L10

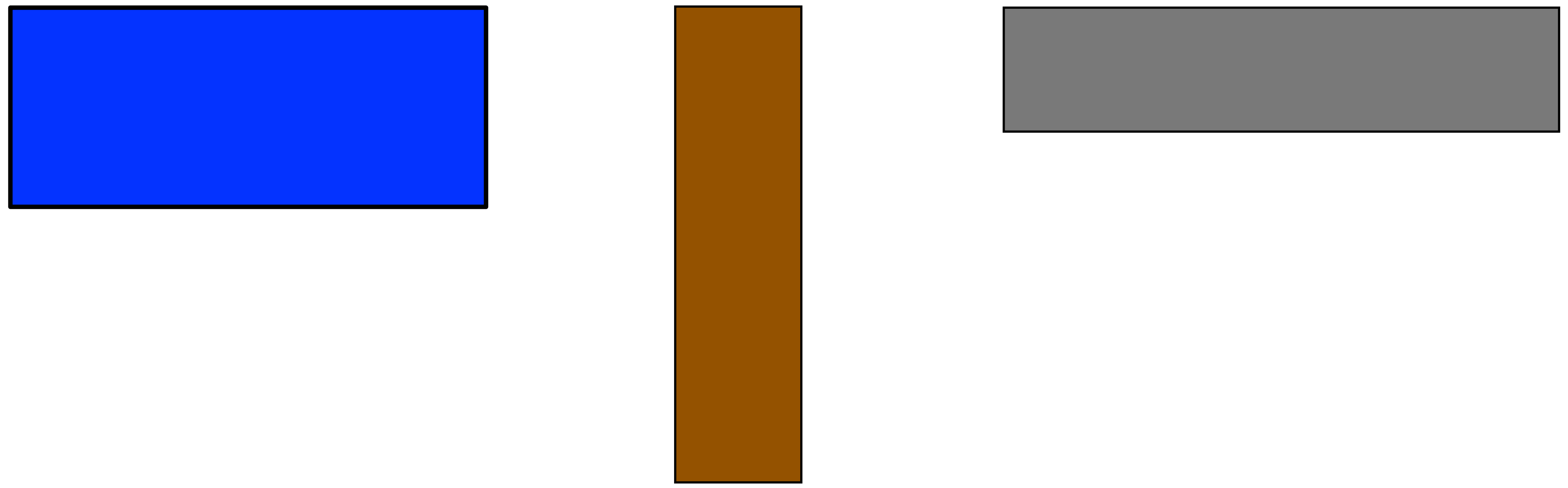
4102

Sep 26 2013

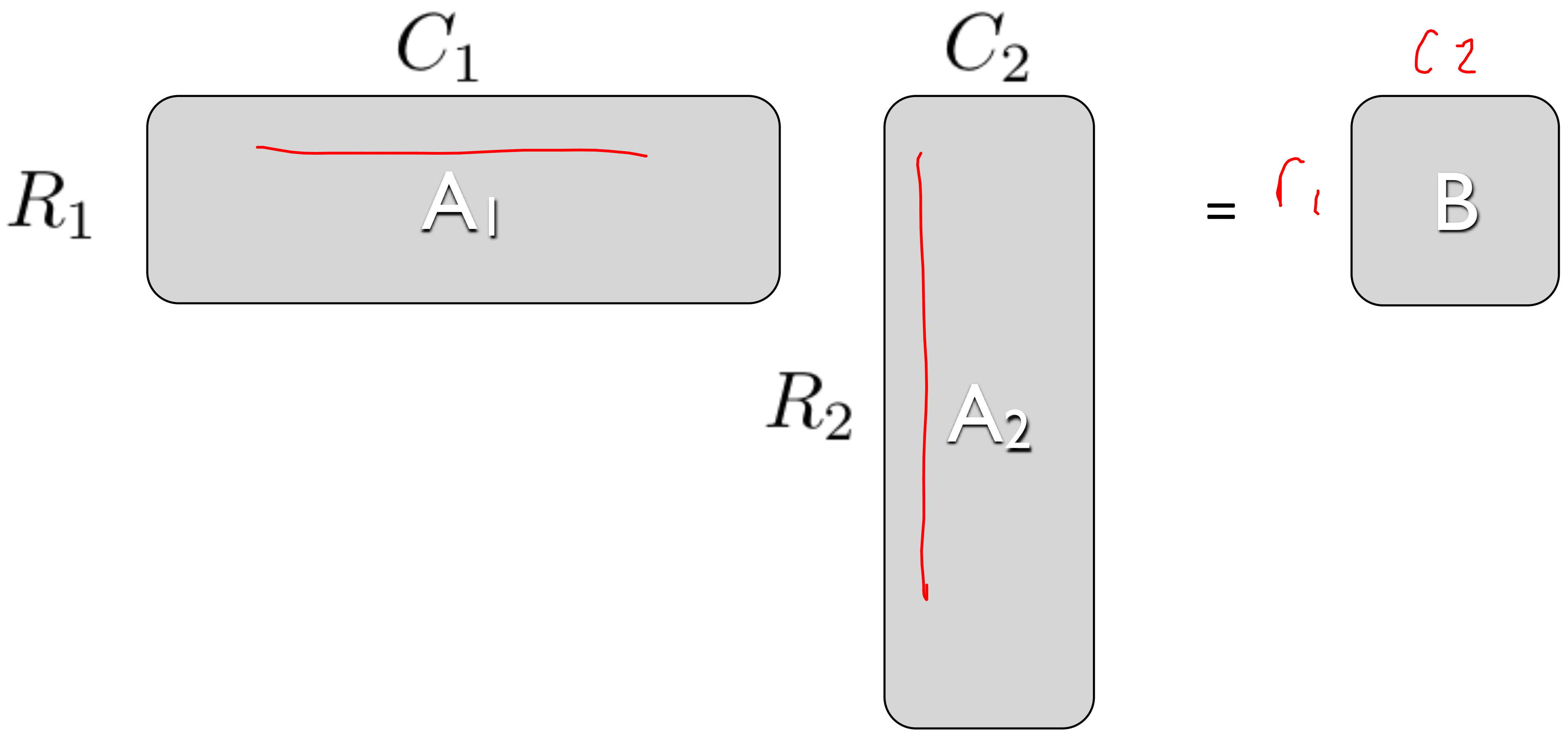
abhi shelat

Dynamic programming: matrix chains, typesetting

Matrix



$c_1 = r_2$

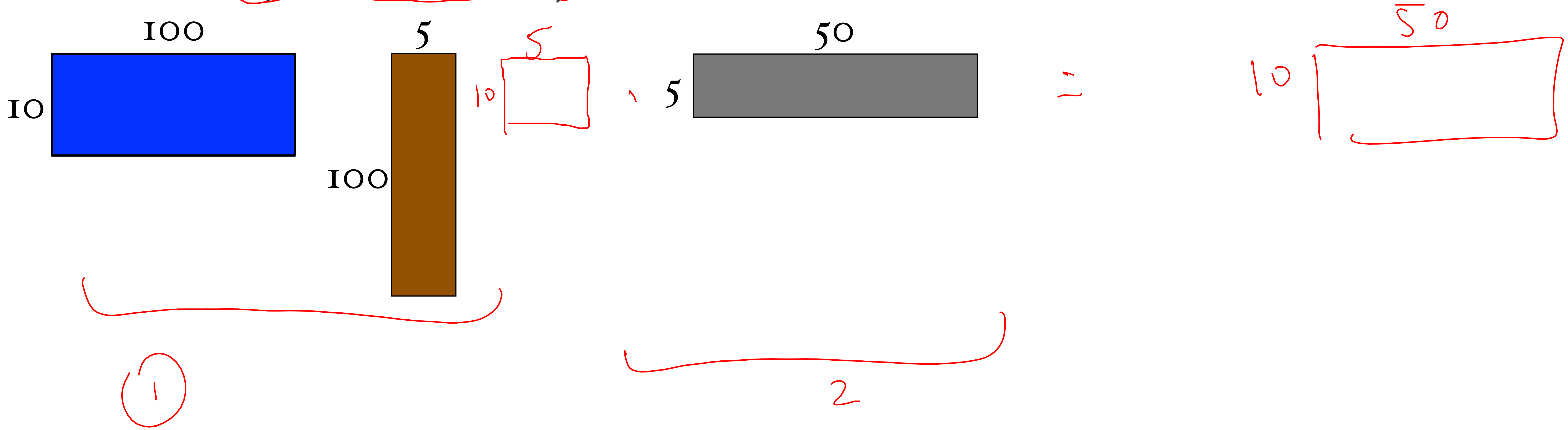


\# operations = $R_1 \cdot C_1 \cdot C_2$

$$A_1 \cdot A_2 \cdot A_3$$

$$\underbrace{(A_1 \cdot A_2)} \cdot \underbrace{A_3} \qquad A_1 \cdot (\underbrace{A_2 \cdot A_3})$$

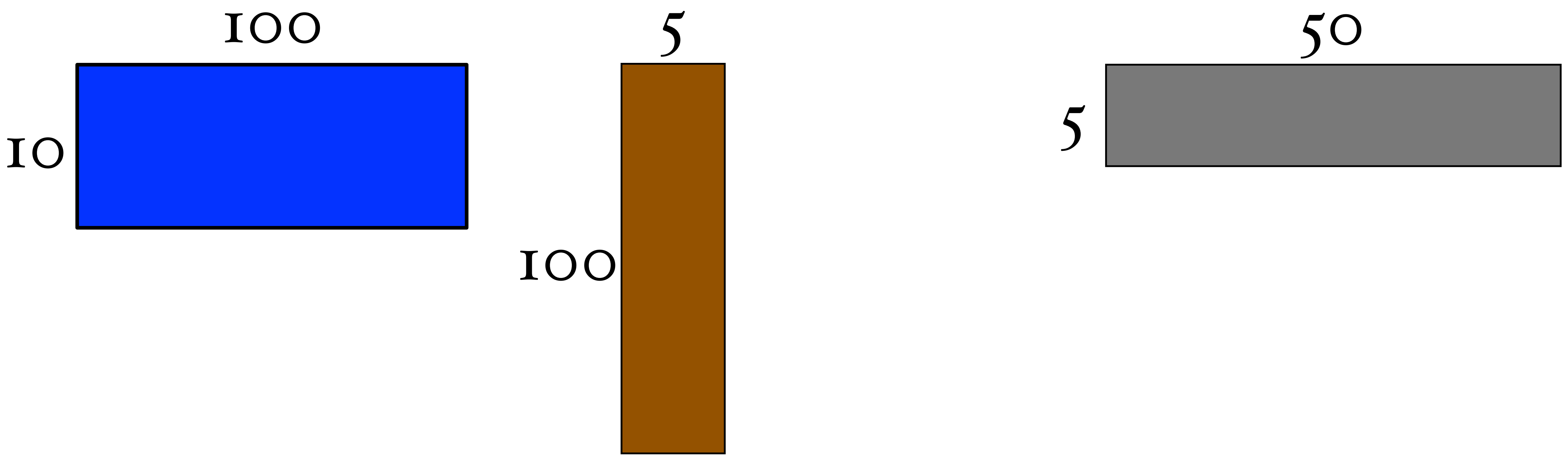
$$(A_1 \cdot A_2) \cdot A_3$$



$$10 \cdot 100 \cdot 5 + 10 \cdot 5 \cdot 50 = 7500$$

operations

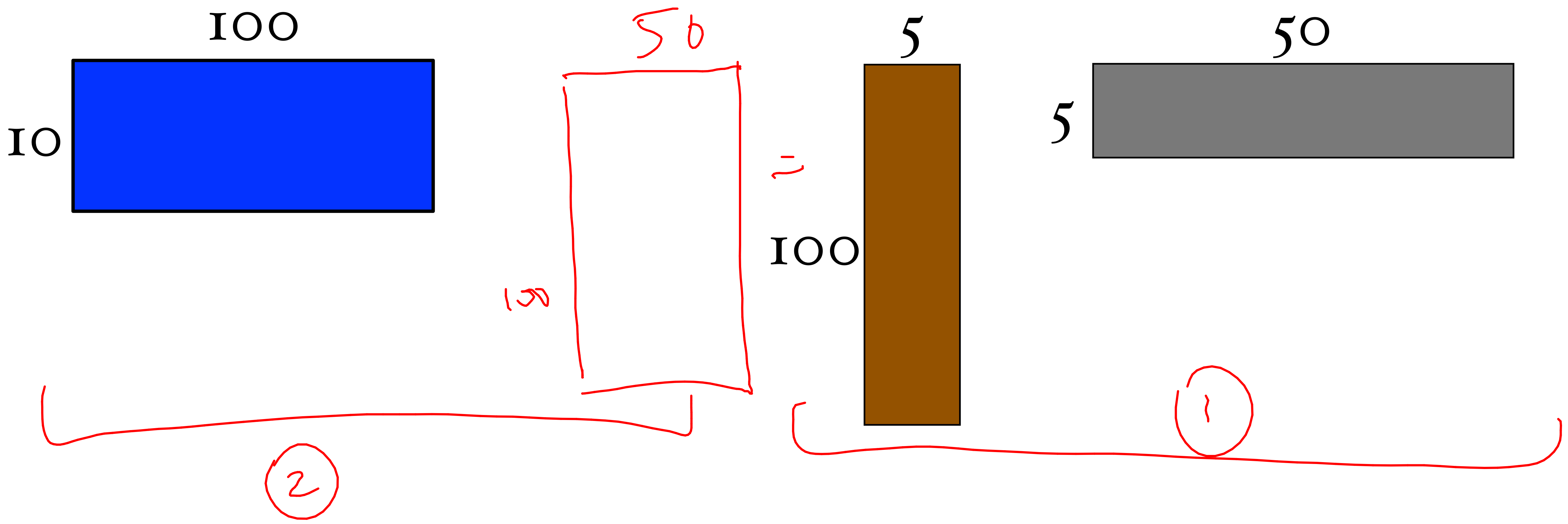
$$(A_1 \cdot A_2) \cdot A_3$$



$$10 \cdot 100 \cdot 5 + 10 \cdot 5 \cdot 50$$

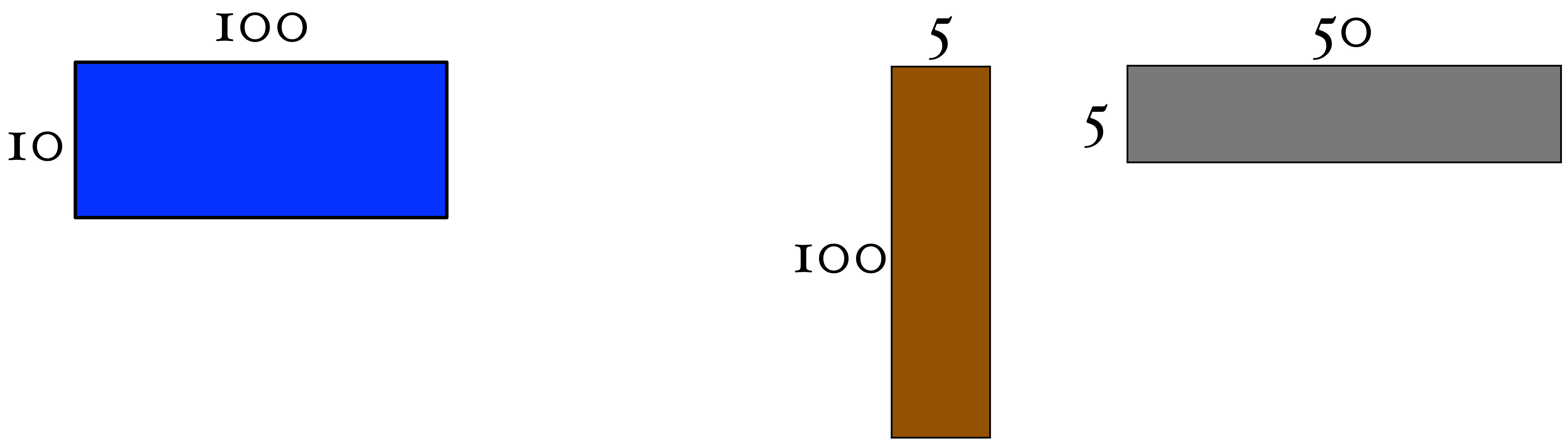
operations

$$A_1 \cdot A_2 \cdot A_3$$



10 · 100 · 50 + 100 · 5 · 50 = 75,000 operations !!

$$A_1 \cdot A_2 \cdot A_3$$



$$100 \cdot 5 \cdot 50 + 10 \cdot 100 \cdot 50$$

operations

order matters

(for efficiency)

determine the order that minimizes the work required to multiply a sequene of matricies.

how many ways to compute?

$$A_1 A_2 A_3 \ldots A_n$$

P(n): # of ways to multiply
n matricies.

how many ways to compute?

$A_1 A_2 A_3 \ldots A_n$

$P(1) \cdot P(n-1)$

+

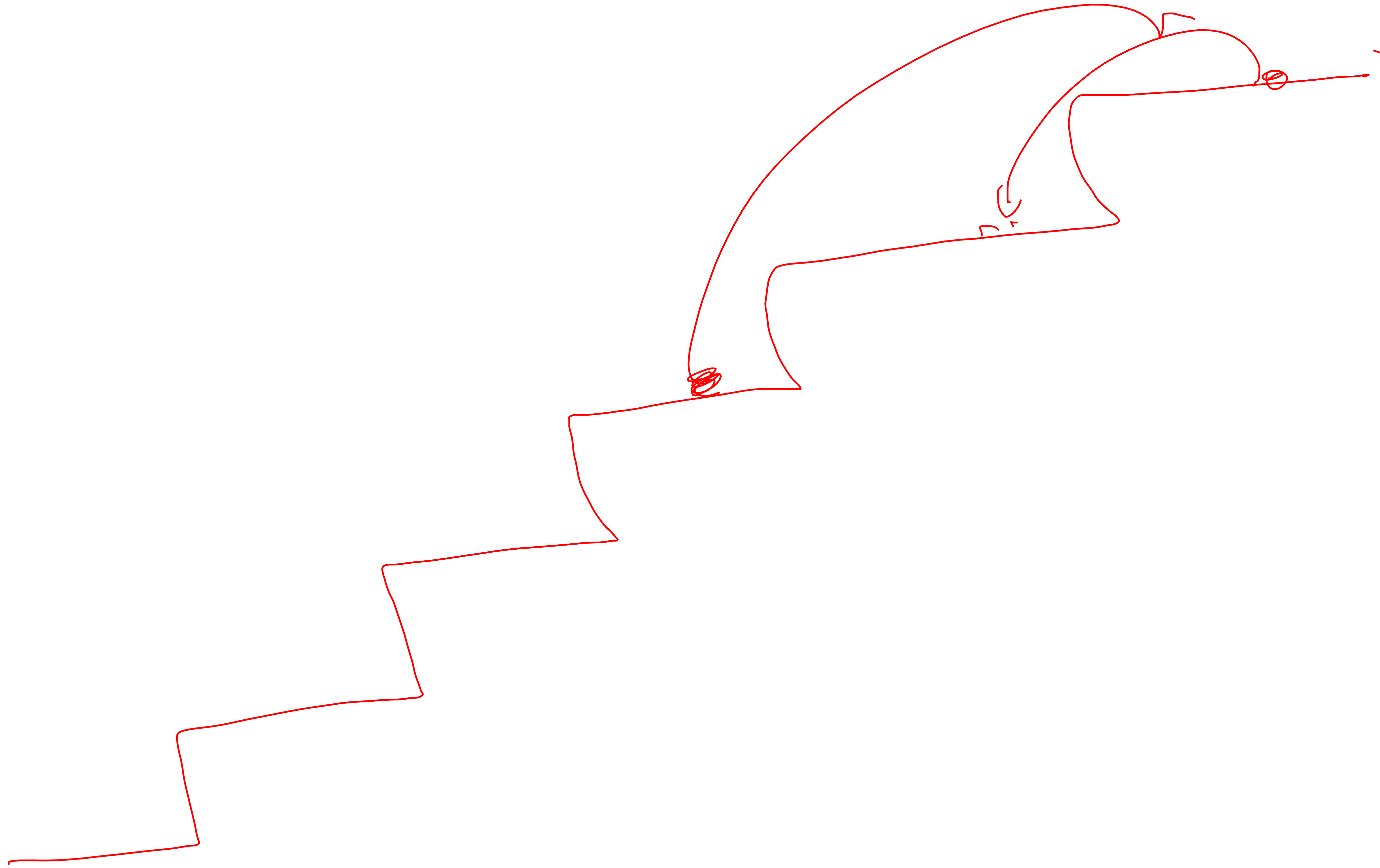
$A_1 A_2 A_3 \ldots A_n$

$P(2) \cdot P(n-2)$

+

...

