L22

4102

4.12.2016

abhi shelat

Max flow

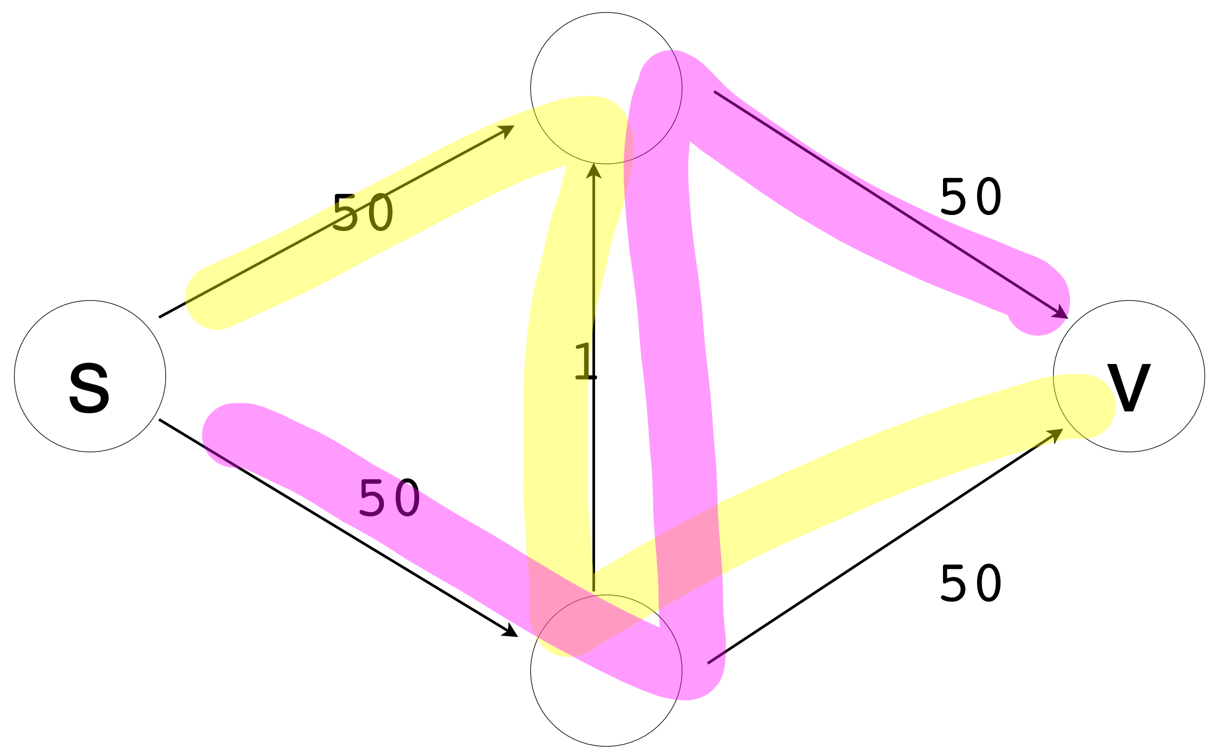
Min Cut

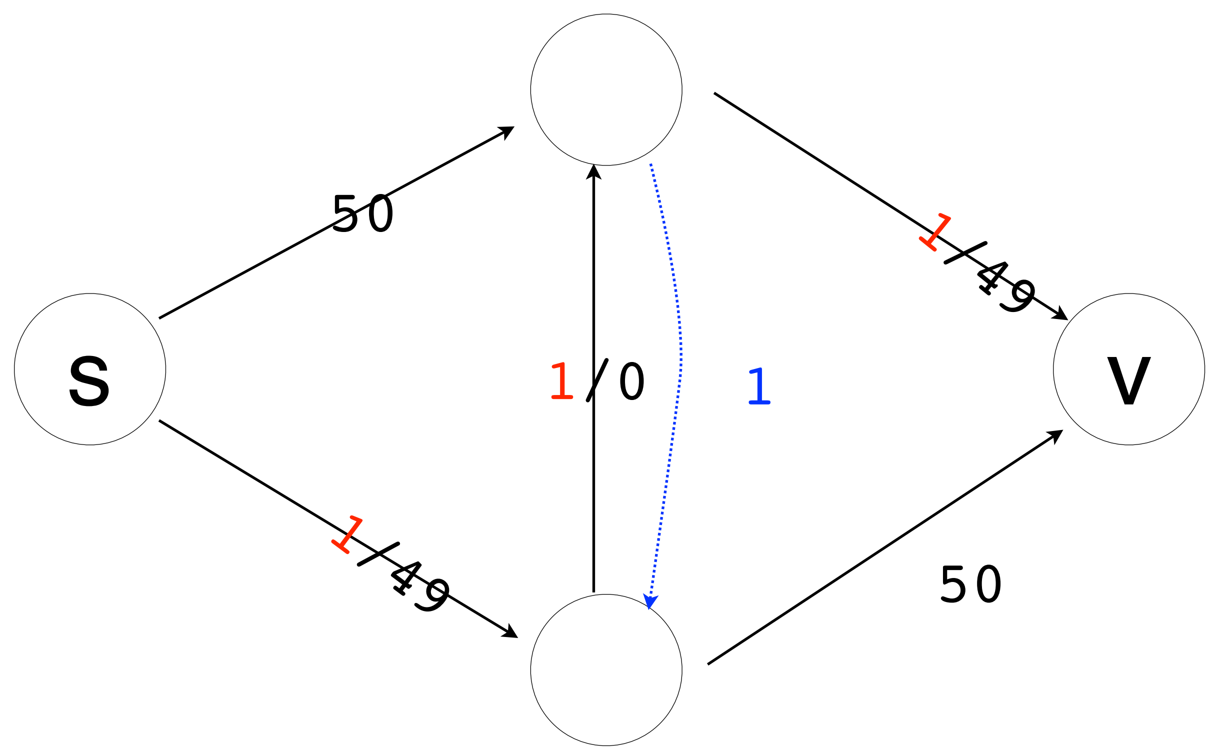
Ford-Fulkerson

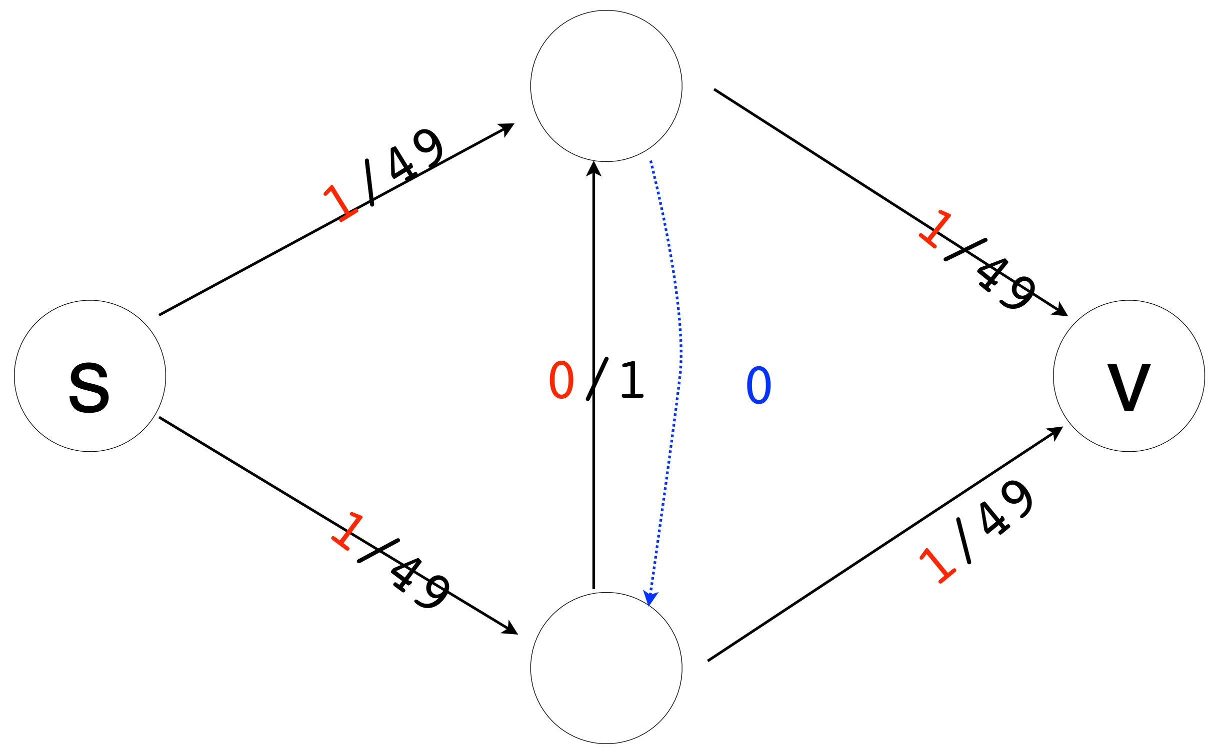
initialize $f(u,v) \leftarrow 0 \; \forall u,v$
while exists an augmenting path *p* in G_f
augment *f* with $c_f(p) = \min_{(u,v)\in p} c_f(u,v)$

