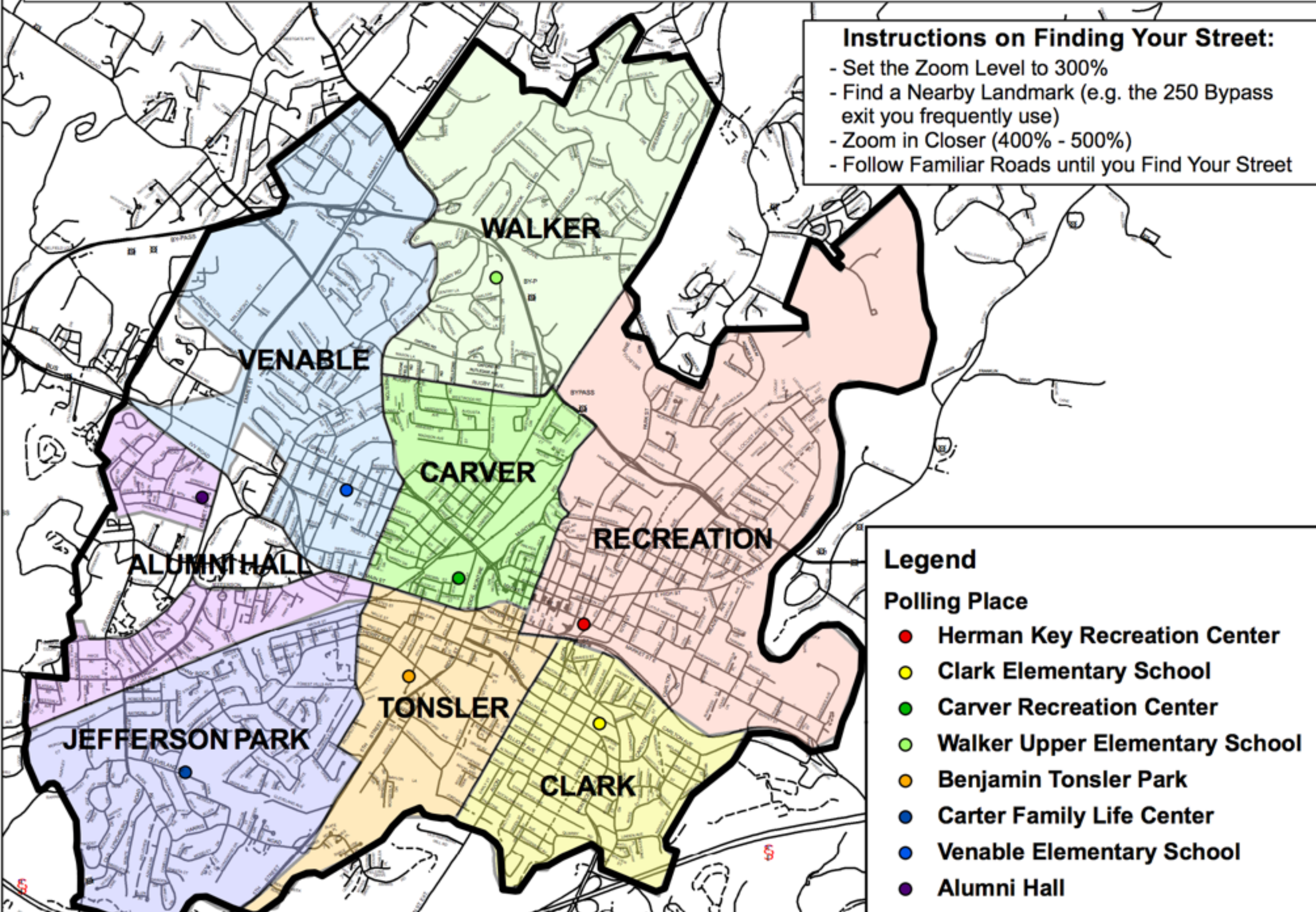
Virgin



# Map of Charlottesville Precincts and Polling Places

## Instructions on Finding Your Street:

- Set the Zoom Level to 300%
- Find a Nearby Landmark (e.g. the 250 Bypass exit you frequently use)
- Zoom in Closer (400% - 500%)
- Follow Familiar Roads until you Find Your Street