$P(n) =$



$$T(n) = T(n-1) + T(n-2)$$

how many ways to compute?

$$A_1 \, A_2 A_3 \ldots A_n$$

$$A_1 A_2 \, A_3 \ldots A_n$$

$$A_1 A_2 A_3 \, \ldots A_n$$

$$P(n) = P(1) \cdot P(n-1) + P(2)P(n-2) + P(3)P(n-3) + \ldots + P(n-2)P(2) + P(n-1)P(1)$$

$$= \sum_{i=1}^{n-1} P(i)P(n-i) \quad \sim 4^n$$

how many ways to compute?

$$(A_1)(A_2 A_3 \ldots A_n)$$

$$(A_1 A_2)(A_3 \ldots A_n)$$

$$(A_1 A_2 A_3)(\ldots A_n)$$

how do we solve it?

identify smaller instances of the problem

devise method to combine solutions

small # of different subproblems

solved them in the right order

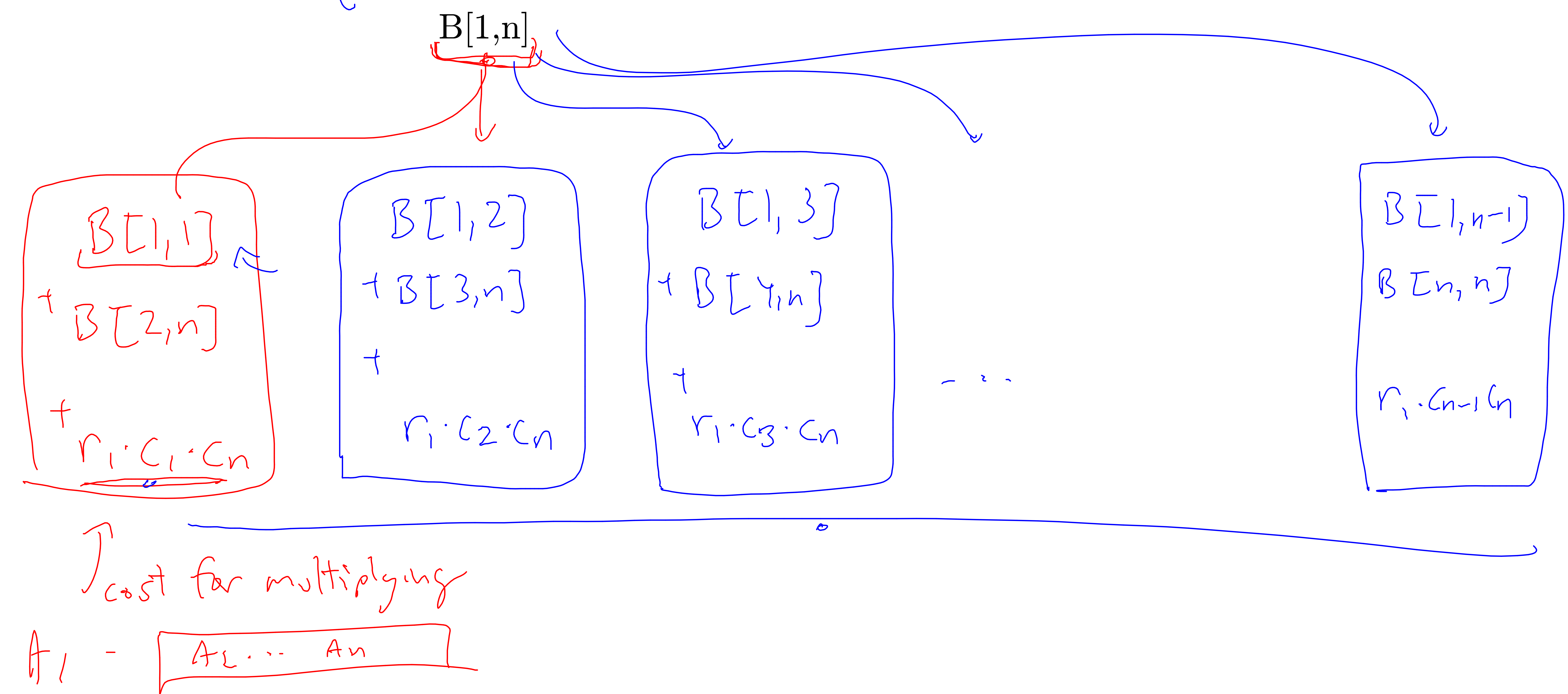
optimal way to compute

$$A_1 A_2 A_3 A_4 \ldots A_n$$

$B[1,n]$ = cost of the optimal order for computing the product $A_1 \ldots A_n$

optimal way to compute

$$A_1 A_2 A_3 A_4 \ldots A_n$$



optimal way to compute

$$A_1 A_2 A_3 A_4 \ldots A_n$$

B[1,n]

B[1,1]
B[2,n]

$R_1 C_1 C_n$

optimal way to compute

$$A_1 A_2 A_3 A_4 \ldots A_n$$

B[1,n]

B[1,1]	B[1,2]	...	B[1,n-2]	B[1,n-1]
B[2,n]	B[3,n]	...	B[n-1,n]	B[n,n]
$R_1 C_1 C_n$	$R_1 C_2 C_n$		$R_1 C_{n-2} C_n$	$R_1 C_{n-1} C_n$

$$B(i,i) = 0$$

$$B(1,n) = \min \begin{cases} B(1,1) + B(2,n) + r_1 \cdot c_1 \cdot c_n \\ B(1,2) + B(3,n) + r_1 \cdot c_2 \cdot c_n \\ \vdots \\ B(1,n-1) + B(n,n) + r_1 \cdot c_{n-1} \cdot c_n \end{cases}$$

$$B(i,j) = \begin{cases} 0 & \text{if } i=j \\ \min_{k=i}^{j-1} \left\{ \underbrace{B[i,k]}_{\text{prefix}} + \underbrace{B[k+1,j]}_{\text{suffix}} + \underbrace{r_i \cdot c_k \cdot c_j}_{\text{combo}} \right\} \end{cases}$$

$$B(i,i) = 1$$

$$B(1,n) = \min \begin{cases} B(1,1) + B(2,n) + r_1 c_1 c_n \\ B(1,2) + B(3,n) + r_1 c_2 c_n \\ \vdots \\ B(1,n-1) + B(n,n) + r_1 c_{n-1} c_n \end{cases}$$

$$B(i,j) =$$

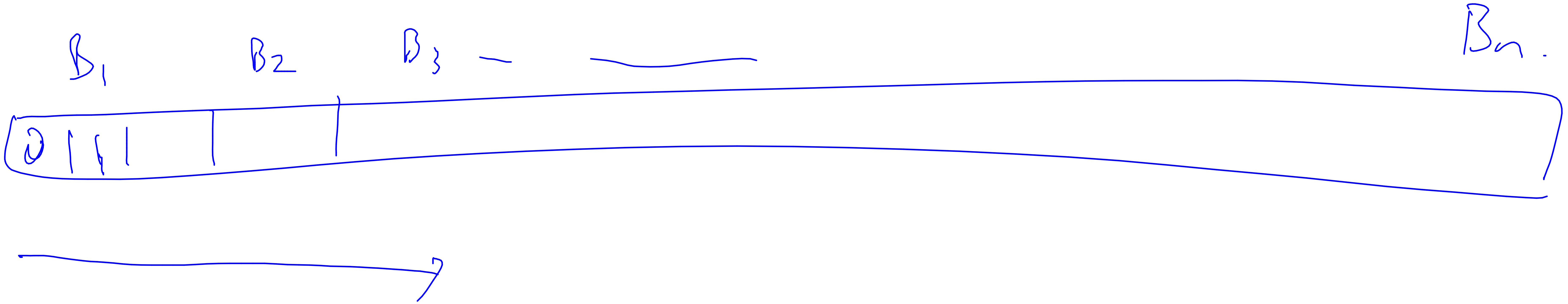
$$\begin{cases} 0 \text{ if } i = j \\ \min_k\{B(i,k) + B(k+1,j) + r_i c_k c_j \end{cases}$$

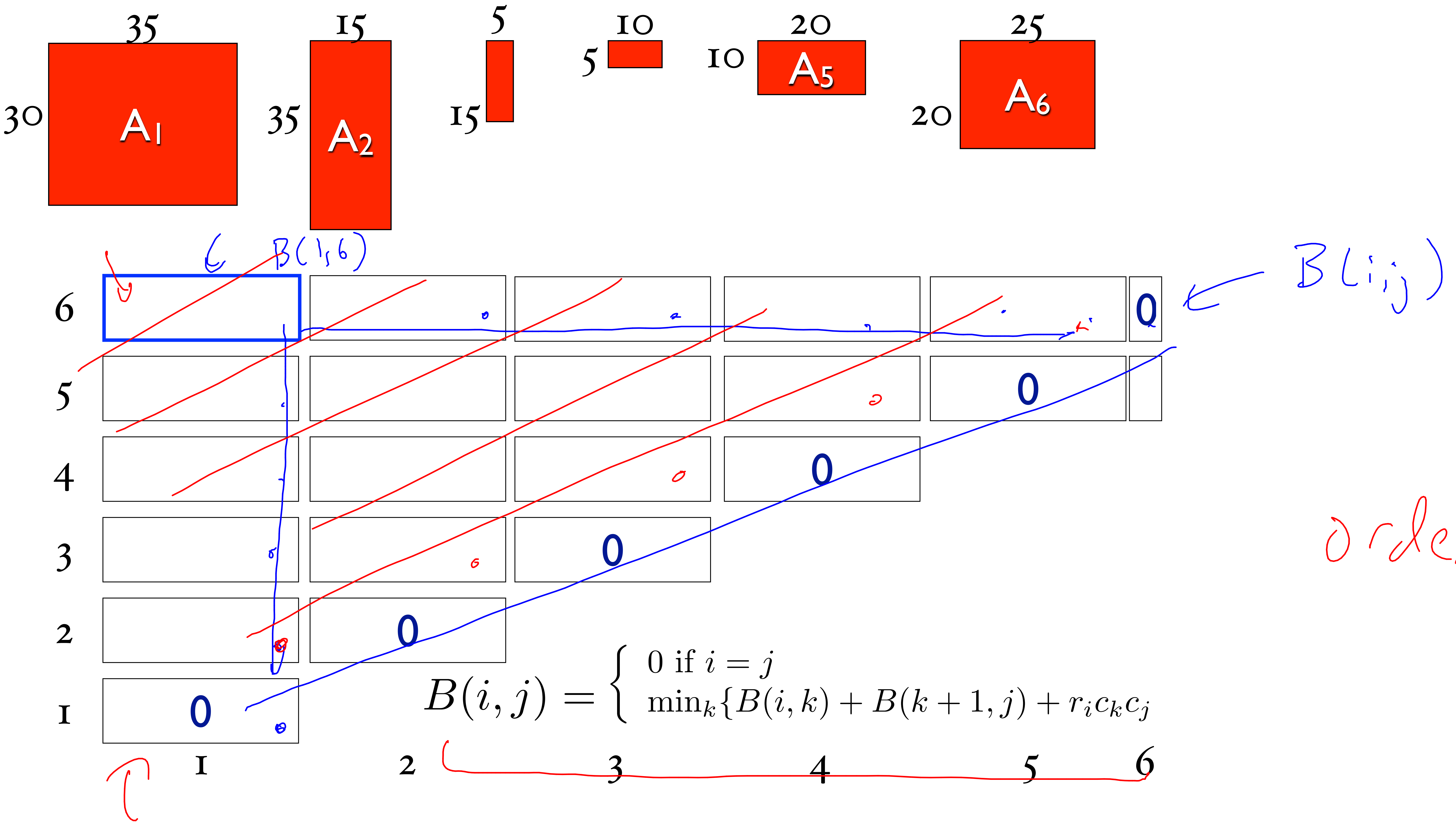
i $k = i \rightarrow j-1$

$$B(i,j) = \begin{cases} 0 \text{ if } i = j \\ \min_k \{B(i,k) + B(k+1,j) + r_i c_k c_j \end{cases}$$

which order to solve?

Logcutter

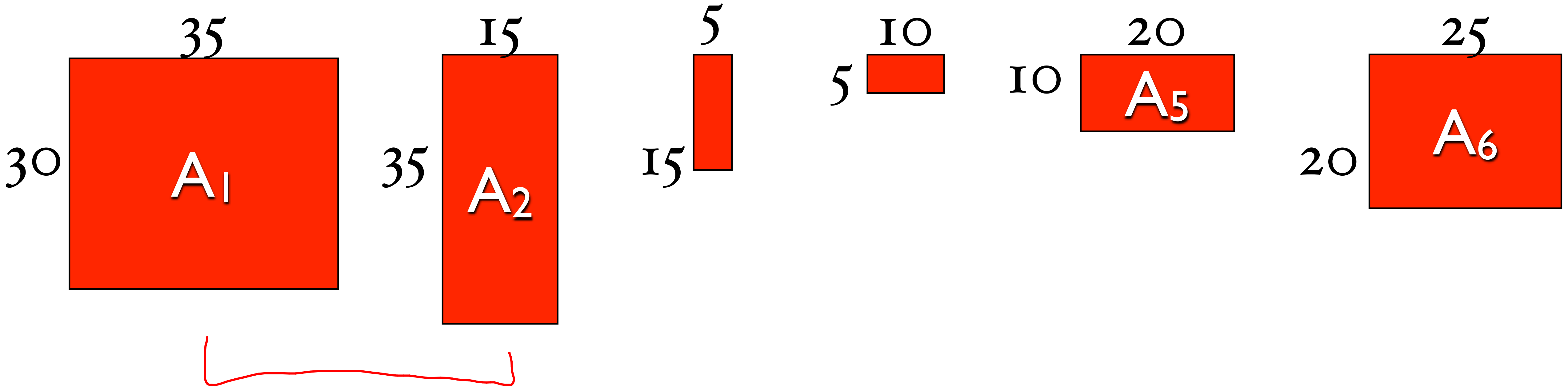




$$B(i,j) = \begin{cases} 0 \text{ if } i = j \\ \min_k \{B(i,k) + B(k+1,j) + r_i c_k c_j \end{cases}$$

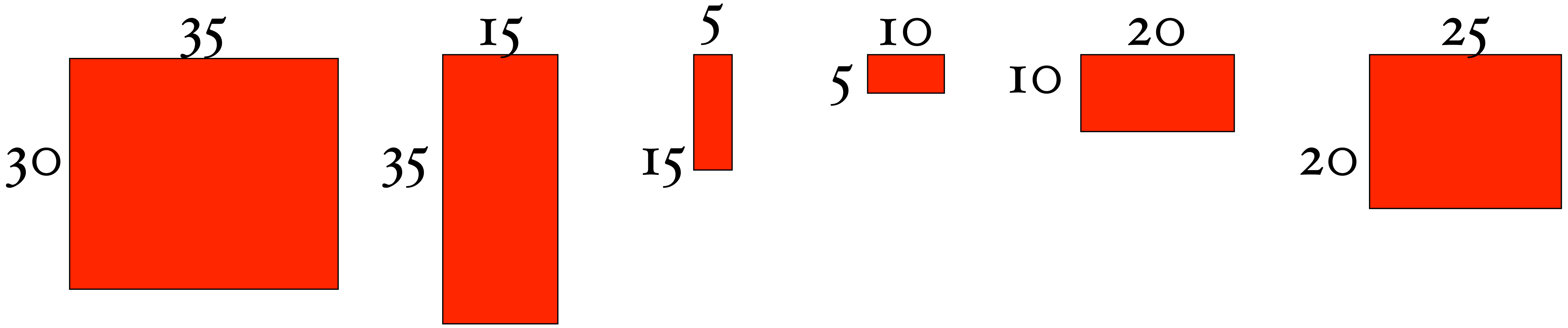
order: solve the diagonals!