time to find an augmenting path: $O(V+E)$

number of iterations of while loop: $O(E|f|)$

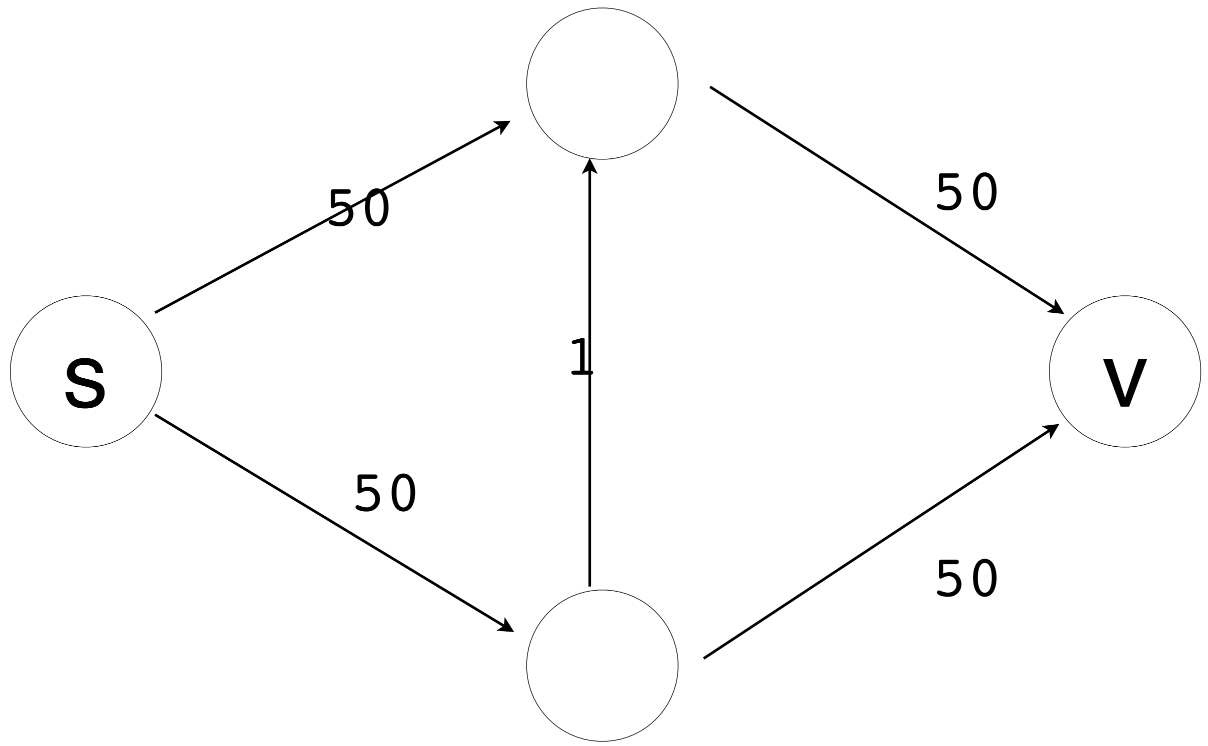

50
50
1
50
50
S
V


S
V
50
1/49
1/0
1
1/49
50



S
V
1/49
1/49
1/49
1/49
0/1
0

root of the problem



Edmonds-Karp 2

choose path with fewest edges first.

$\delta_f(s, v)$: smallest number of edges in a path from s to v in residual graph G_f.