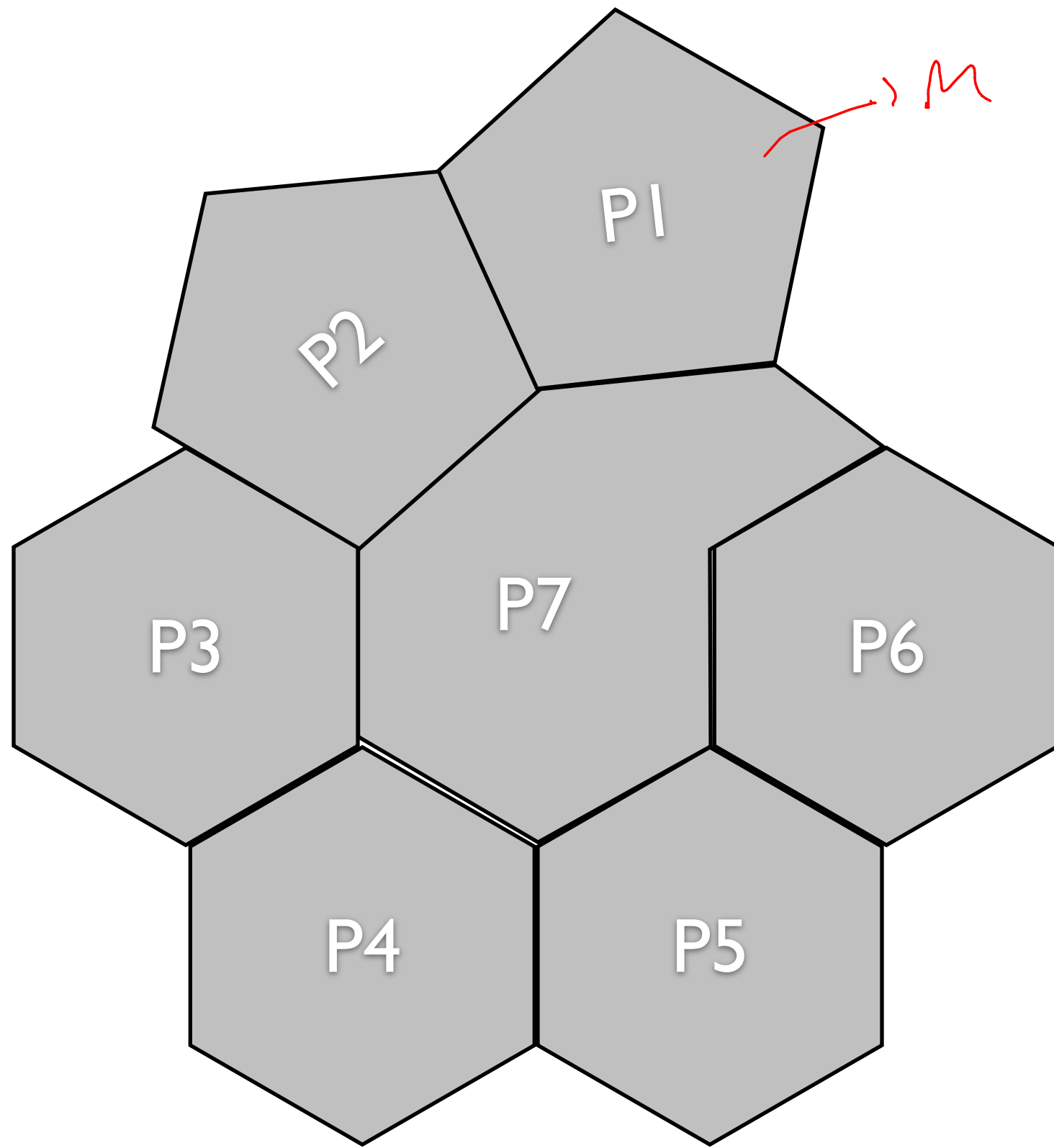


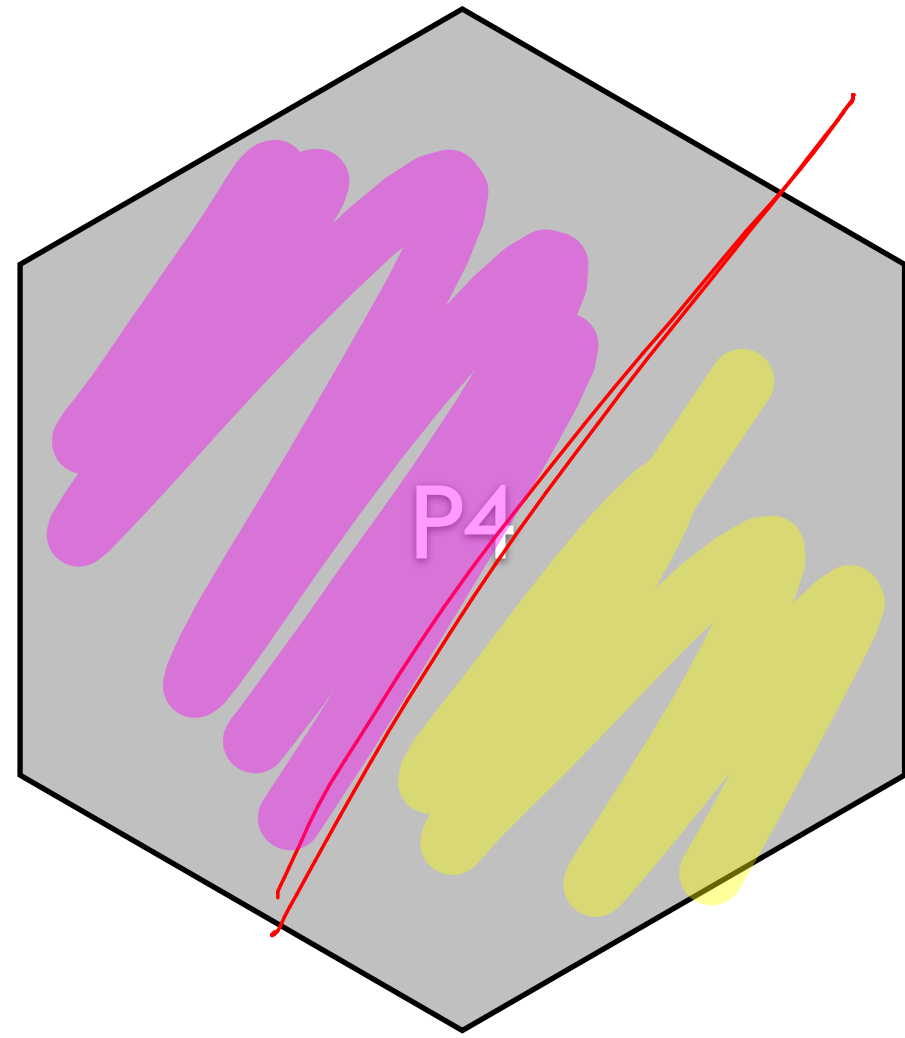
## Legend

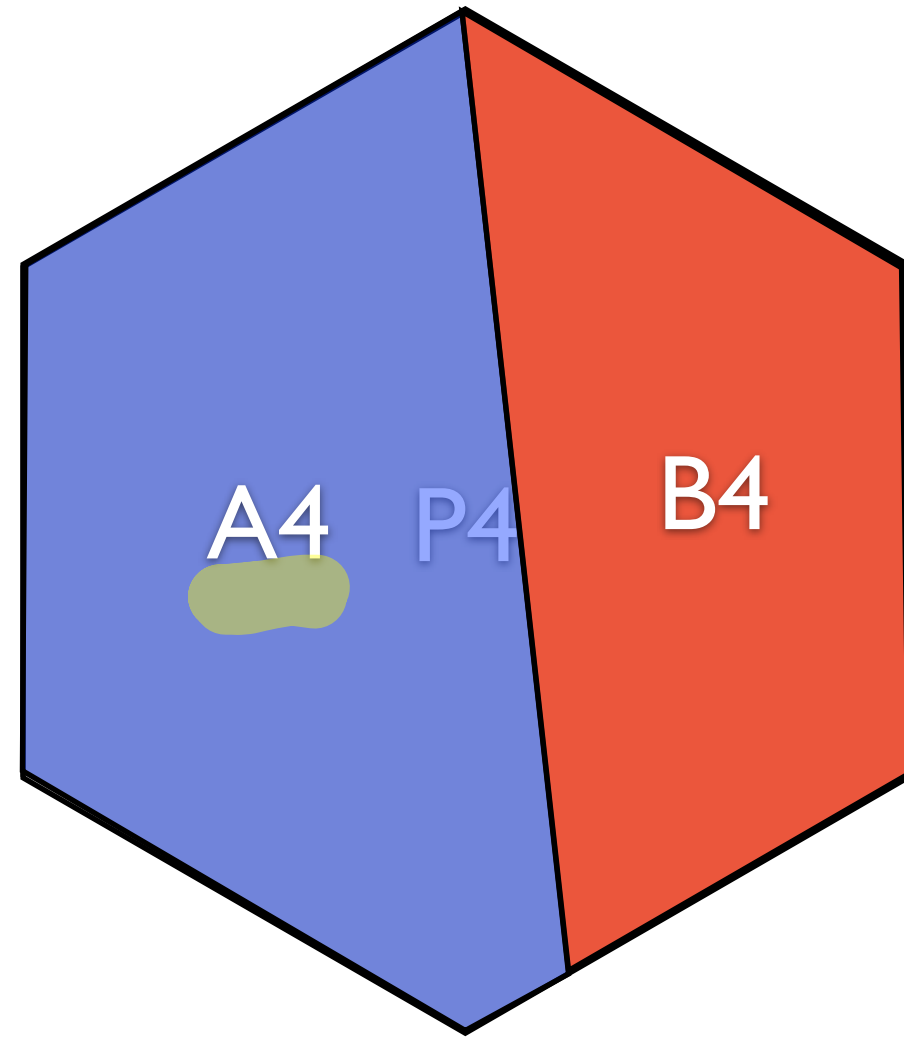
### Polling Place

- Herman Key Recreation Center
- Clark Elementary School
- Carver Recreation Center
- Walker Upper Elementary School
- Benjamin Tonsler Park
- Carter Family Life Center
- Venable Elementary School
- Alumni Hall









# gerrymander problem

given:

output:



# gerrymander problem

given:  $m$   $A_1, A_2, \dots, A_n$   $n$  is even

# people

fraction of  $m$  that votes for  $A$

output:  $D_1, D_2$

such that  $|D_1| = |D_2|$