B(1,6) = $B_{11} + B_{26}$ +
$B_{12} + B_{36}$ +



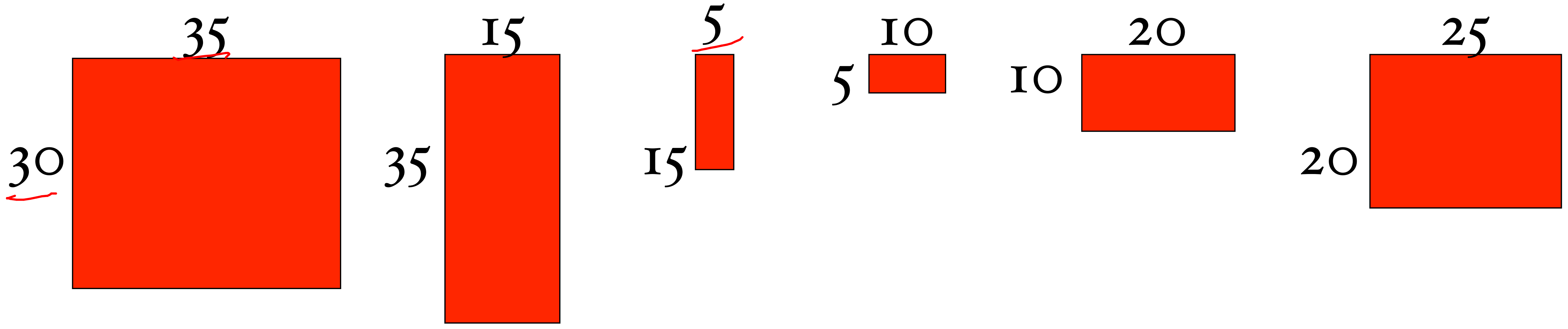
$B(1,2) = B_{11} + B_{22} + 30 \cdot 35 \cdot 15 = 15\,750$

1050



	1	2	3	4	5	6
6					10*20*25 = 5000	0
5				5*10*20 = 1000	0	
4			15*5*10 = 750	0		
3		35*15*5 = 2625	0			
2	30*35*15 = 15750	0				
1	0					

$$B(i,j) = \begin{cases} 0 \text{ if } i = j \\ \min_k \{ B(i,k) + B(k+1,j) + r_i c_k c_j \end{cases}$$



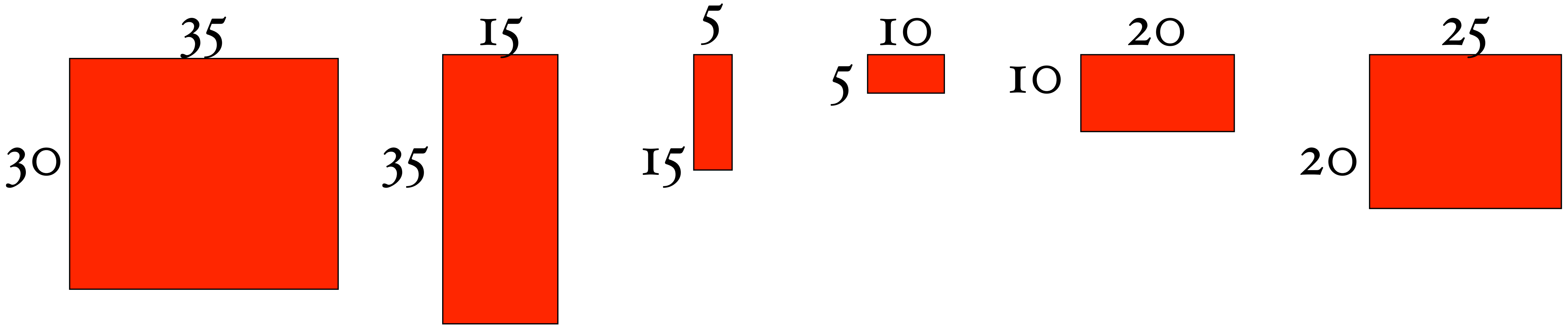
3	7875	35*15*5 = 2625	0
2	30*35*15 = 15750	0	
1	0		
	1	2	3

4 5 6

$$B(i,j) = \begin{cases} 0 \text{ if } i = j \\ \min_k \{B(i,k) + B(k+1,j) + r_i c_k c_j \end{cases}$$

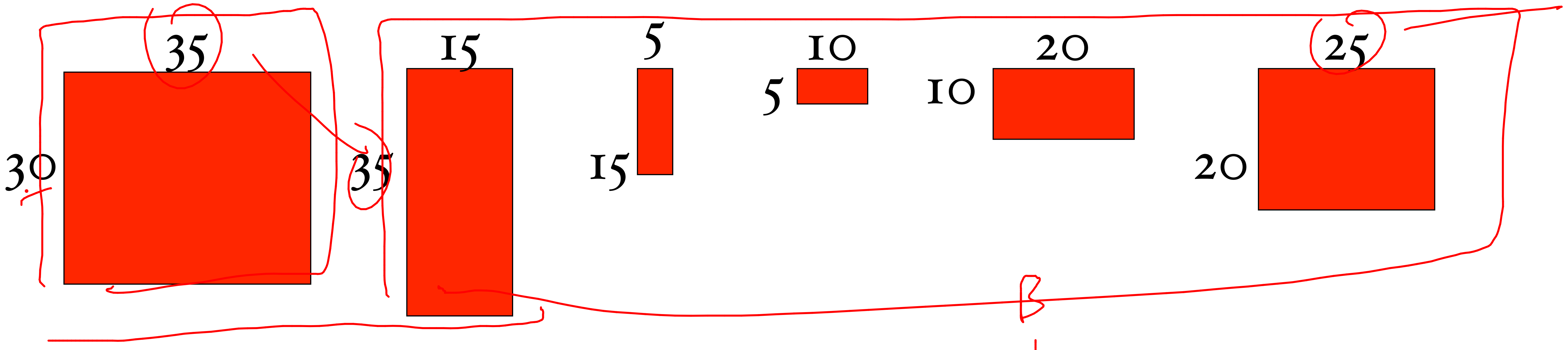
$$B_{13} = \min \begin{cases} \overset{0}{B_{11}} + \overset{2625}{B_{23}} + 30 \cdot 35 \cdot 5 \\ B_{12} + B_{33} + 30 \cdot 15 \cdot 5 \\ 15750 + 0 \end{cases}$$

$$= 7875$$



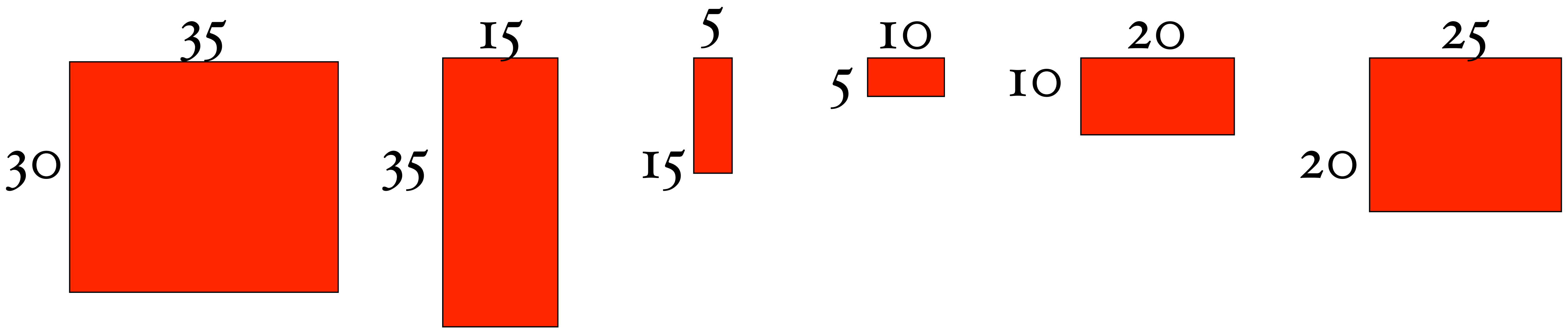
6		10500	5375	3500	10*20*25 = 5000	0
5	11875	7125	2500	5*10*20 = 1000	0	
4	9375	4375	15*5*10 = 750	0		
3	7875	35*15*5 = 2625	0			
2	30*35*15 = 15750	0				
1	0					
	1	2	3	4	5	6

$$B(i,j) = \begin{cases} 0 \text{ if } i = j \\ \min_k \{B(i,k) + B(k+1,j) + r_i c_k c_j \end{cases}$$



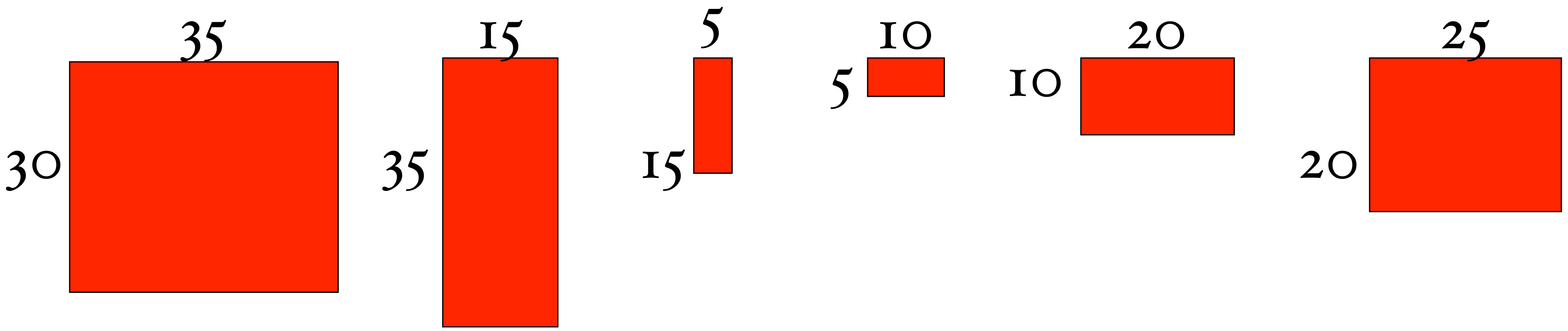
6 

$$C(1,6) = \min \begin{cases} k=1 & C(1,1) + C(2,6) + r_1 c_1 c_6 \\ k=2 & C(1,2) + C(3,6) + r_1 c_2 c_6 \\ k=3 & C(1,3) + C(4,6) + r_1 c_3 c_6 \\ k=4 & C(1,4) + C(5,6) + r_1 c_4 c_6 \\ k=5 & C(1,5) + C(6,6) + r_1 c_5 c_6 \end{cases}$$



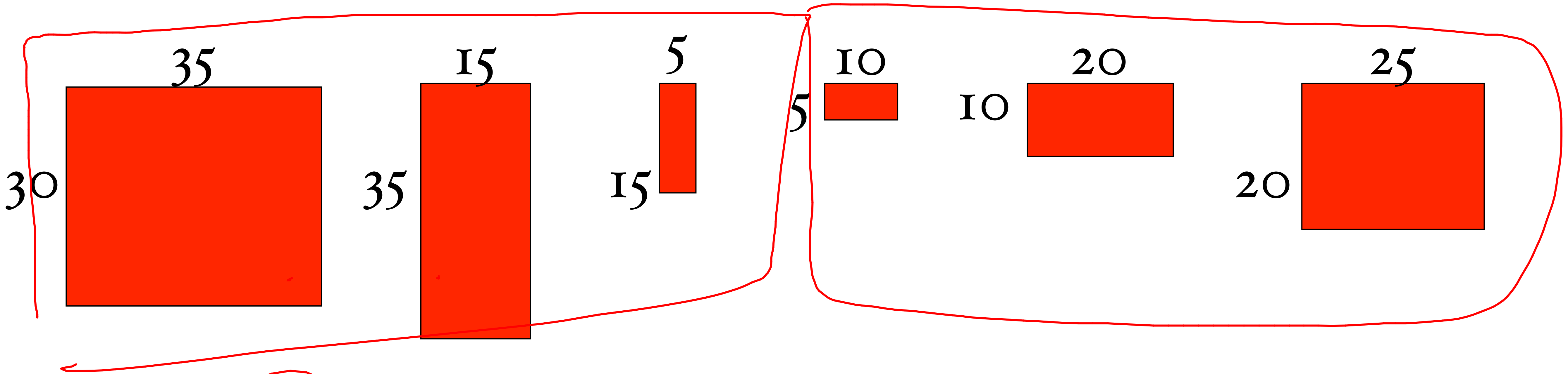
6

$$C(1,6) = \min \begin{cases} k=1 & 0 + 10500 + 30 \cdot 35 \cdot 25 \\ k=2 & 15750 + 5375 + 30 \cdot 15 \cdot 25 \\ k=3 & 7875 + 3500 + 30 \cdot 5 \cdot 25 \\ k=4 & 9375 + 5000 + 30 \cdot 10 \cdot 25 \\ k=5 & 11875 + 0 + 30 \cdot 20 \cdot 25 \end{cases}$$



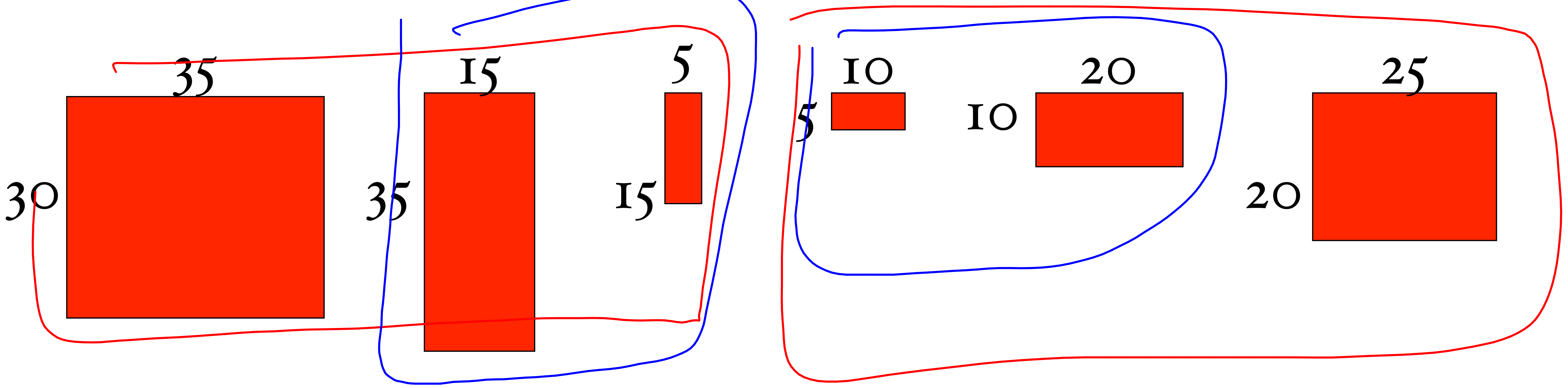
6 

$$C(1,6) = \min \begin{cases} k=1 & 0 + 10500 + 26250 \\ k=2 & 15750 + 5375 + 11250 \\ k=3 & 7875 + 3500 + 3750 \\ k=4 & 9375 + 5000 + 7500 \\ k=5 & 11875 + 0 + 15000 \end{cases}$$



	1	2	3	4	5	6
6	15125	10500	5375	3500	10*20*25 = 5000	0
5	11875	7125	2500	5*10*20 = 1000	0	
4	9375	4375	15*5*10 = 750	0		
3	7875	35*15*5 = 2625	0			
2	30*35*15 = 15750	0				
1	0					

$B_{11} \cdot B_{23}$



6	15125	10500	5375	3500	10*20*25 = 5000	0
5	11875	7125	2500	5*10*20 = 1000	0	
4	9375	4375	15*5*10 = 750	0		
3	7875	35*15*5 = 2625	0			
2	30*35*15 = 15750	0				
1	0					
	1	2	3	4	5	6

matrix-chain-mult(p)

initialize array m[x,y] to zero

matrix-chain-mult(p)

initialize array m[x,y] to zero

starting at diagonal, working towards upper-left

compute m[i,j] according to

$$\begin{cases} 0 \text{ if } i = j \\ \min_k \{B(i,k) + B(k+1,j) + r_i c_k c_j \end{cases}$$

how many cells are there $\Theta(n^2)$

how much time per cell

$\Theta(n)$

$\Rightarrow \Theta(n^3)$

running time?

initialize array m[x,y] to zero

starting at diagonal, working towards upper-left

compute m[i,j] according to

$$\begin{cases} 0 \text{ if } i = j \\ \min_k\{B(i,k) + B(k+1,j) + r_i c_k c_j \end{cases}$$

Typesetting

It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair, we had everything before us, we had nothing before us, we were all going direct to heaven, we were all going direct the other way - in short, the period was so far like the present period, that some of its noisiest authorities insisted on its being received, for good or for evil, in the superlative degree of comparison only.

It was the best of times, it was the
worst of times, it was the age of wisdom, it was
the age of foolishness, it was the epoch of belief,
it was the epoch of incredulity, it was the season
of Light, it was the season of Darkness, it was the
spring of hope, it was the winter of despair, we
had everything before us, we had nothing before us,
we were all going direct to heaven, we were all
going direct the other way - in short, the period
was so far like the present period, that some of
its noisiest authorities insisted on its being
received, for good or for evil, in the superlative
degree of comparison only.

margin M

First rule of typesetting

never print in the margin!

⟷ are simply not allowed

It was the best of times, it was the worst
of times, it was the age of wisdom, it was
the age of foolishness, it was the epoch
of belief, it was the epoch of
incredulity, it was the season of Light,
it was the season of Darkness, it was the
spring of hope, it was the winter of
despair, we had everything before us, we
had nothing before us, we were all going
direct to heaven, we were all going direct
the other way - in short, the period was
so far like the present period, that some of its
noisiest authorities insisted on its being
received, for good or for evil, in the superlative
degree of comparison only.

whitespace

______ is....

2nd rule: minimize screen space over the entire paragraph.

```
It was the best of times, it was the worst       0     0
of times, it was the age of wisdom, it was       0     0
the age of foolishness, it was the epoch         2     4
of belief, it was the epoch of                  12   144
incredulity, it was the season of Light,         2     4
it was the season of Darkness, it was the        1     1
spring of hope, it was the winter of             6    36
despair, we had everything before us, we         2     4
had nothing before us, we were all going         2     4
direct to heaven, we were all going direct       0     0
the other way - in short, the period was
so far like the present period, that some of its      197
noisiest authorities insisted on its being
received, for good or for evil, in the superlative
degree of comparison only.
```

greedy typeset has penalty 197

It was the best of times, it was the	6	36
worst of times, it was the age of wisdom,	1	1
it was the age of foolishness, it was the	1	1
epoch of belief, it was the epoch of	6	36
incredulity, it was the season of Light,	2	4
it was the season of Darkness, it was the	1	1
spring of hope, it was the winter of	6	36
despair, we had everything before us, we	2	4
had nothing before us, we were all going	2	4
direct to heaven, we were all going direct	0	0
the other way - in short, the period was		
so far like the present period, that some		123
of its noisiest authorities insisted on		
its being received, for good or for evil,		
in the superlative degree of comparison		
only.		