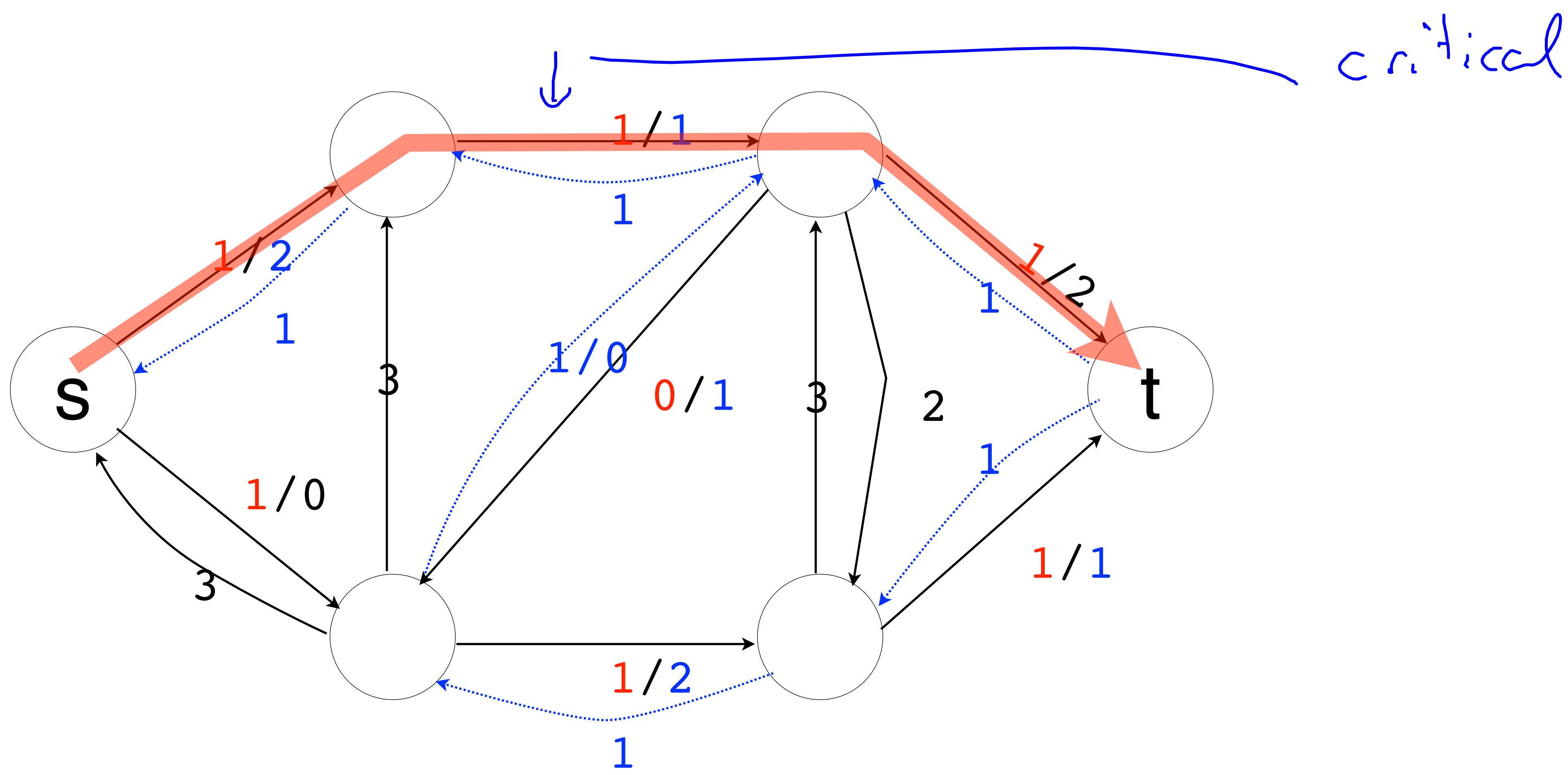
↑

G_f

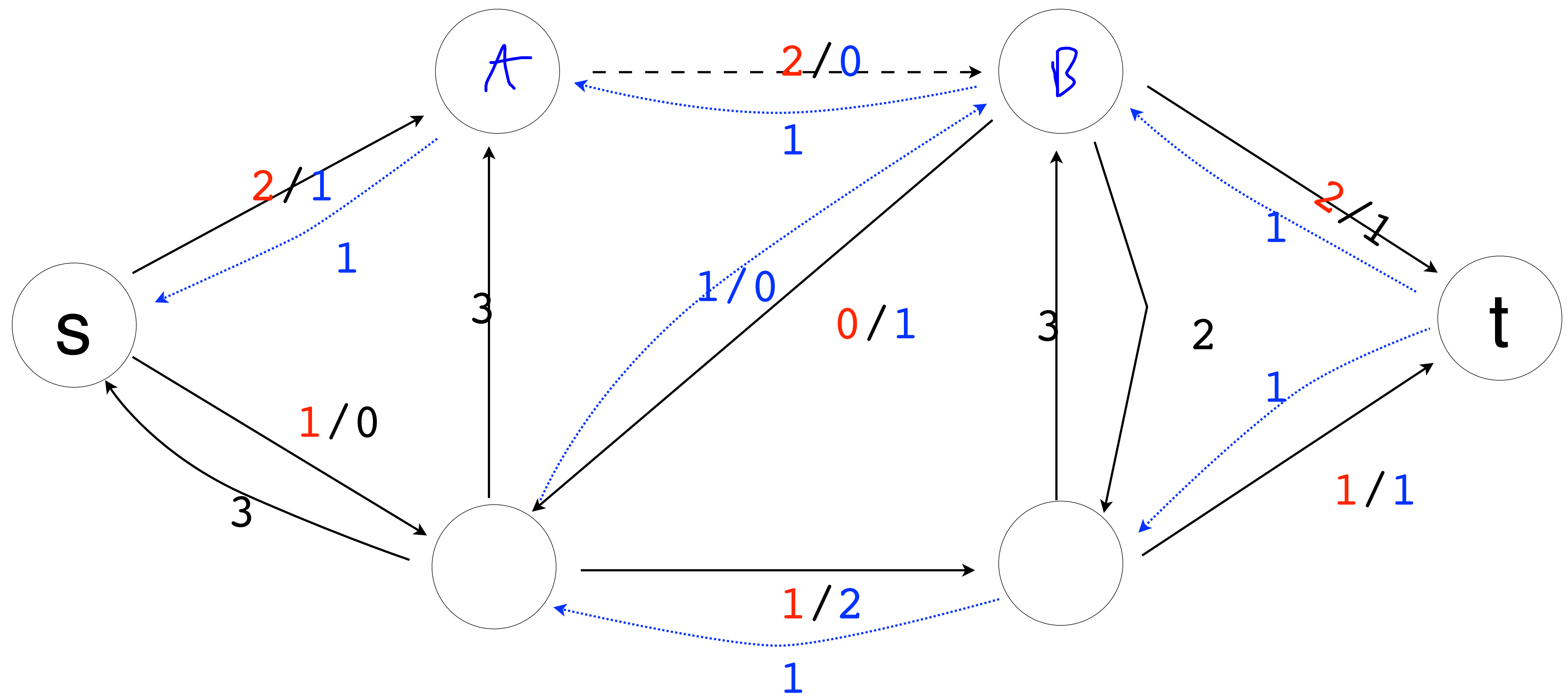
$\delta_f(s,v)$ increases monotonically thru exec