$$A(D_1) > \frac{mn}{4}$$

$$A(D_2) > \frac{mn}{4}$$

or “failure” if no such solution is possible

# Example

# Gerrymander

imagine very last precinct and how it is assigned:

$D_1$

$j, x, y$   
 $e$

$P_n$

assign to  $D_1$

this means that  $A_n$  people votes for  $A$  in  $D_1$

$D_2$

$n-j, x, y$

assign to  $D_2$

"  $A_n$  people vote for  $A$  in  $D_2$

# Gerrymander

$$\underline{\underline{S_{j,k,x,y} =}}$$

# Gerrymander

$S_{j,k,x,y}$  = there is a split of first  $j$  precincts  
in which  $|D_1| = k$  and  
 $x$  people in  $D_1$  vote A  
 $y$  people in  $D_2$  vote A

true or false

$$S_{j,k,x,y} = S_{j-1, k, x, y - a_j} \quad \text{or}$$

$$S_{j-1, k-1, x - a_j, y}$$

true if placing  $p_j$  into  
 $D_2$

place  $p_j$  into  
 $D_1$

$$S_{j,k,x,y} = S_{j-1,k-1,x-A_j,y} \vee S_{j-1,k,x,y-A_j}$$

Gerrymander(P,A,m)

initialize array S[o,o,o,o]

$$S_{j,k,x,y} = S_{j-1,k-1,x-A_j,y} \vee S_{j-1,k,x,y-A_j}$$

Gerrymander(P,A,m)

initialize array  $S[0,0,0,0]$

for  $j=1, \dots, n$

  for  $k=1, \dots, j$

    for  $x=0, \dots, jm$

      for  $y=0, \dots, jm$

        fill table according to equation

search for true entry at  $S[n, n/2, >mn/4, >mn/4]$