Typesetting problem

input: $w_1, w_2 \ldots w_n$, and a margin M

output: line breaks

such that

length of line i

$c_i \leq M$

minimize $\sum (M - c_i)^2$

all lines of the paragraph

whitespace on line i

Typesetting problem

input: $W = \{w_1, w_2, w_3, \ldots, w_n\}$ M

output: $L = (w_1, \ldots, w_{\ell_1}), (w_{\ell_1+1}, \ldots, w_{\ell_2}), \ldots, (w_{\ell_{x+1}, \ldots, w_n})$

such that

Typesetting problem

the x^{th} line starts @ Word ℓ_{x+1}

input: $W = \{w_1, w_2, w_3, \ldots, w_n\}$ $\quad M$

output: $L = (w_1, \ldots, w_{\ell_1}), (w_{\ell_1+1}, \ldots, w_{\ell_2}), \ldots, (w_{\ell_x+1}, \ldots, w_n)$ ⟸ line breaks

$$c_i = \left(\sum_{j=\ell_i+1}^{\ell_{i+1}} |w_j|\right) + (\ell_{i+1} - \ell_i - 1)$$

↻ space between each word

such that $c_i \leq M \quad \forall i$

$$\min \sum (M - c_i)^2$$

penalty for the paragraph

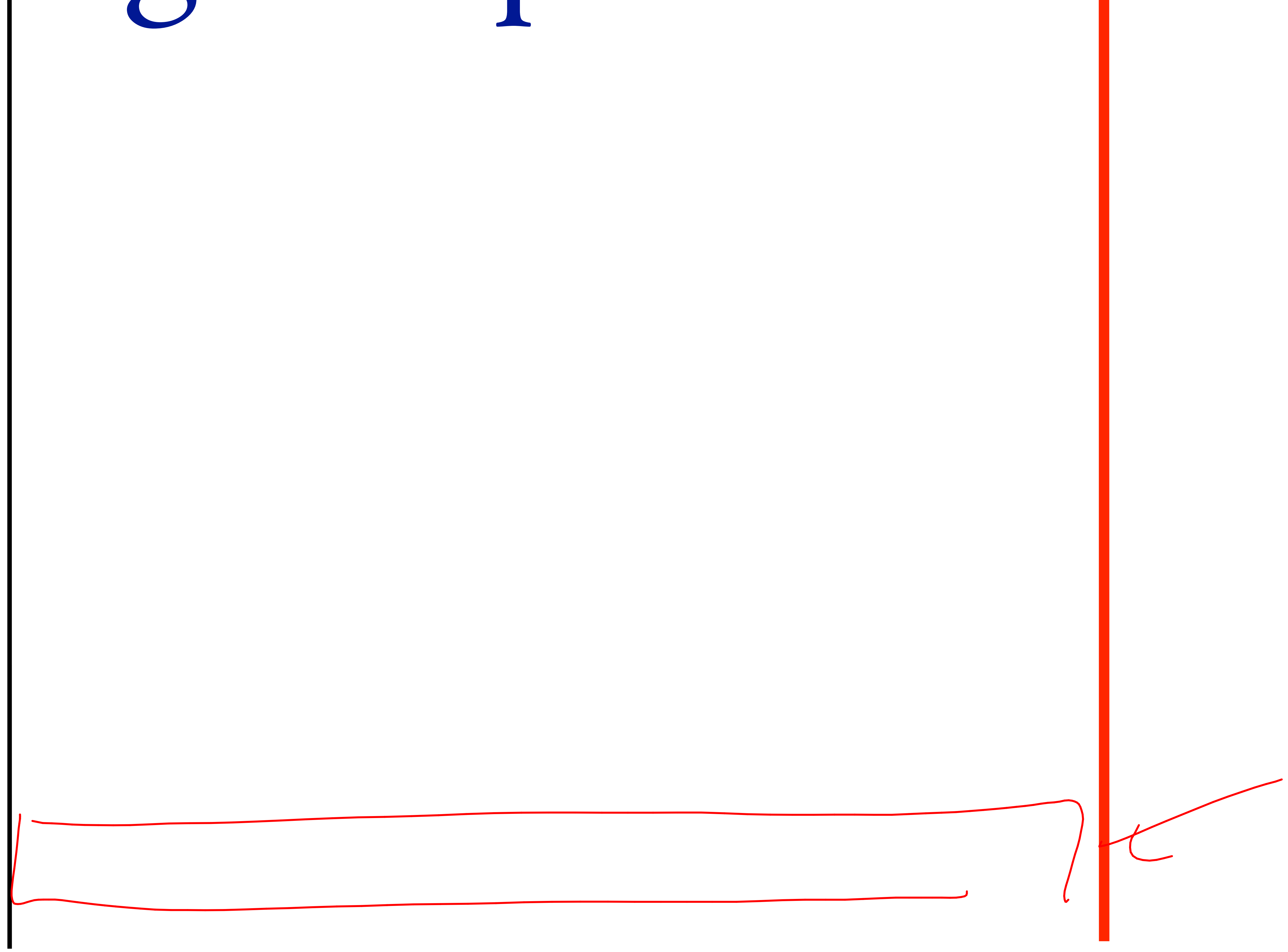
how to solve

define the right variable:

$Best_n$ = smallest penalty for typesetting n words

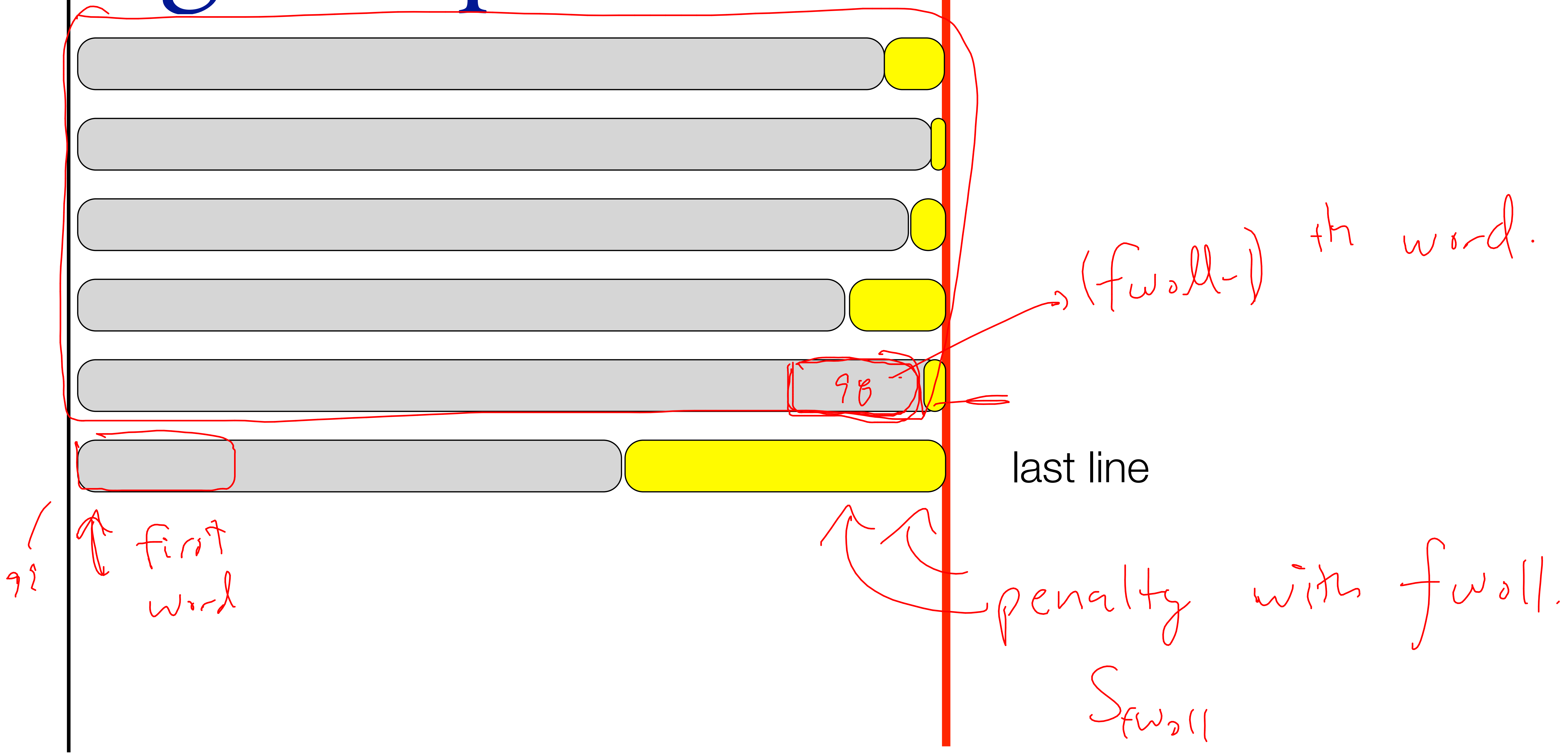
$Best_n$ = ___ + ___

imagine optimal solution



consider the very last line of this paragraph when optimally typeset.

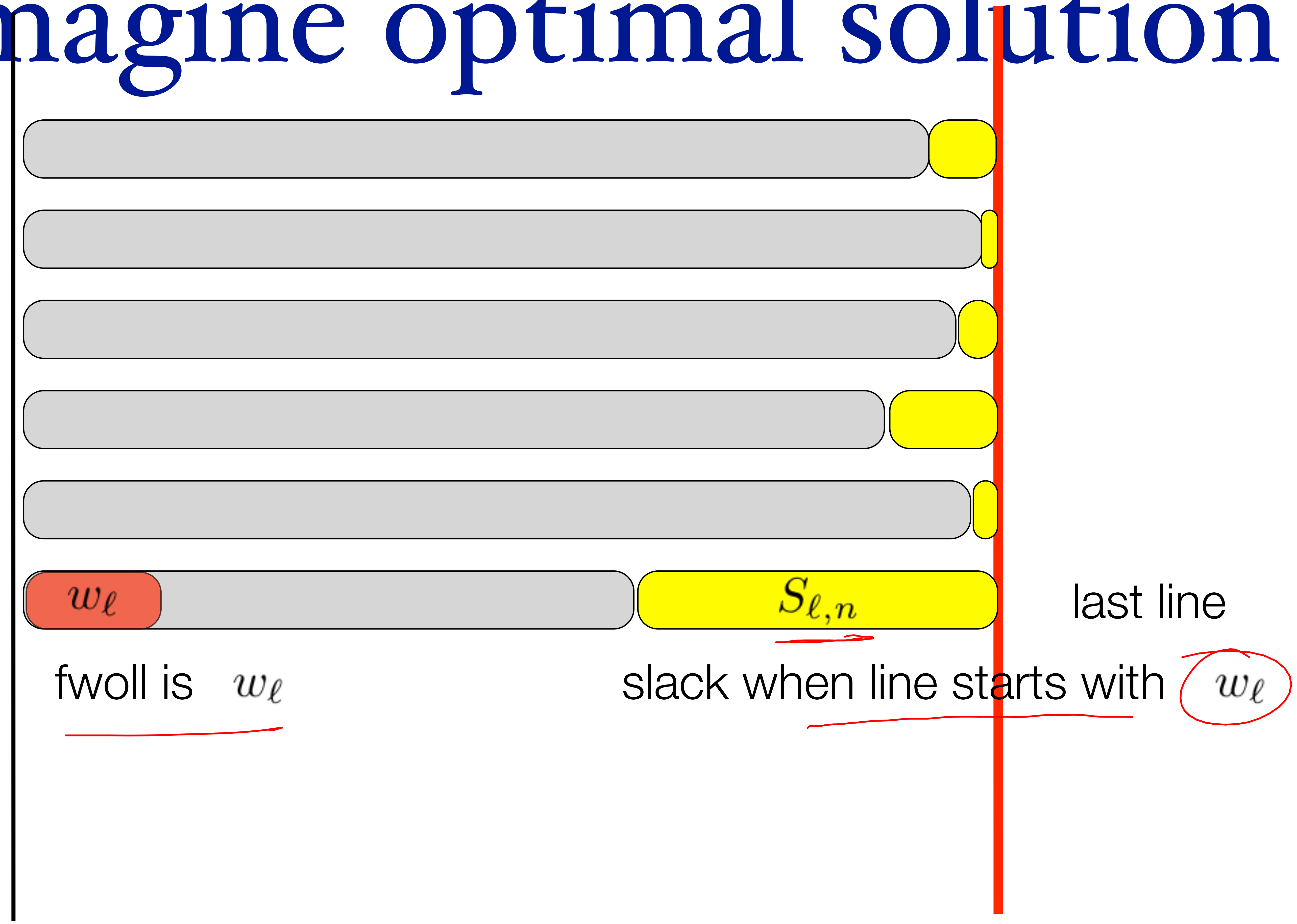
imagine optimal solution



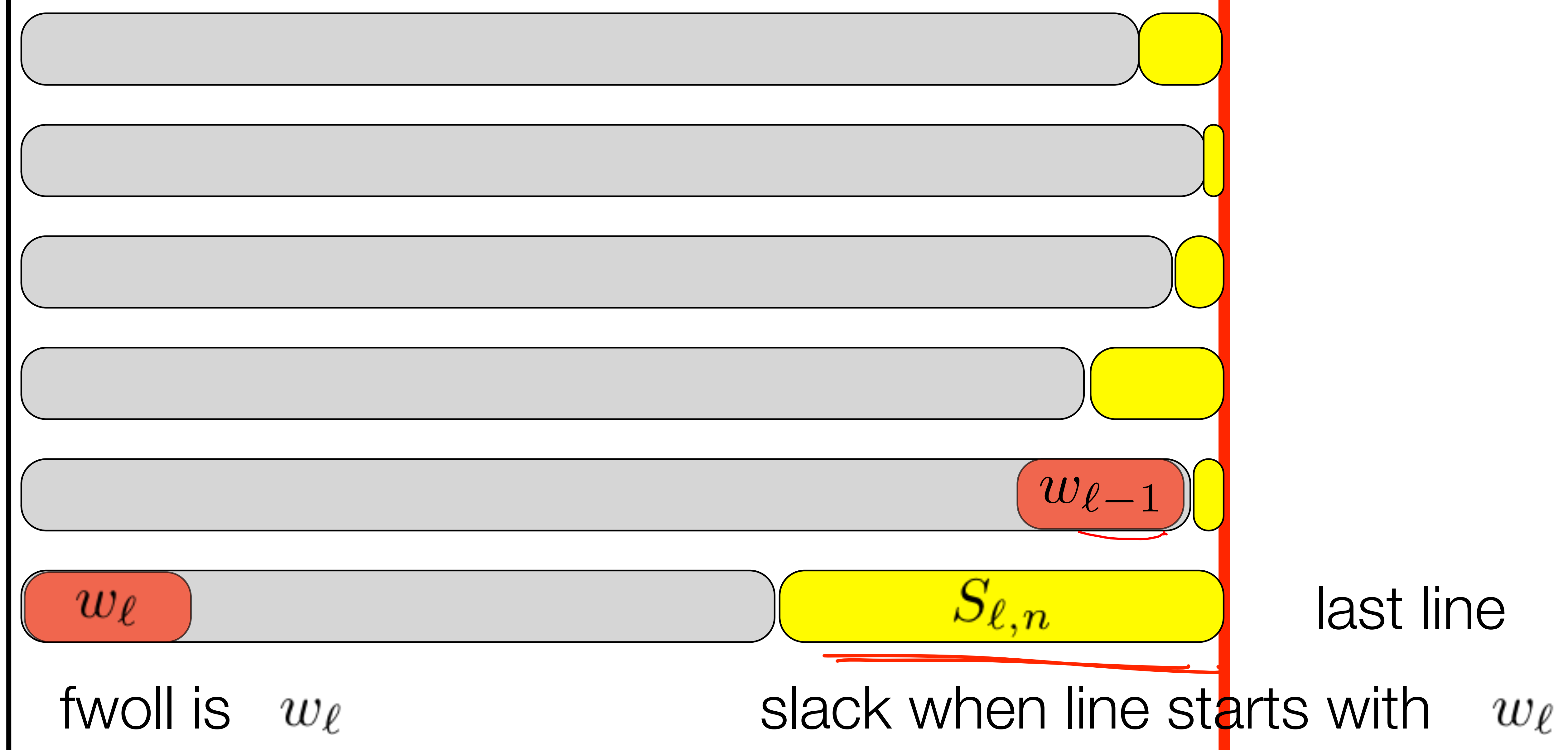
$$Best_n = Best_{fwall-1} + (S_{fwall})^2$$

some word has to
be the first-word-of-
last-line
(fwoll)

imagine optimal solution



imagine optimal solution



$$\mathrm{BEST}_n = \mathrm{BEST}_{\ell-1} + S_{\ell,n}^2$$

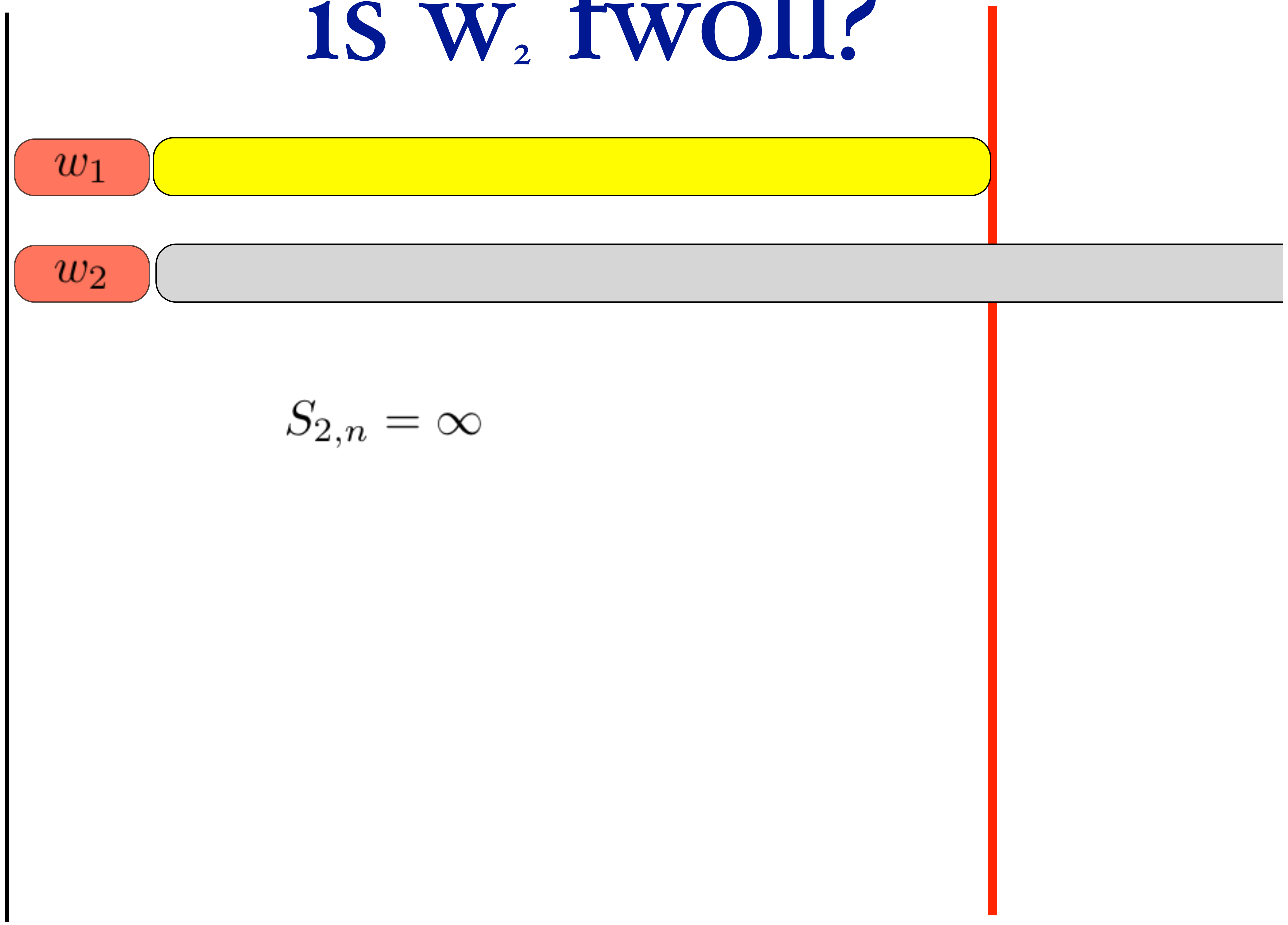
how many candidates
are there for the fwoll?

is w_1 fwoll?

there is no slack (no solution even)
because words go beyond edge!