$\delta_{i+1}(v) \geq \delta_i(v)$

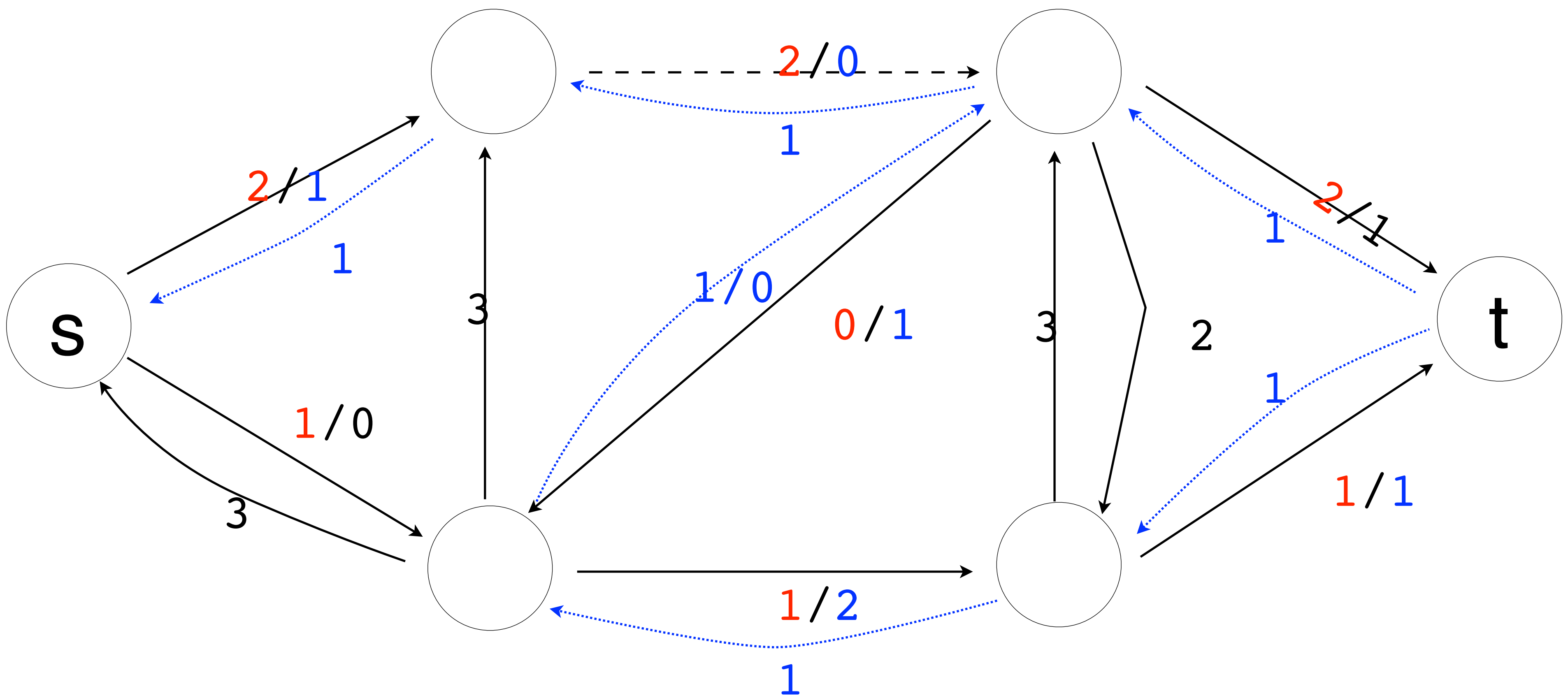
EK2

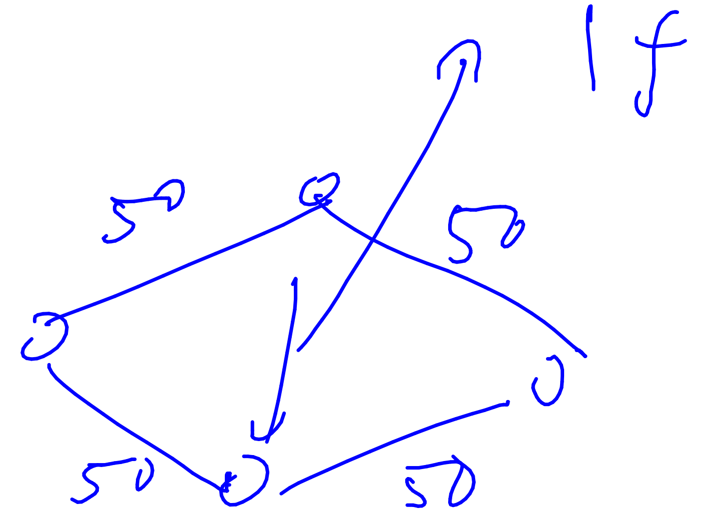


for every augmenting path, some edge is critical.



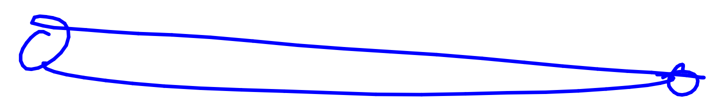
critical edges are removed in next residual graph.

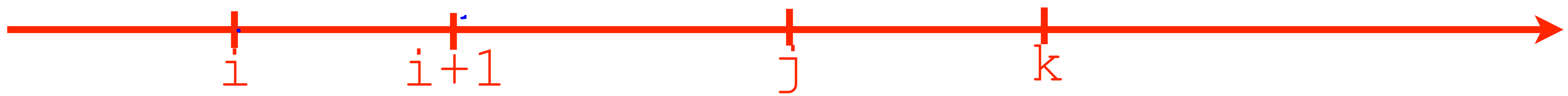




key idea: how many times can an edge be critical?

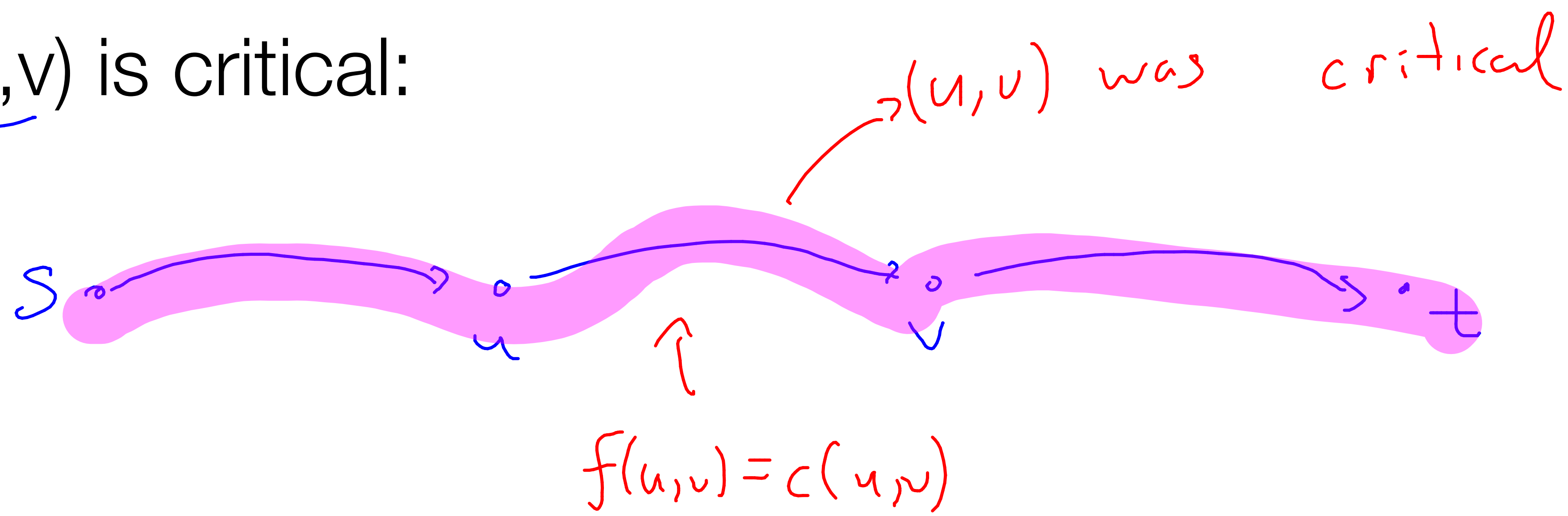
ANY edge becomes critical $\leq \frac{V}{2}$ times.





first time (u,v) is critical:

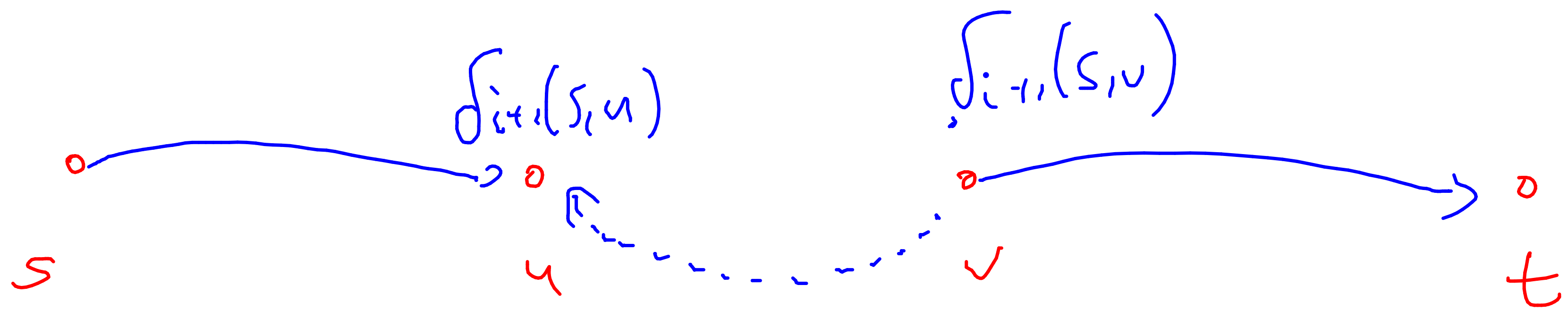
time i



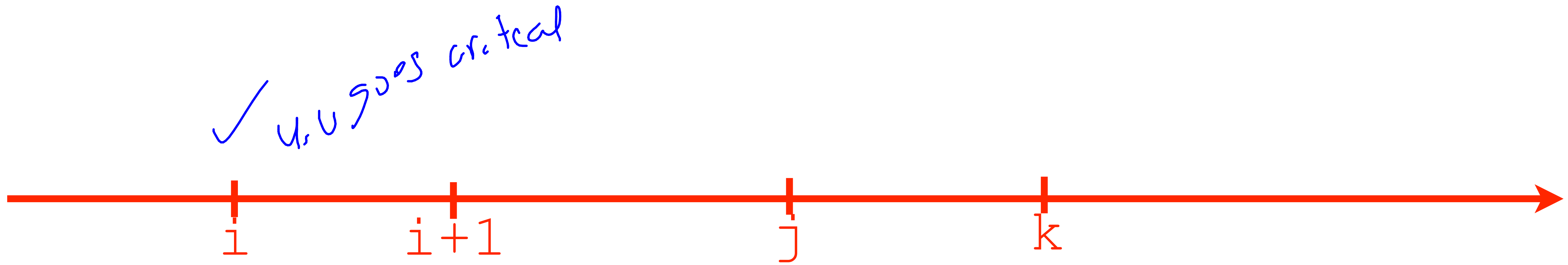
$$\delta_i(s,v) = \delta_i(s,u) + 1$$

why?? we are picking sp from s to t.