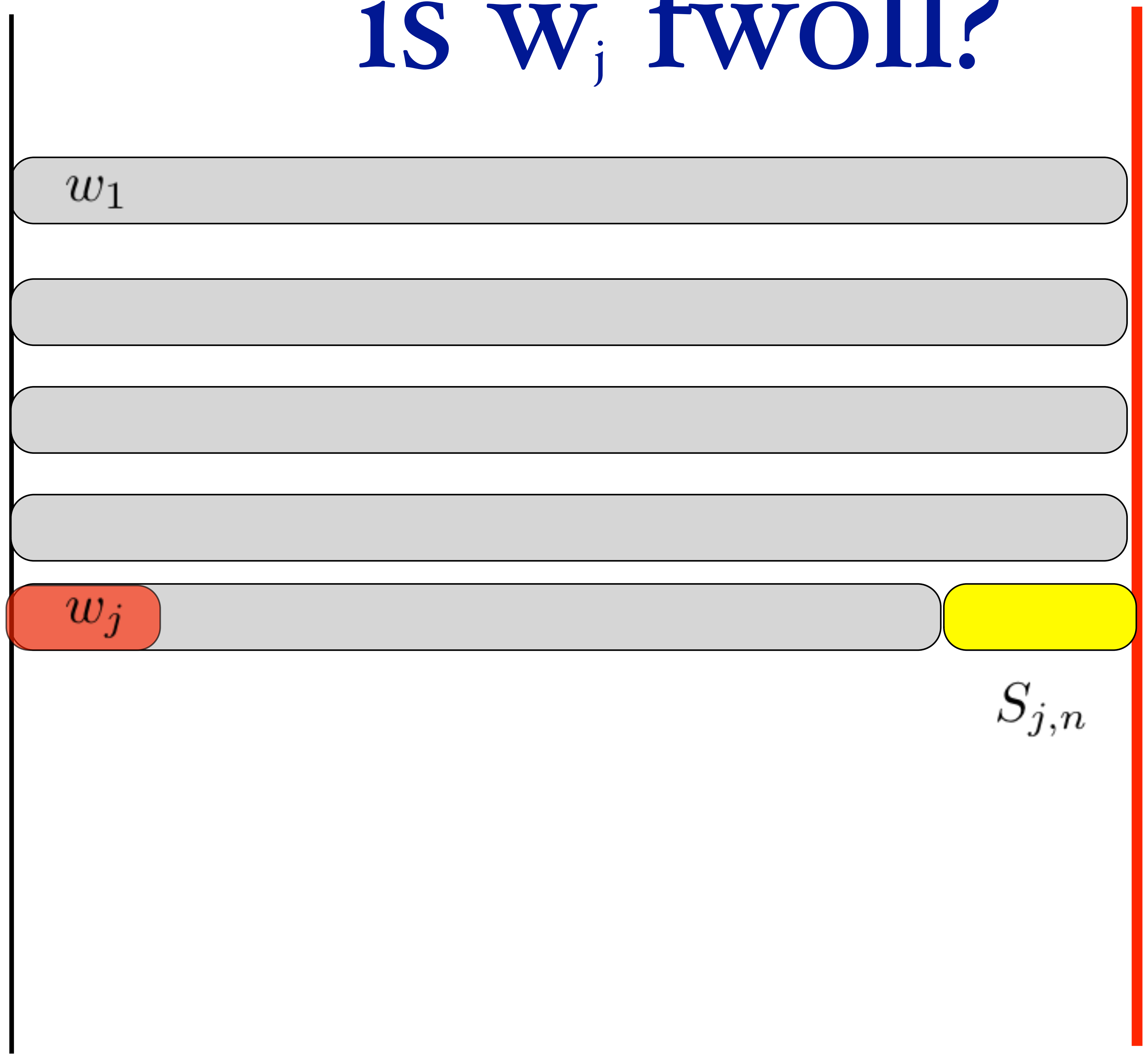
define $S_{1,n} = \infty$ if this happens

is w₂ fwoll?



$$S_{2,n} = \infty$$

is w_j fwoll?



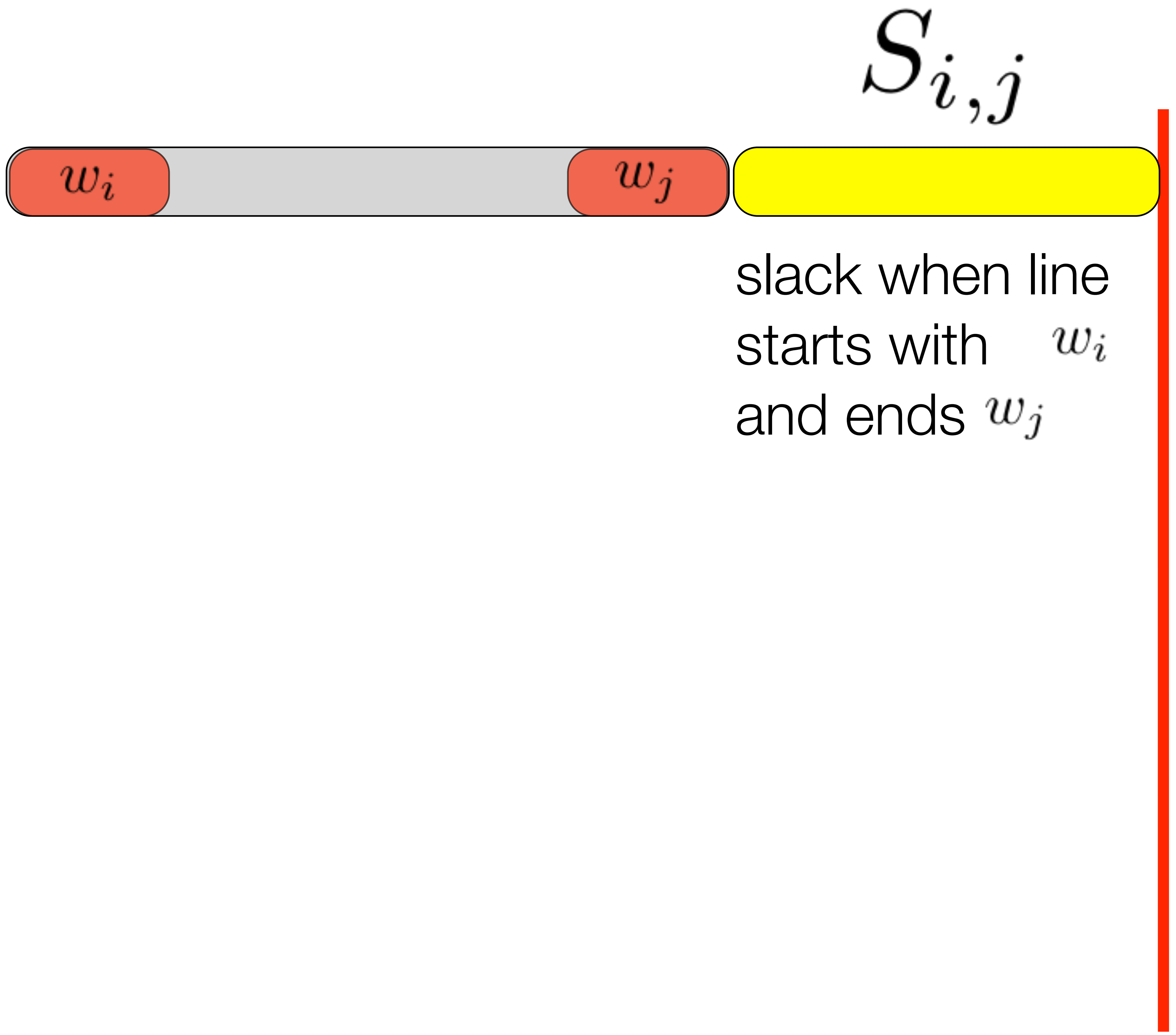
which word is fwoll?

$$\mathrm{BEST}_n = \min \left\{ \right.$$

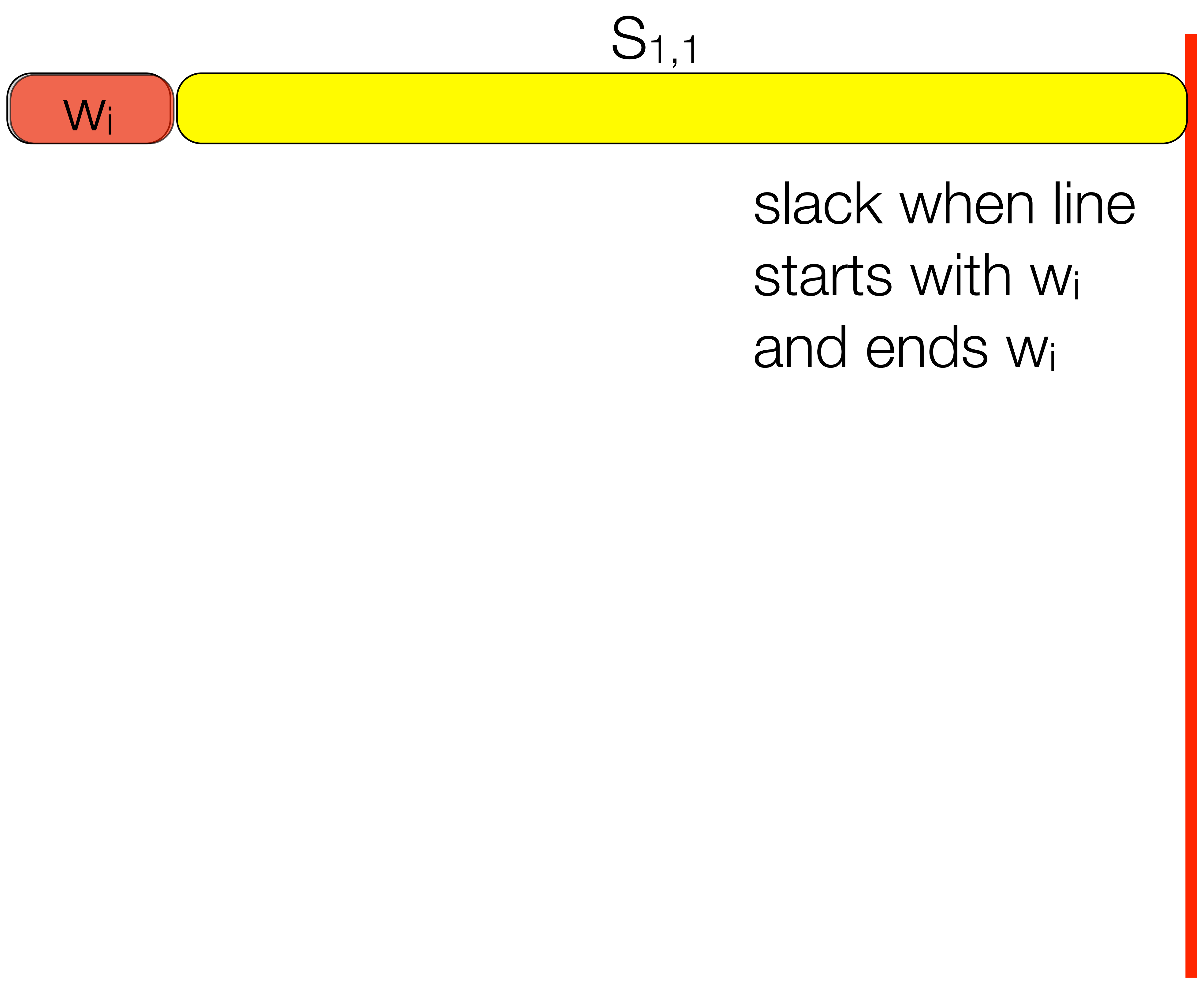
which word is fwoll?

$$\text{BEST}_n = \min \begin{cases} \text{BEST}_0 + S^2_{1,n} \\ \text{BEST}_1 + S^2_{2,n} \\ \text{BEST}_2 + S^2_{3,n} \\ \quad\cdots \\ \text{BEST}_{\ell-1} + S^2_{\ell,n} \\ \\ \quad\cdots \\ \text{BEST}_{n-1} + S^2_{n,n} \end{cases}$$

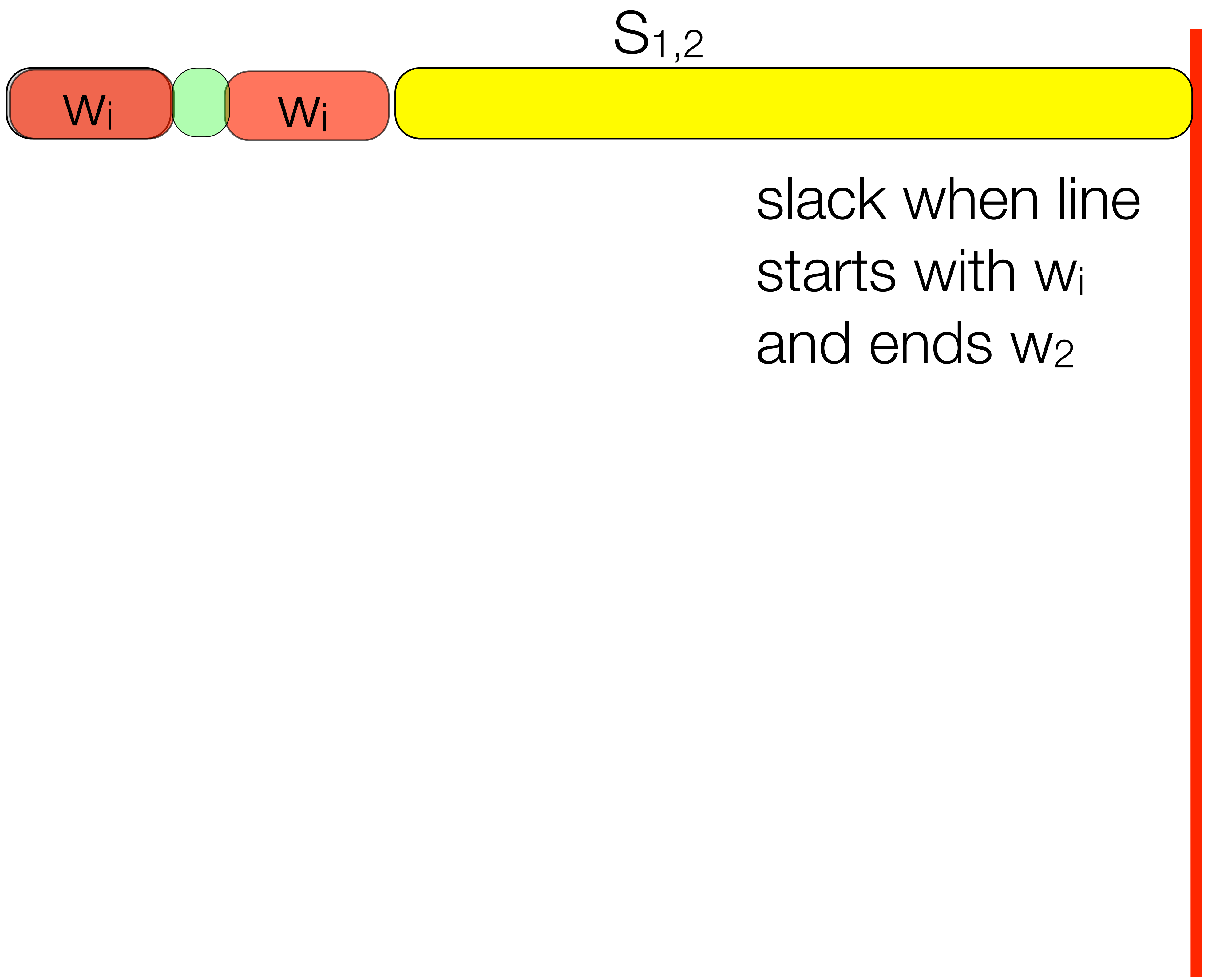
how to compute $S_{i,j}$



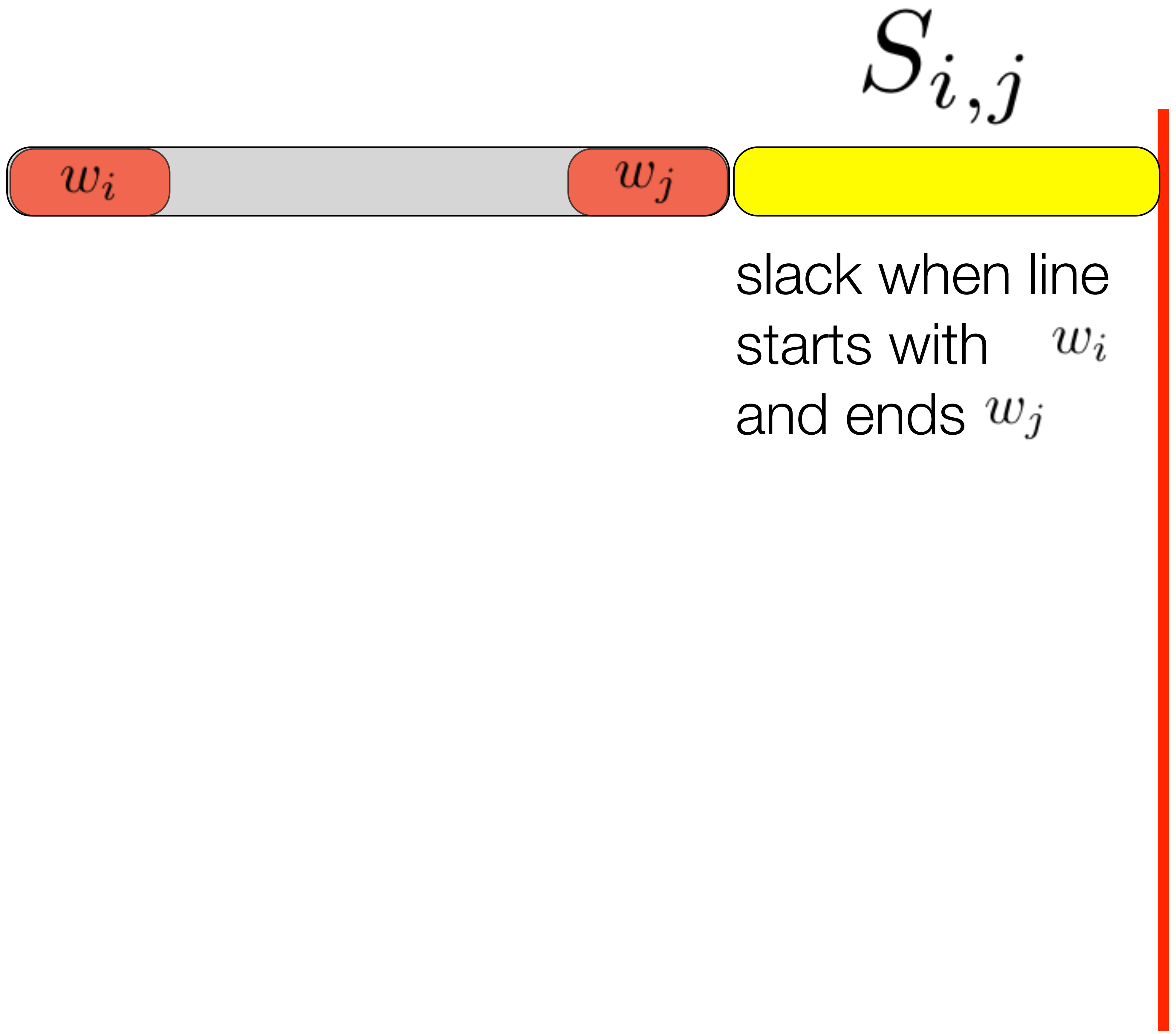
Simplest case

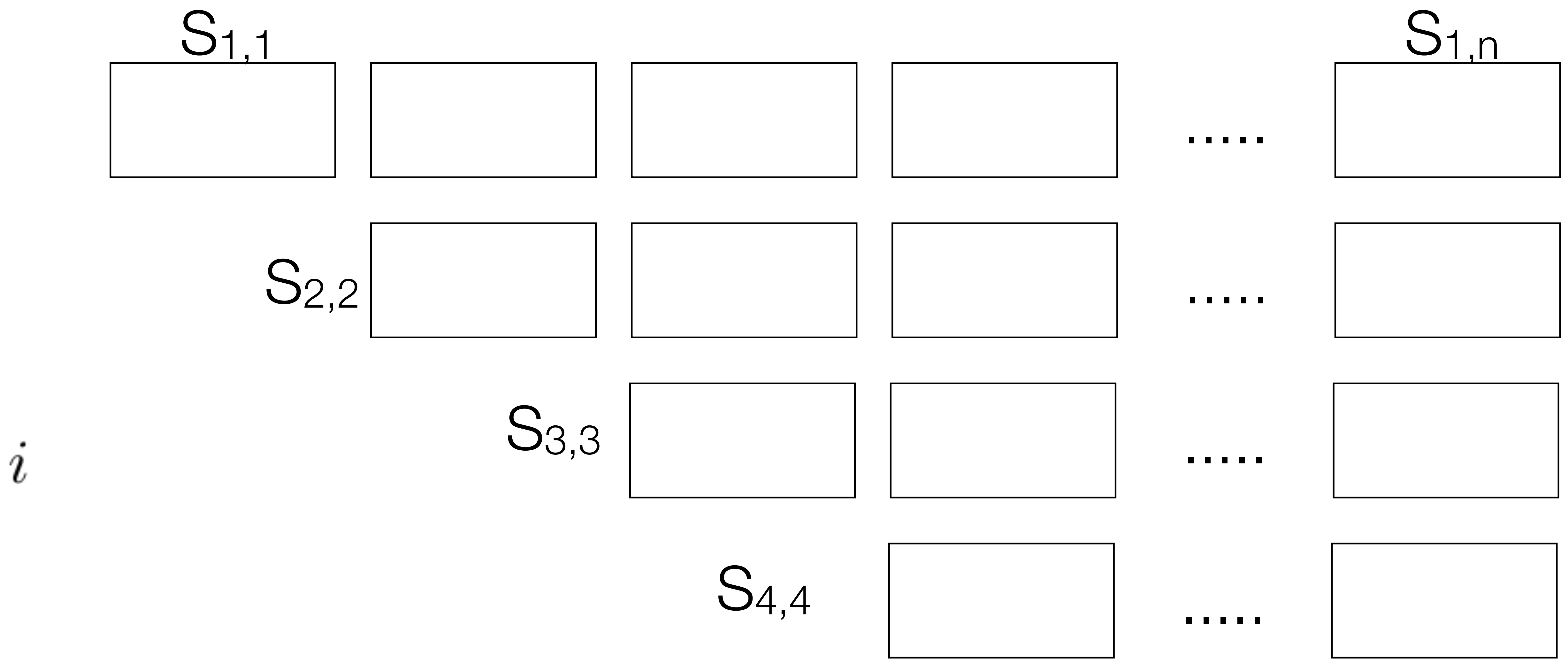


Simplest case



how to compute $S_{i,j}$





S1,1
S1,n
S2,2
S3,3
S4,4
i

typesetting algorithm

make table for $S_{i,j}$

typesetting algorithm

make table for $S_{i,j}$

for `i=1` to `n`

best[`i`] = min{ best[`j`] + s[`j+1`][`i`]^2 }

```
// compute best_0,...,best_n
  int best[] = new int[n+1];
  int choice[] = new int[n+1];
  best[0] = 0;
  for(int i=1;i<=n;i++) {
      int min = infty;
      int ch  = 0;
      for(int j=0;j<i;j++) {
          int t = best[j] + S[j+1][i]*S[j+1][i];
          if (t<min) { min = t; ch = j;}
      }
      best[i] = min;
      choice[i] = ch;
  }
```

example

```
It was the best of times, it was the worst of times; it was the age o
wisdom, it was the age of foolishness; it was the epoch of belief, it
was the epoch of incredulity; it was the season of
```

```
  2 3 3 4 2 6 2 3 3 5 2 6 2 3 3 3 2 7 2 3 3
  3 2 12 2 3 3 5 2 7 2 3 3 5 2 12 2 3 3 6 2
```

first step: make $S_{i,j}$

	1	2	3	4	5	6	7	8	9	10	11	12	...
1	?												

2 3 3 4 2 6 2 3 3 5 2 6 2 3 3 3 2 7 2 3 3
3 2 12 2 3 3 5 2 7 2 3 3 5 2 12 2 3 3 6 2

$M = 42$

$$S_{i,i} = M - |w_i|$$

$$S_{i,j} = S_{i,j-1} - 1 - |w_j|$$

first step: make $S_{i,j}$

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2													

2 3 3 4 2 6 2 3 3 5 2 6 2 3 3 3 2 7 2 3 3
3 2 12 2 3 3 5 2 7 2 3 3 5 2 12 2 3 3 6 2

$M = 42$

first step: make $S_{i,j}$

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30	27	20	17	13	9	3	0	99	99

2 3 3 4 2 6 2 3 3 5 2 6 2 3 3 3 2 7 2 3 3
3 2 12 2 3 3 5 2 7 2 3 3 5 2 12 2 3 3 6 2

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30	27	20	17	13	9	3	0	99	99
3													

2 3 3 4 2 6 2 3 3 5 2 6 2 3 3 3 2 7 2 3 3
3 2 12 2 3 3 5 2 7 2 3 3 5 2 12 2 3 3 6 2

second step: compute

	0	1	2	3	4	5	6	7	8	9	10	...
best	0											

$$\text{BEST}_i = \min_{j=0}^{i-1} \left\{ \text{BEST}_j + S^2_{j+1,i} \right\}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30	27	20	17	13	9	3	0	99	99

second step: compute

	0	1	2	3	4	5	6	7	8	9	10	...
best	0	1600										

$$\mathrm{BEST}_i = \min_{j=0}^{i-1} \left\{ \mathrm{BEST}_j + S^2_{j+1,i} \right\}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30	27	20	17	13	9	3	0	99	99

second step: compute

	0	1	2	3	4	5	6	7	8	9	10	...
best	0	1600	1296									

$$\mathrm{BEST}_i = \min_{j=0}^{i-1} \left\{ \mathrm{BEST}_j + S^2_{j+1,i} \right\}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	40	36	32	27	24	17	14	10	6	0	99	99	99
2		39	35	30	27	20	17	13	9	3	0	99	99

Running time

make table for $S_{i,j}$

for `i=1` to `n`

best[`i`] = min{ best[`j`] + s[`j+1`][`i`]^2 }

PROBLEM: REDUCE IMAGE

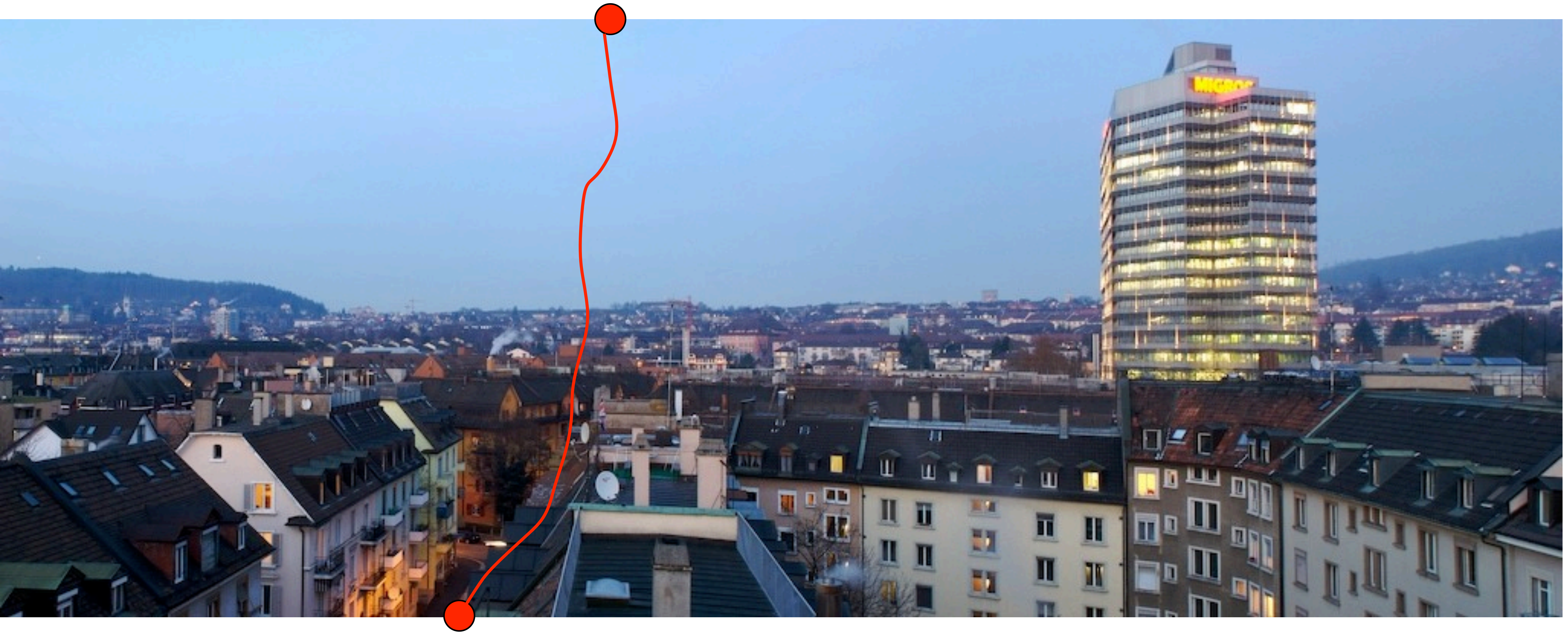
scaling: distortion
deleting column: distortion
delete the most invisible seam

Shai Avidan
Mitsubishi Electric Research Lab
Ariel Shamir
The interdisciplinary Center & MERL

DEMO?

http://rsizr.com/

WHICH SEAM TO DELETE?

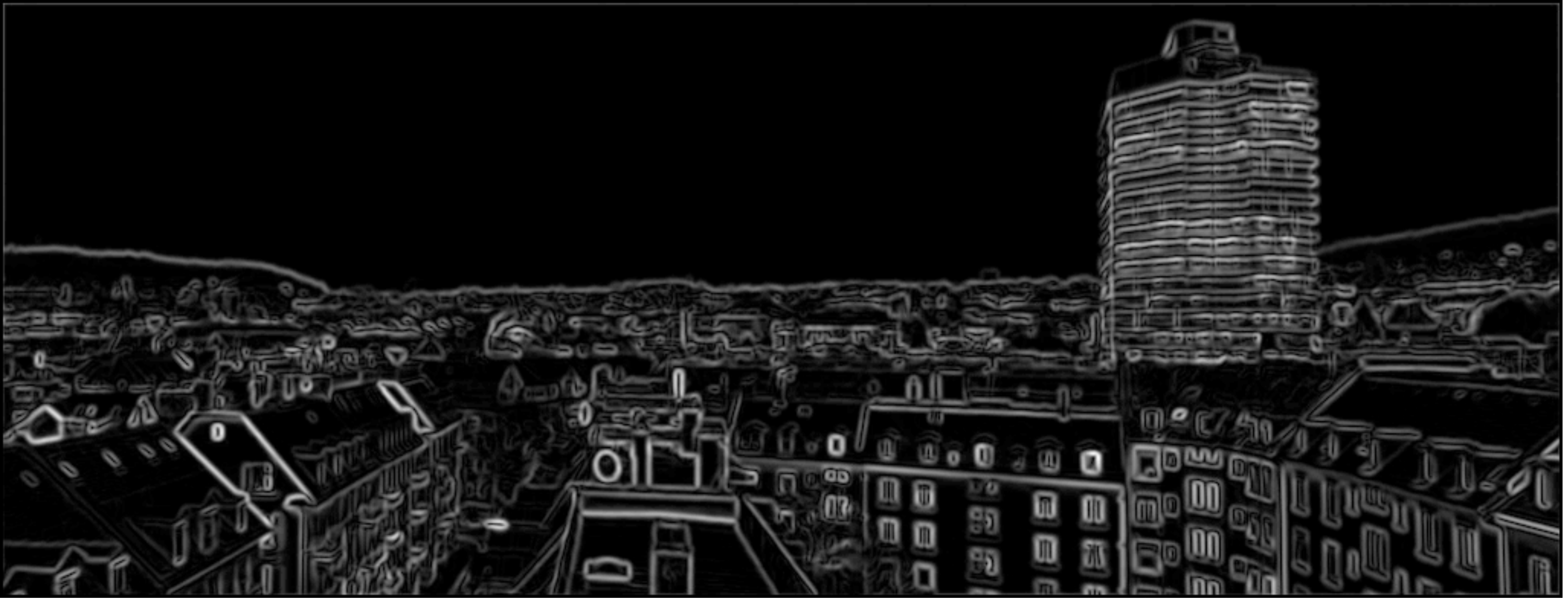


ENERGY OF AN IMAGE

$$e(\mathbf{I}) = \left|\frac{\partial}{\partial x}\mathbf{I}\right| + \left|\frac{\partial}{\partial y}\mathbf{I}\right|$$

"magnitude of gradient at a pixel"

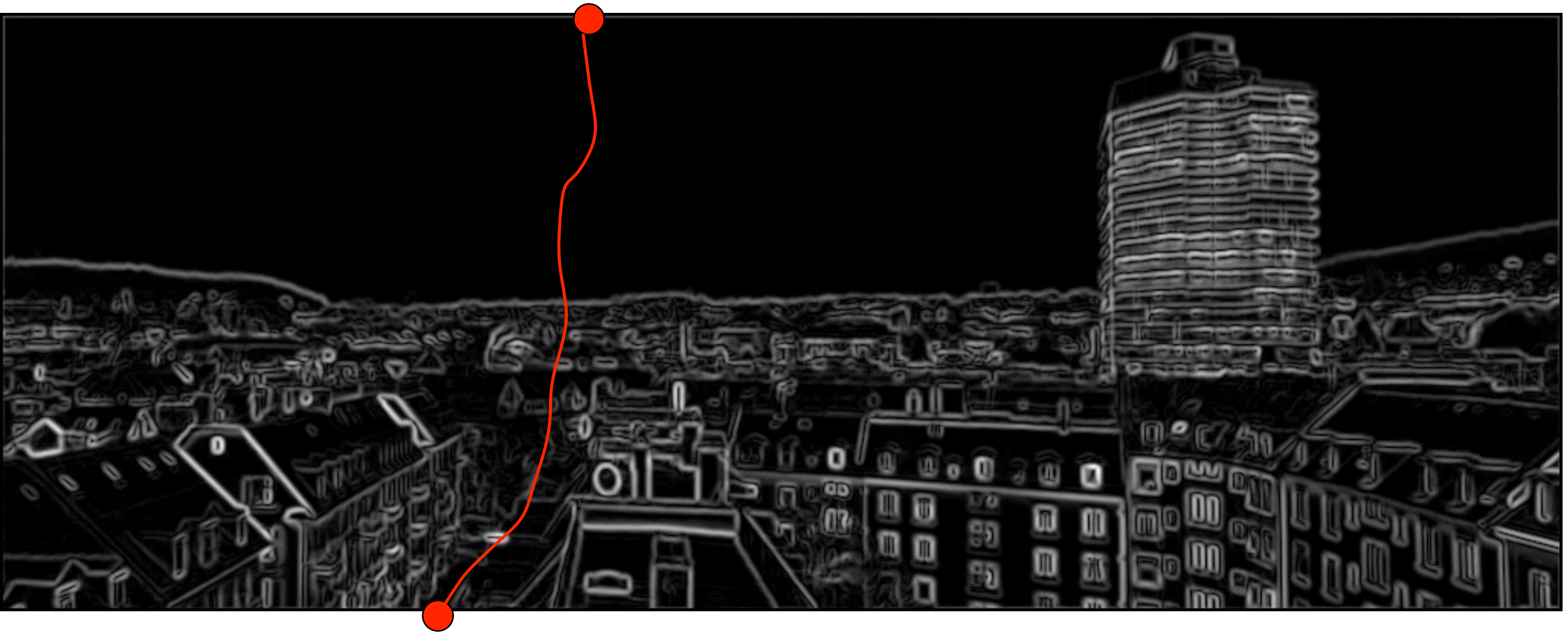
$$\frac{\partial}{\partial x} I_{x,y} = I_{x-1,y} - I_{x+1,y}$$