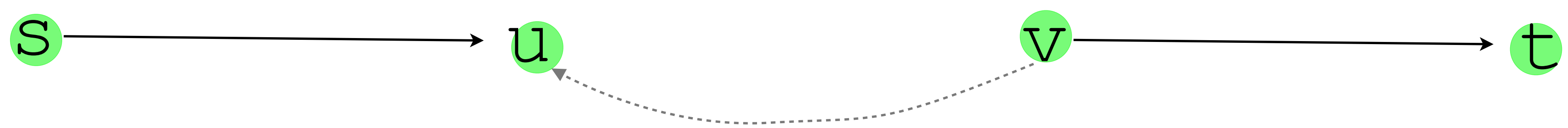
time i+1 G_f



$$\delta_{i+1}(s,v) \geq \delta_i(s,u) + 1$$



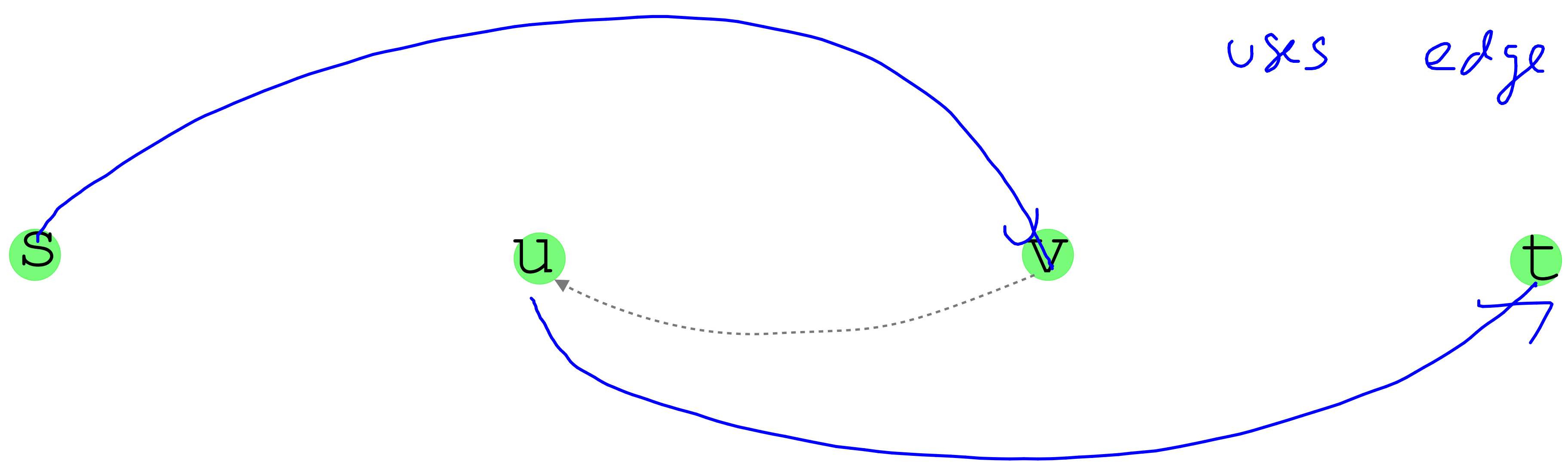
time i+1: (u,v) is critical: $\delta_{i+1}(s,v) \geq \delta_i(s,u) + 1$



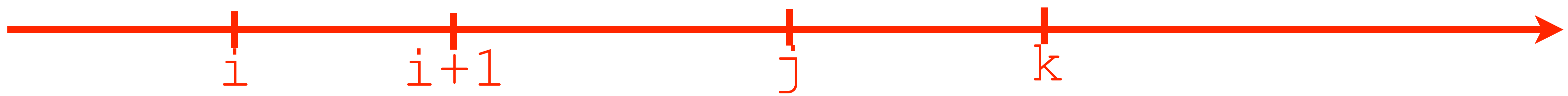
time j: Edge (u,v) STRIKES BACK

Edge (u,v) needs to be added back.
⇒ there must be some aug.path that uses edge v → u.

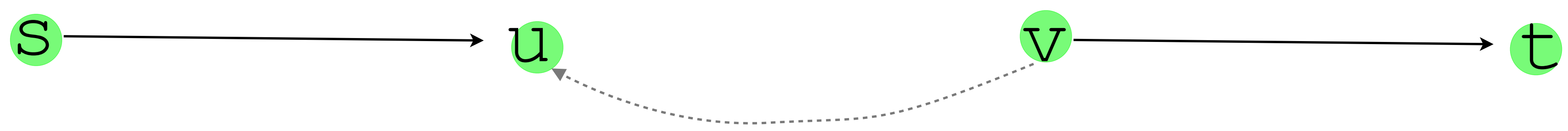
this path is "shortest"



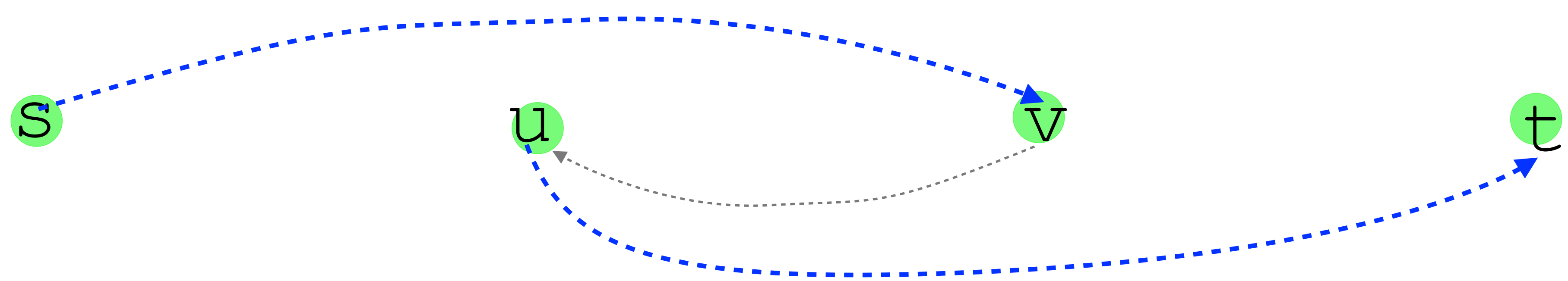
$\delta_j(s,u) = \delta_j(s,v) + 1$



time i+1: (u,v) is critical: $\delta_{i+1}(s,v) \geq \delta_i(s,u) + 1$

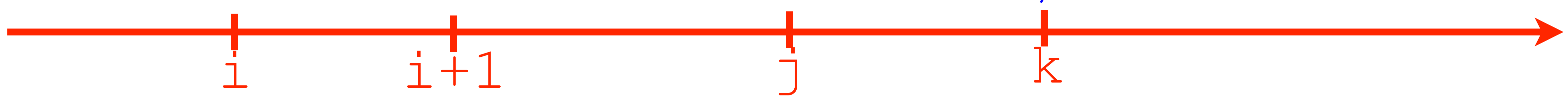


time j: Edge (u,v) STRIKES BACK



$\delta_j(s,u) = \delta_j(s,v) + 1$

edge (u,v) goes critical again.