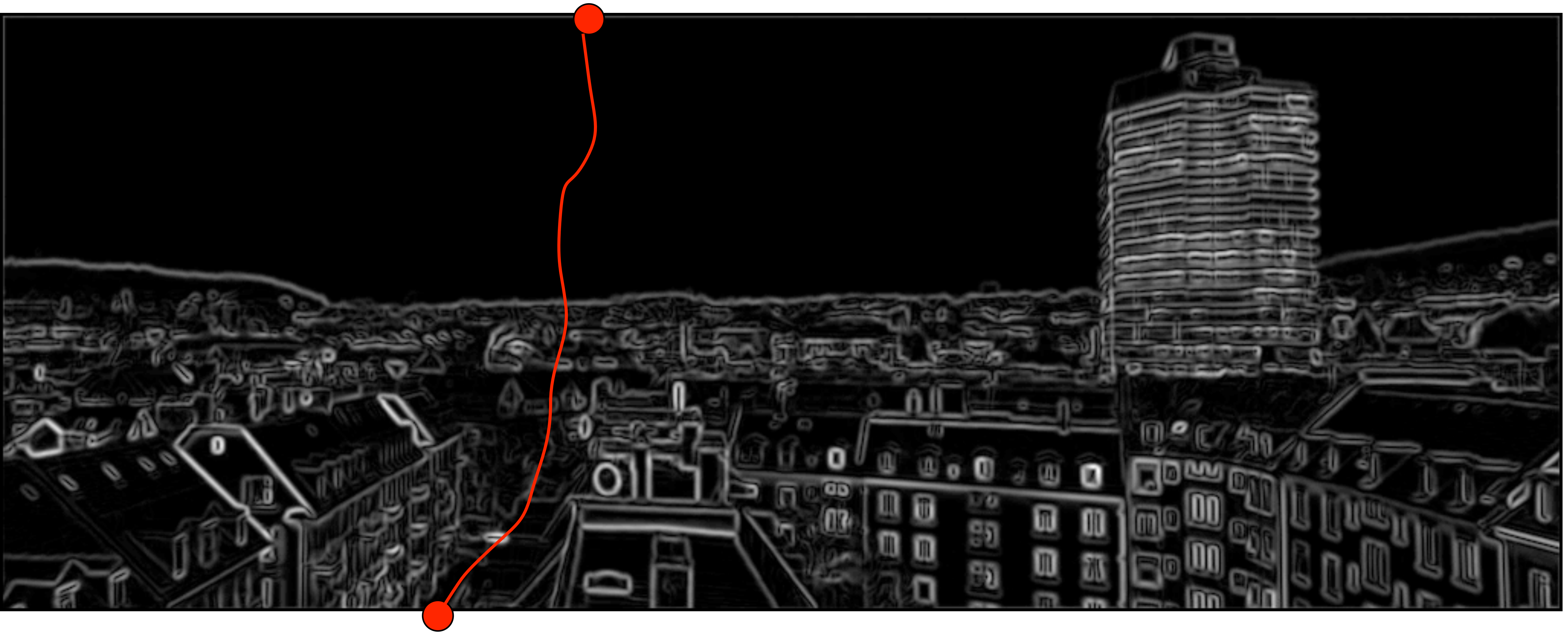
energy of sample image

thanks to Jason Lawrence for gradient software

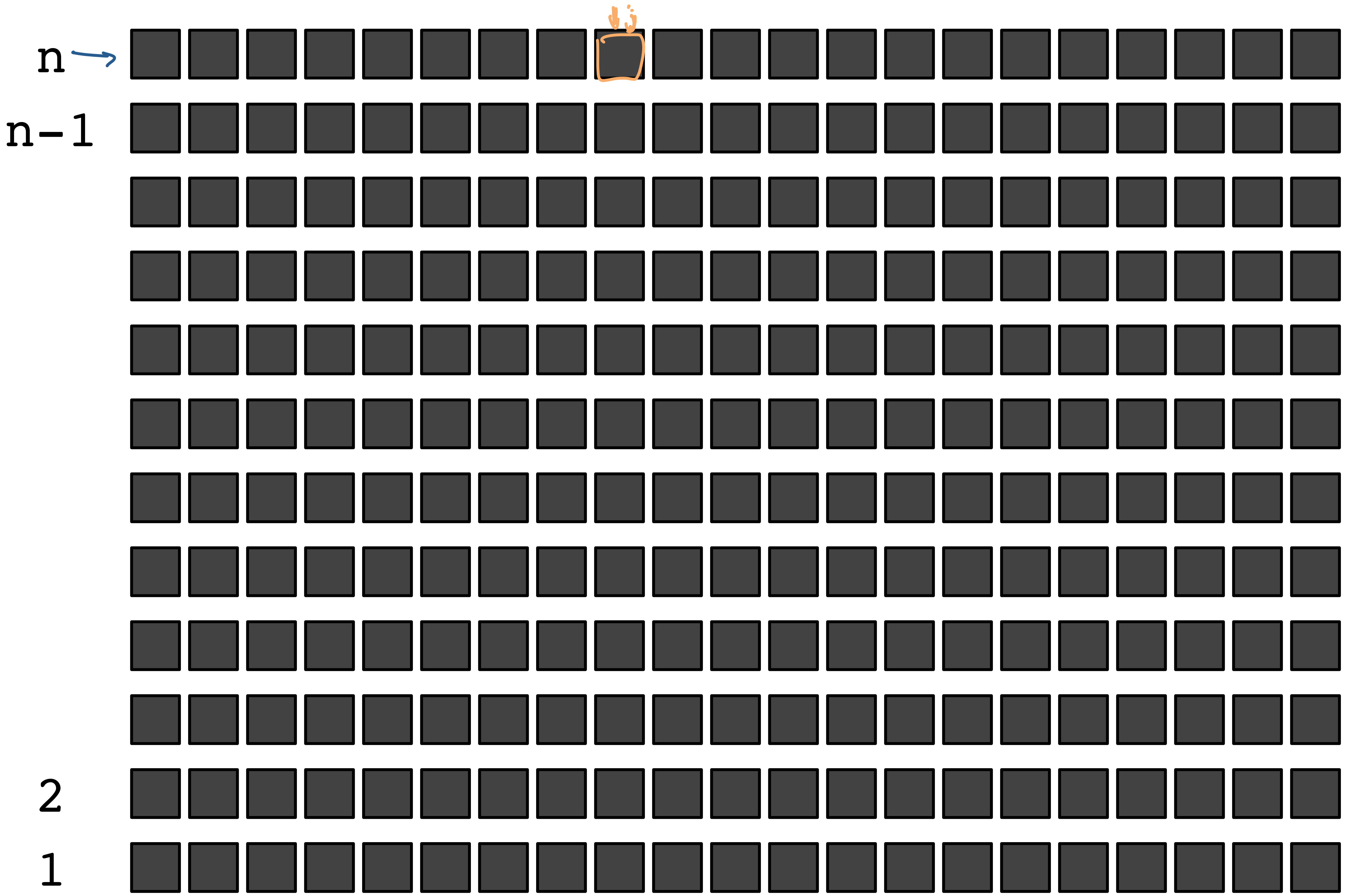
BEST SEAM HAS LOWEST ENERGY



Finding lowest energy seam?

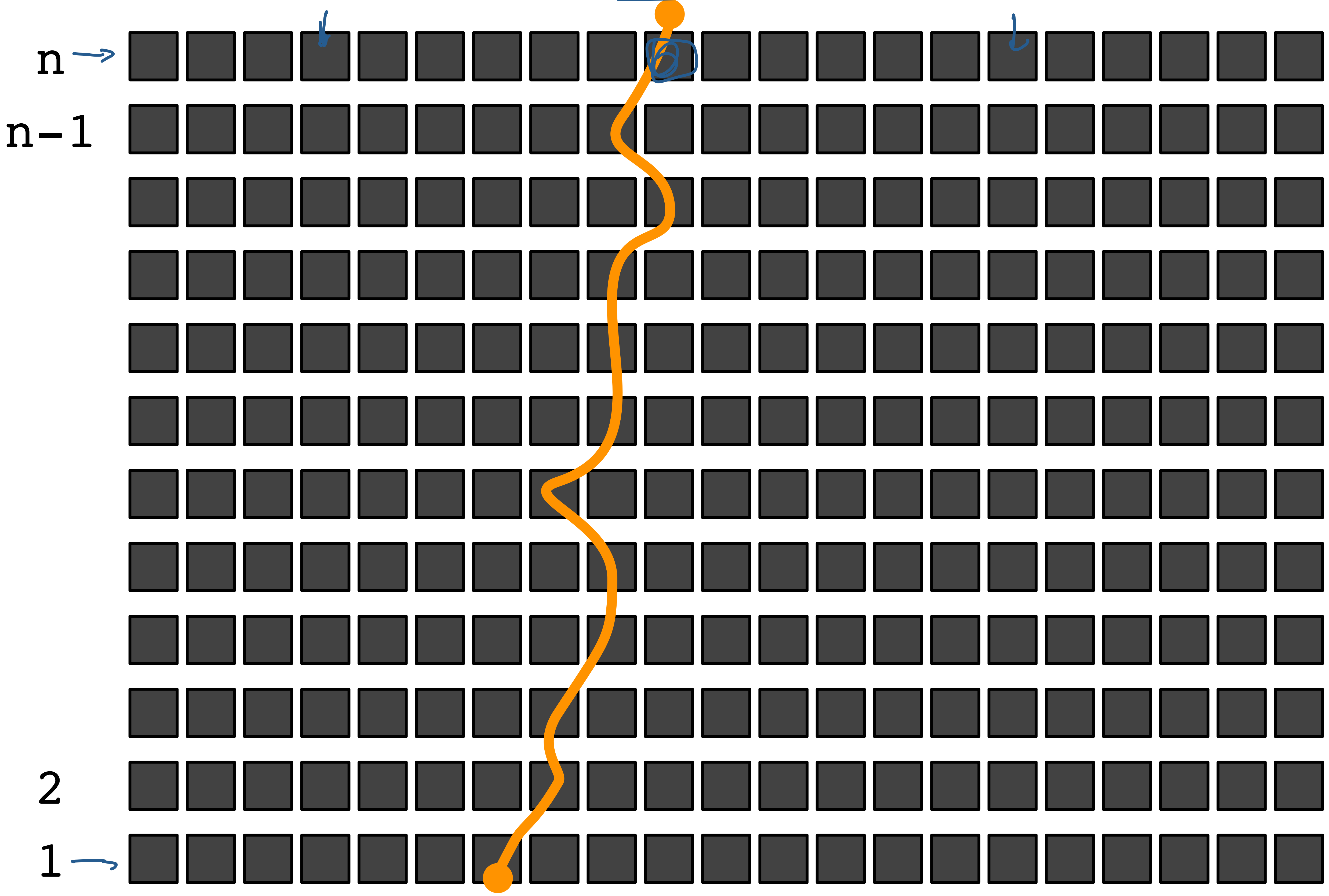


definition: $S_n(j)$



definition:

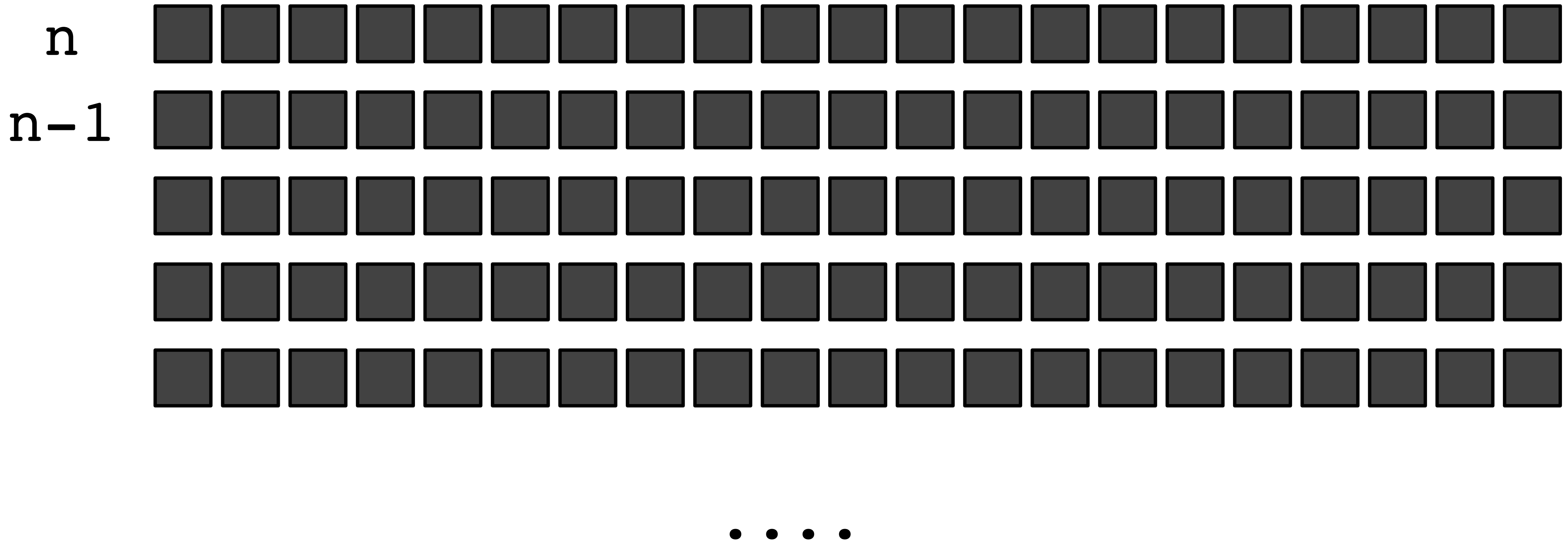
$S_n(j)$ best seam ending at (n,j)