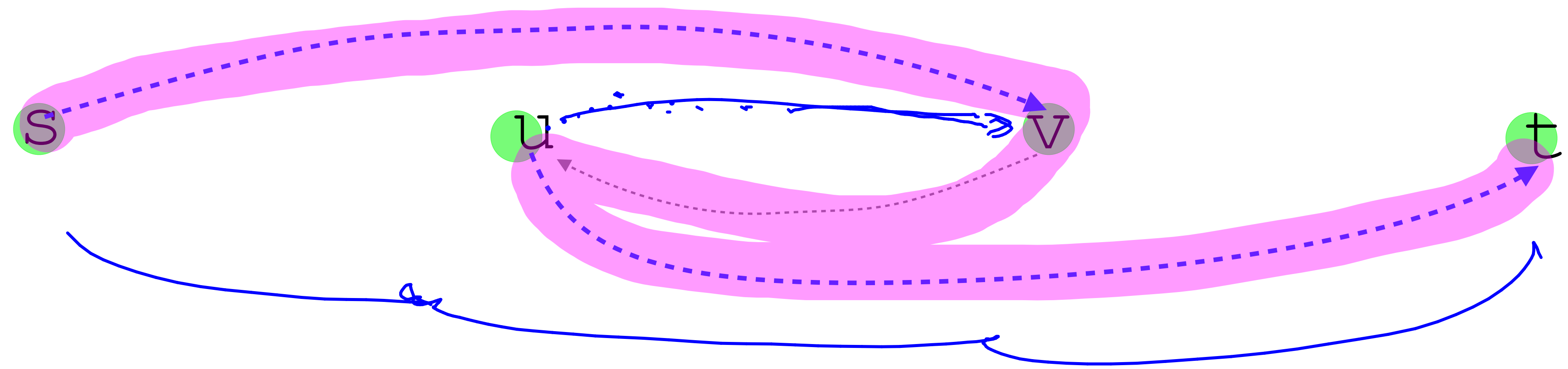


time j: Edge (u,v) STRIKES BACK

$$\delta_{i+1}(s,v) \geq \delta_i(s,u) + 1$$

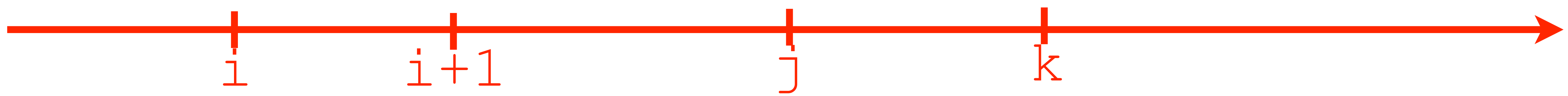
$$\delta_j(s,u) = \delta_j(s,v) + 1$$

$$\Rightarrow \delta_j(s,u) = \delta_j(s,v) + 1 \geq \delta_{i+1}(s,v) + 1 \geq \delta_i(s,u) + 2$$



@ time k:

$$\delta_k(s,u) \geq \delta_j(s,u) \geq \delta_i(s,u) + 2$$



time k: RETURN OF THE (u,v) critical

$$\delta_k(s,u) \geq \delta_i(s,u) + 2$$

QUESTION: How many times can (u,v) be critical?

$\delta_i(s,u)$

$\alpha \quad \alpha+2 \quad \alpha+4 \quad \alpha+6 \quad \leq V-1$

$\Rightarrow \frac{V}{2}$ at most

FF w BFS $\min\{\Theta(VE^2), \Theta(E|f|)\}$

edge critical only $\frac{V}{2}$ times.

there are only E edges.

ergo, total # of augmenting paths: $\leq \Theta(VE)$

time to find an augmenting path: $\Theta(V+E)$ via BFS

total running time of E-K algorithm: $\Theta(V \cdot E^2)$ better than $\Theta(E|f|)$ for many graphs

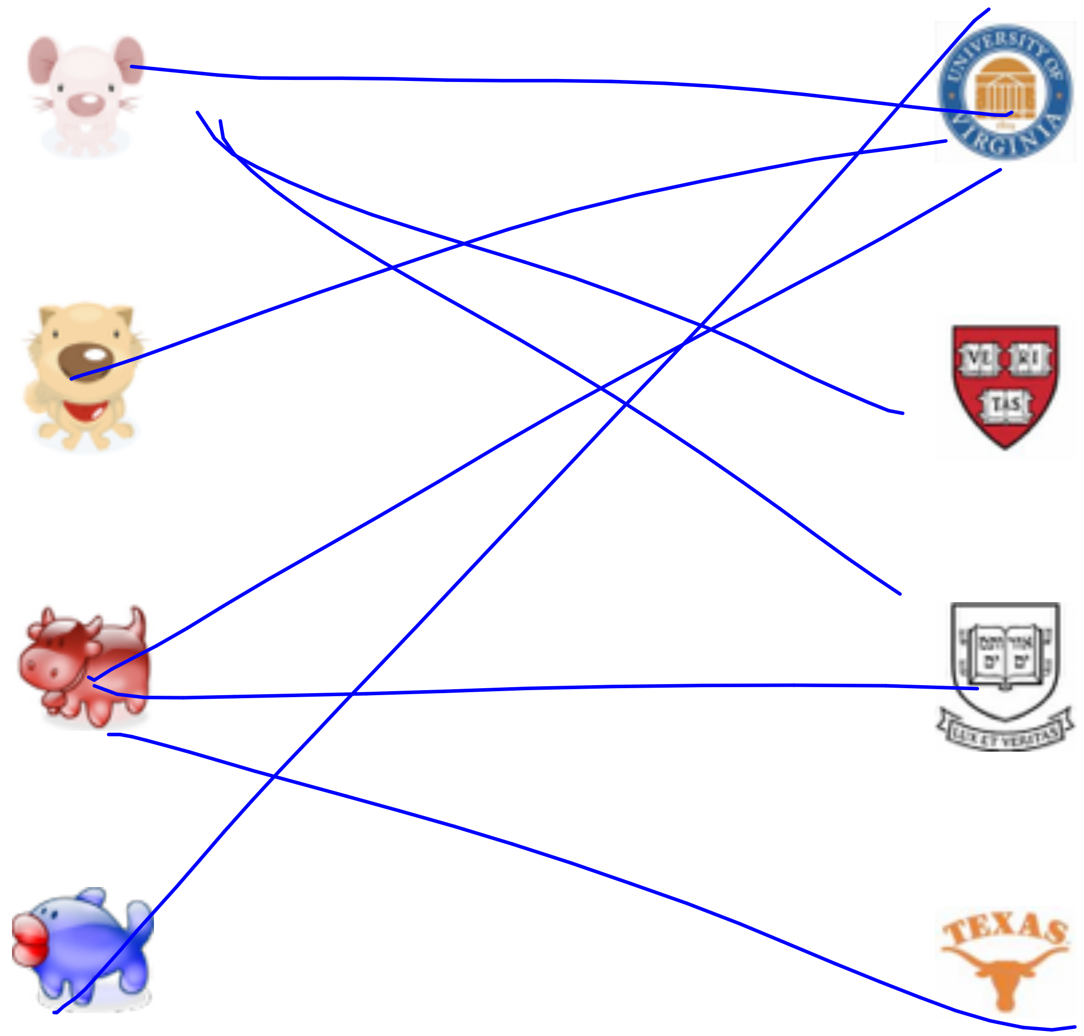
FF	$O(E\|f^*\|)$
EK2	$\Theta(E^2V)$
PUSH-RELABEL	$\Theta(EV^2)$
FASTER PUSH-RELABEL	$\Theta(V^3)$