BEST SEAM TO DELETE HAS TO BE THE BEST AMONG

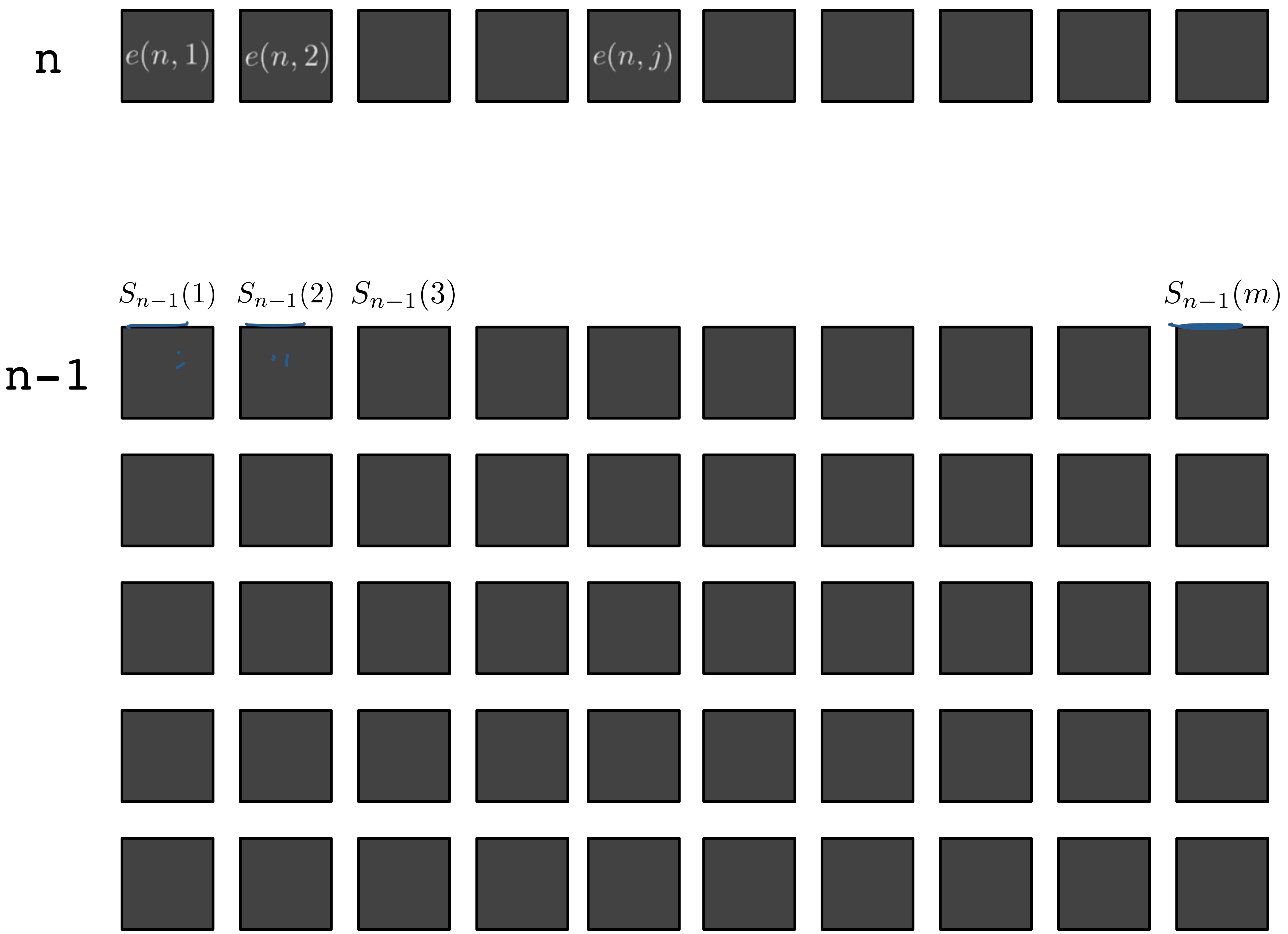
$$S_n(1), S_n(2), \ldots, S_n(m)$$

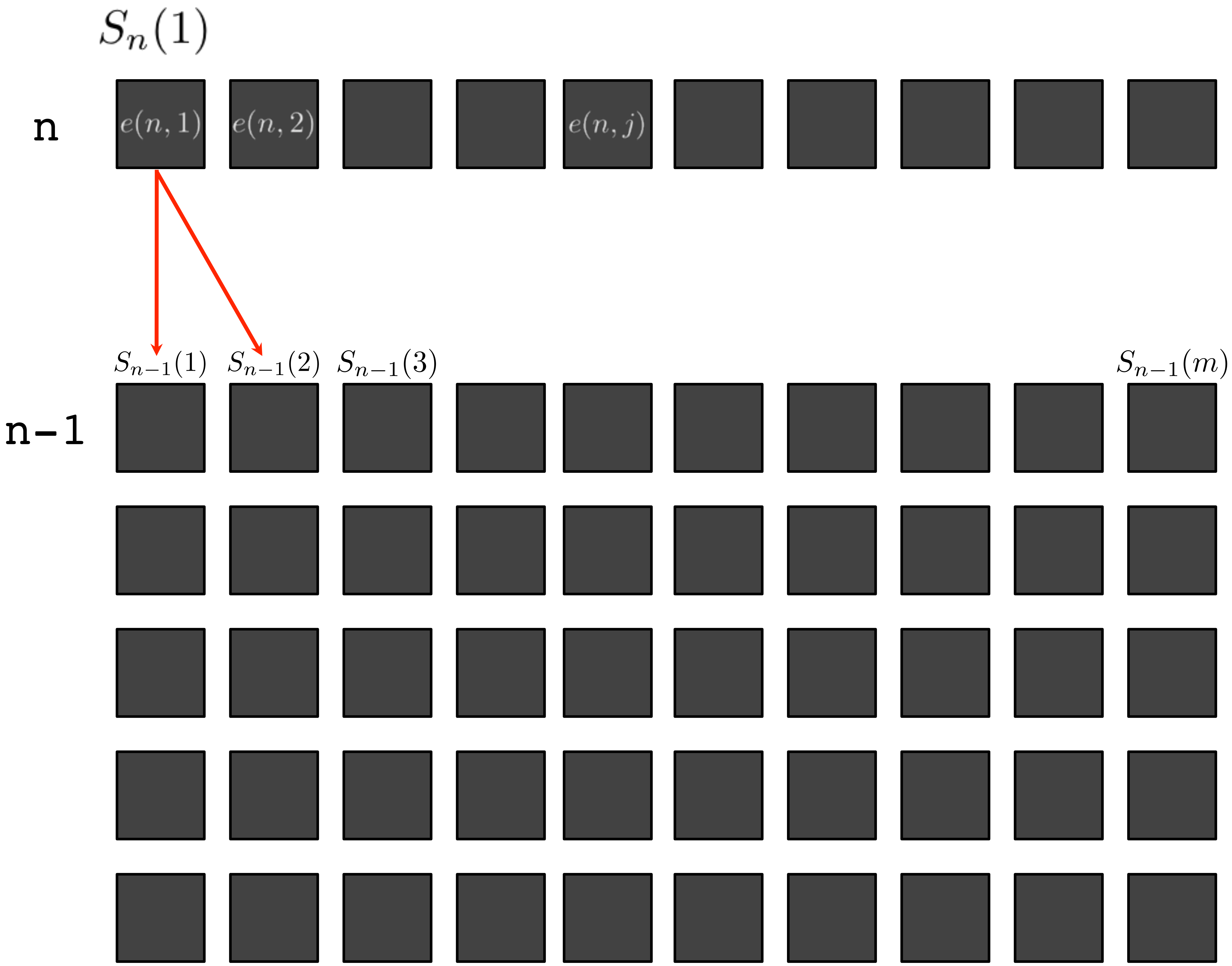
IDEA: COMPUTE + COMPARE



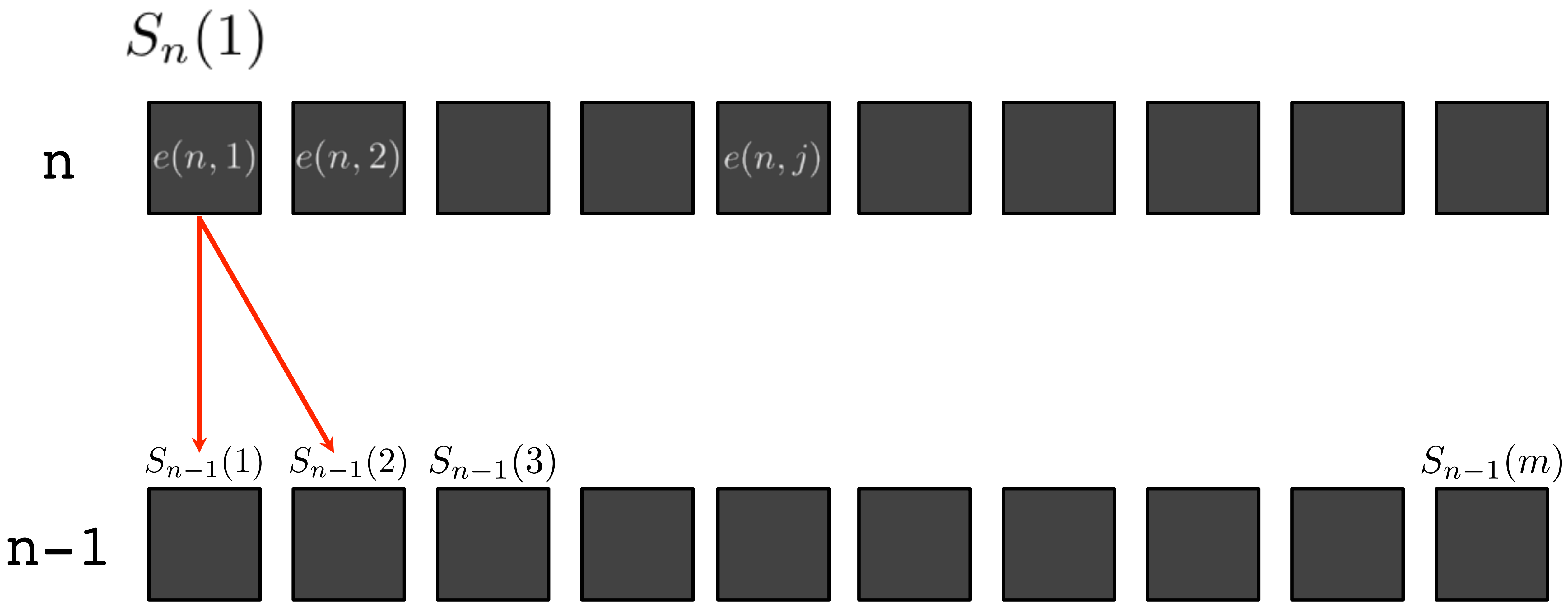
SMALLER PROBLEM APPROACH

IMAGINE YOU HAVE THE SOLUTION TO THE FIRST $n-1$ ROWS

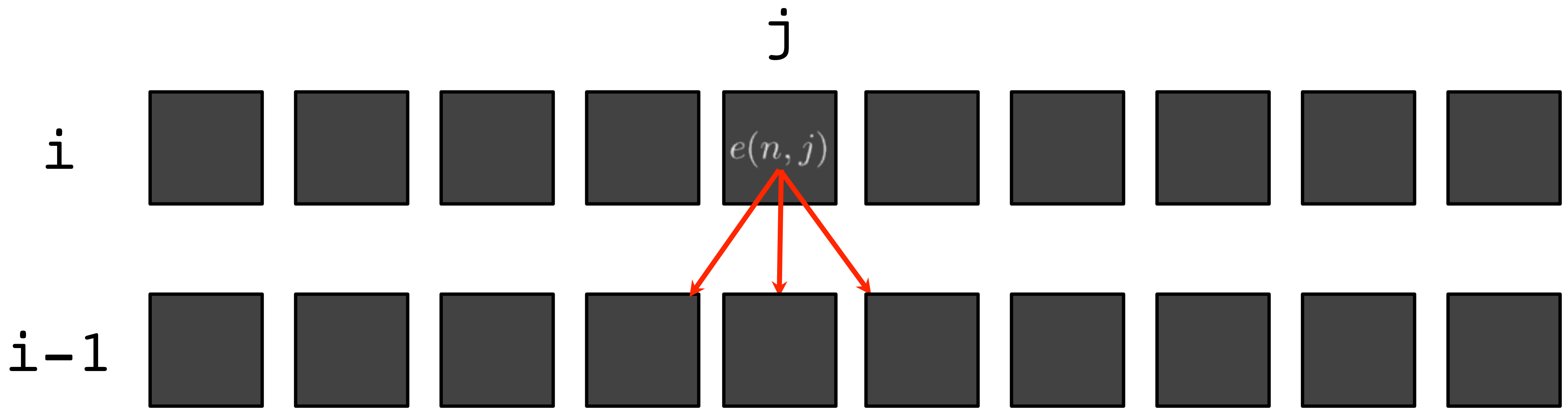

n
$e(n,1)$
$e(n,2)$
$e(n,j)$
$S_{n-1}(1)$
$S_{n-1}(2)$
$S_{n-1}(3)$
$S_{n-1}(m)$
n-1



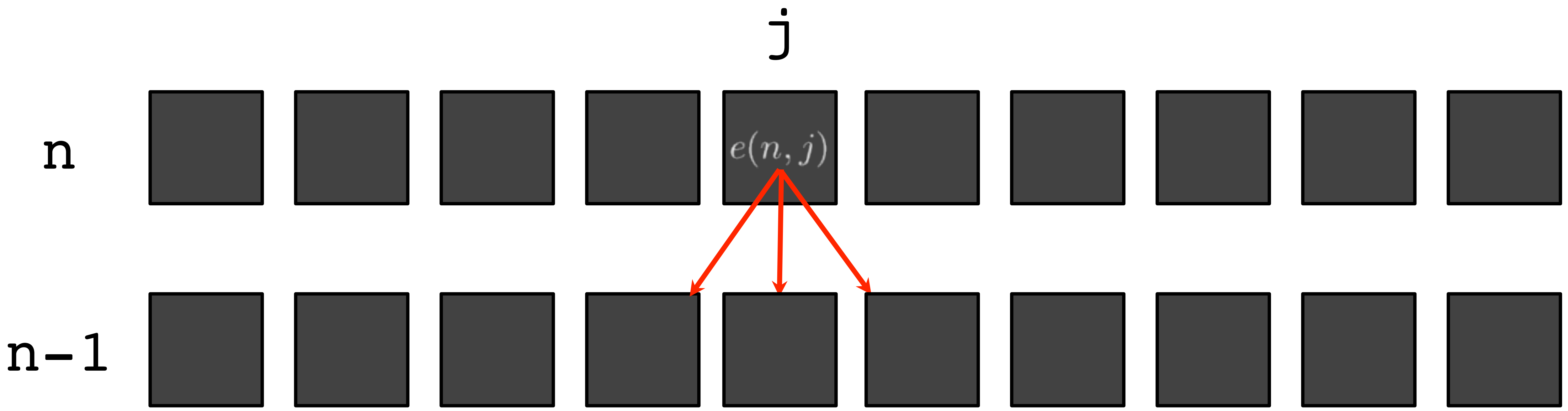
$S_n(1)$
n
$e(n,1)$
$e(n,2)$
$e(n,j)$
$S_{n-1}(1)$
$S_{n-1}(2)$
$S_{n-1}(3)$
$S_{n-1}(m)$
n-1



$$S_n(1) = e(n,1) + \min\{S_{n-1}(1), S_{n-1}(2)\}$$



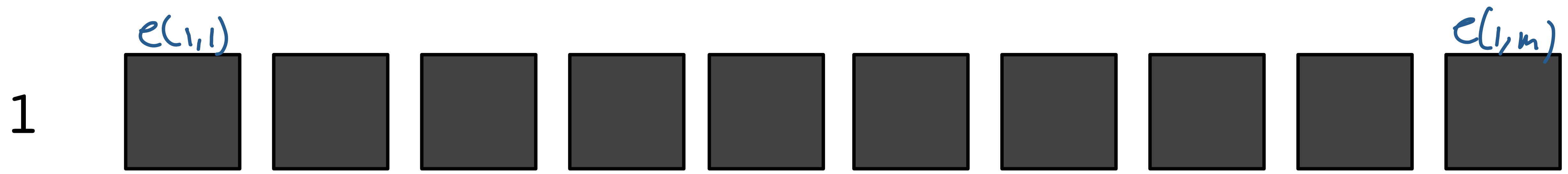
$$S_i(j) =$$



$$S_i(j) = e(i,j) + \min \begin{cases} S_{i-1}(j-1) \\ S_{i-1}(j) \\ S_{i-1}(j+1) \end{cases}$$

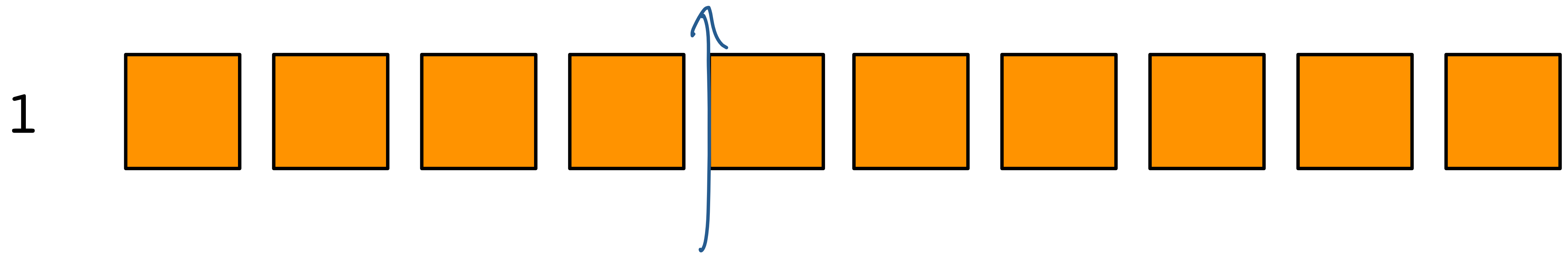
ALGORITHM

start at bottom of picture



ALGORITHM

start at bottom of picture. initialize $S_1(i) = e(1, i)$

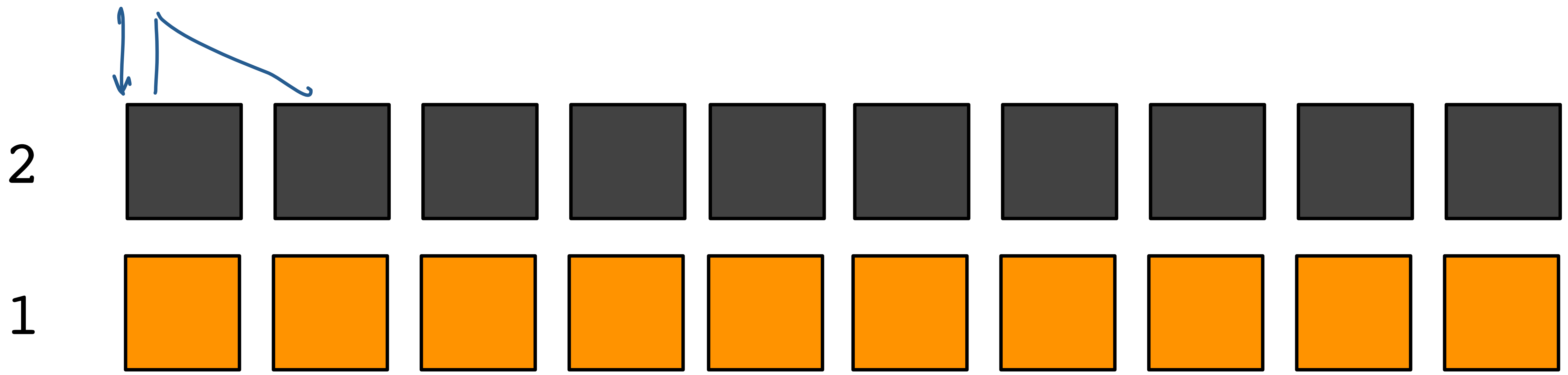


ALGORITHM

start at bottom of picture. initialize $S_1(i) = e(1, i)$

for `i=2,n` use formula to compute $S_{i+1}(\cdot)$

$$S_i(j) = e(i,j) + \min \begin{cases} S_{i-1}(j-1) \\ S_{i-1}(j) \\ S_{i-1}(j+1) \end{cases}$$

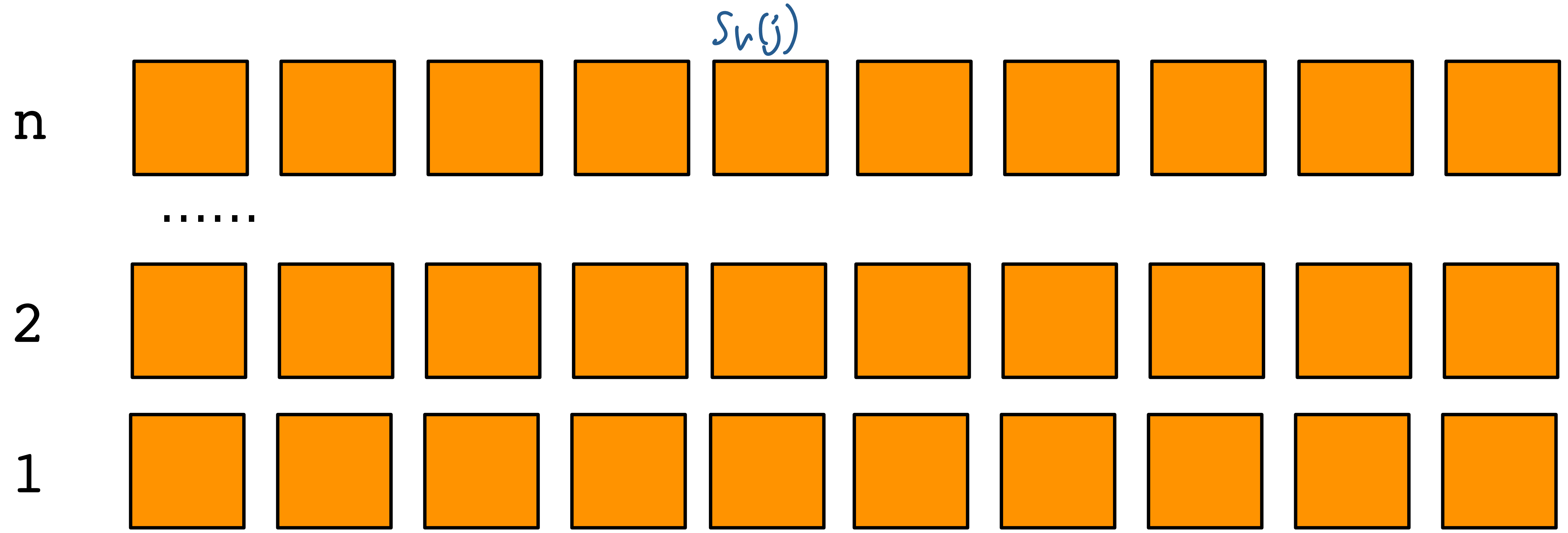


ALGORITHM

start at bottom of picture. initialize $S_1(i) = e(1, i)$

for `i=2,n` use formula to compute $S_{i+1}(\cdot)$

$$S_i(j) = e(i,j) + \min \begin{cases} S_{i-1}(j-1) \\ S_{i-1}(j) \\ S_{i-1}(j+1) \end{cases}$$



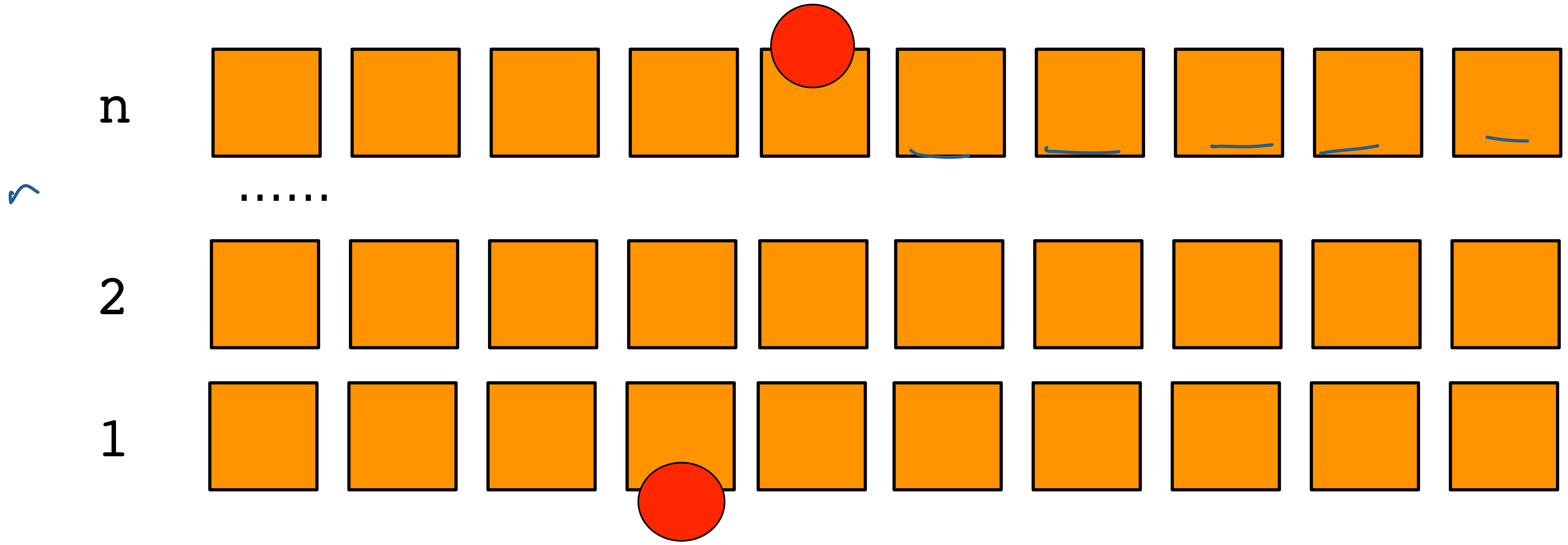
ALGORITHM

start at bottom of picture. initialize $S_1(i) = e(1, i)$

for `i=2,n` use formula to compute $S_{i+1}(\cdot)$

$$S_i(j) = e(i,j) + \min \begin{cases} S_{i-1}(j-1) \\ S_{i-1}(j) \\ S_{i-1}(j+1) \end{cases}$$

pick best among top row, backtrack.



RUNNING TIME

start at bottom of picture. initialize $S_1(i) = e(1, i)$

for `i=2,n` use formula to compute $S_{i+1}(\cdot)$

$$S_i(j) = e(i,j) + \min \begin{cases} S_{i-1}(j-1) \\ S_{i-1}(j) \\ S_{i-1}(j+1) \end{cases}$$

pick best among top row, backtrack.