Tarjan

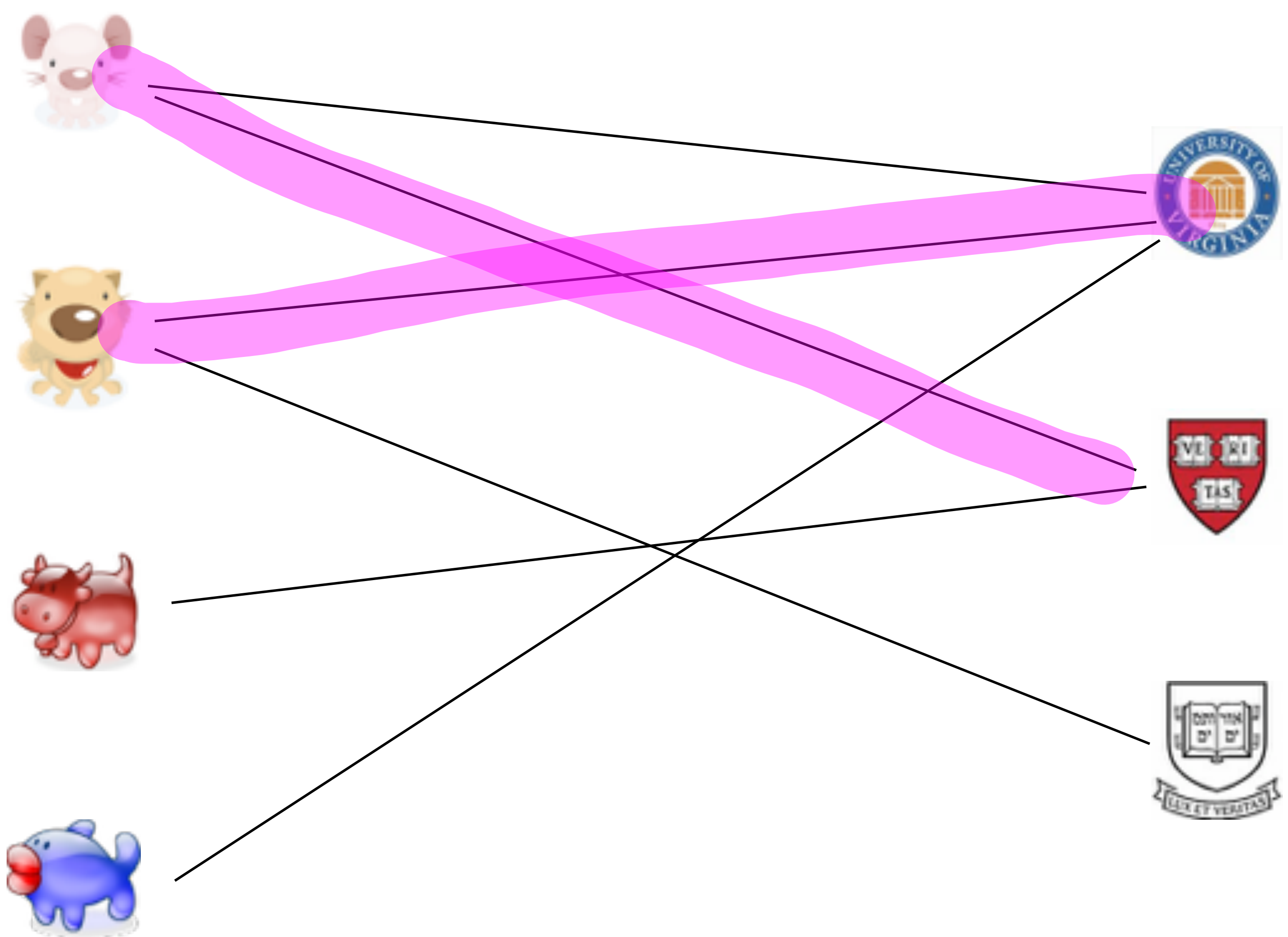
Amortized analysis.

Bipartite

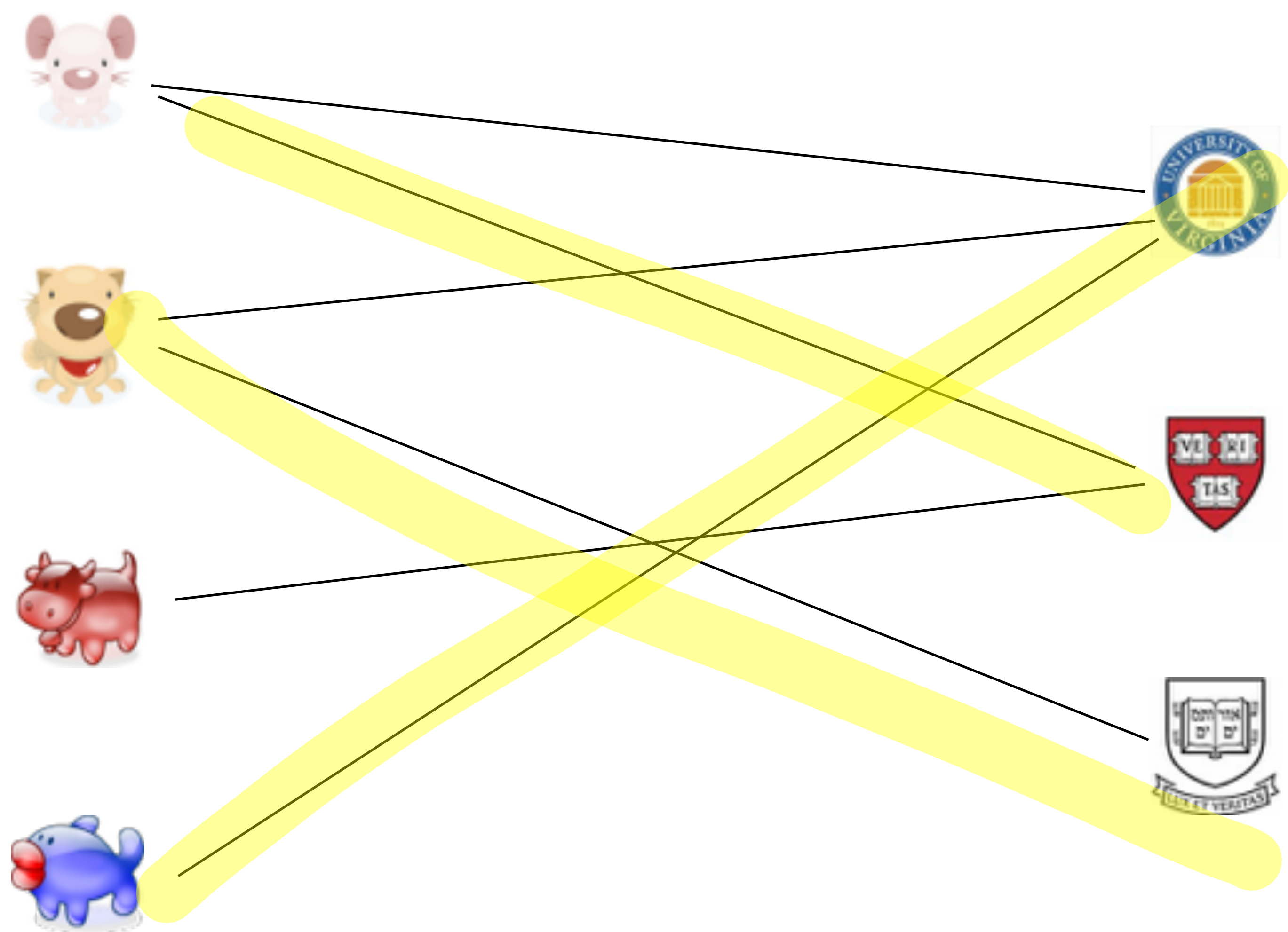
Maximum bipartite matching



Maximum bipartite matching



Maximum bipartite matching



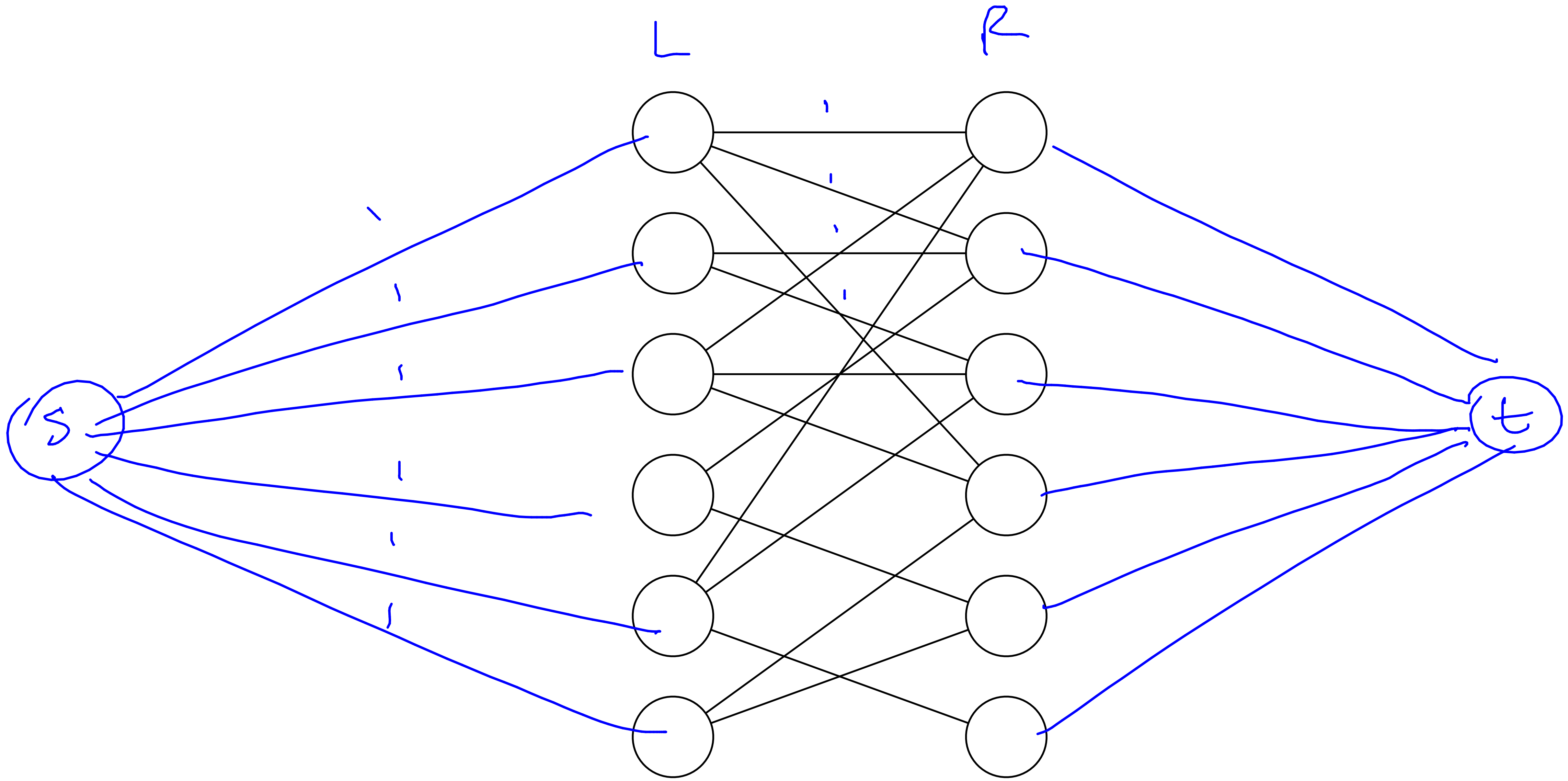
Bipartite matching

problem: Given a graph (L, R, E), find the largest set $M \subseteq E$ such that each node $v \in L$, or $v \in R$ is incident to at most one edge in M.

Algorithm

given $G = (L, R, E)$, construct G'

each edge has $c(e) = 1$.

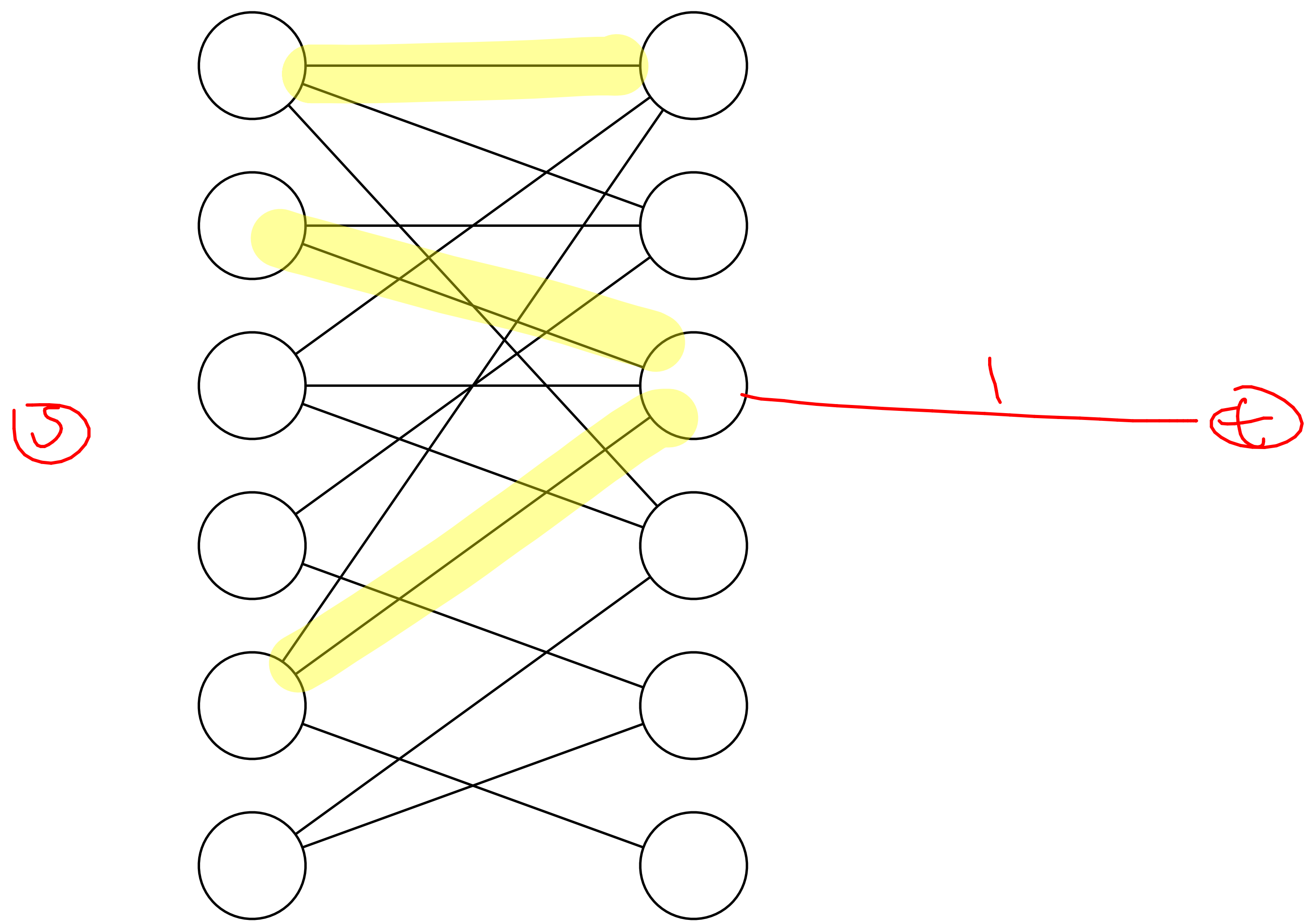


algorithm

1. MAKE NEW G' FROM INPUT G.

2. RUN FF ON G'

3. OUTPUT ALL MIDDLE EDGES WITH FLOW F(E)=1.



Correctness

IF G HAS A MATCHING OF SIZE K, THEN G' has a MAXFLOW of K.

Proof: (outline) Let M^* be the matching of size K for G,

Construct flow f to be

$f(e) = 1$ if $e \in M^*$, and if $e = (x, y)$ then $f(s, x) = 1$, $f(y, t) = 1$

⇒ flow f satisfies ① capacity constraints

② flow constraint

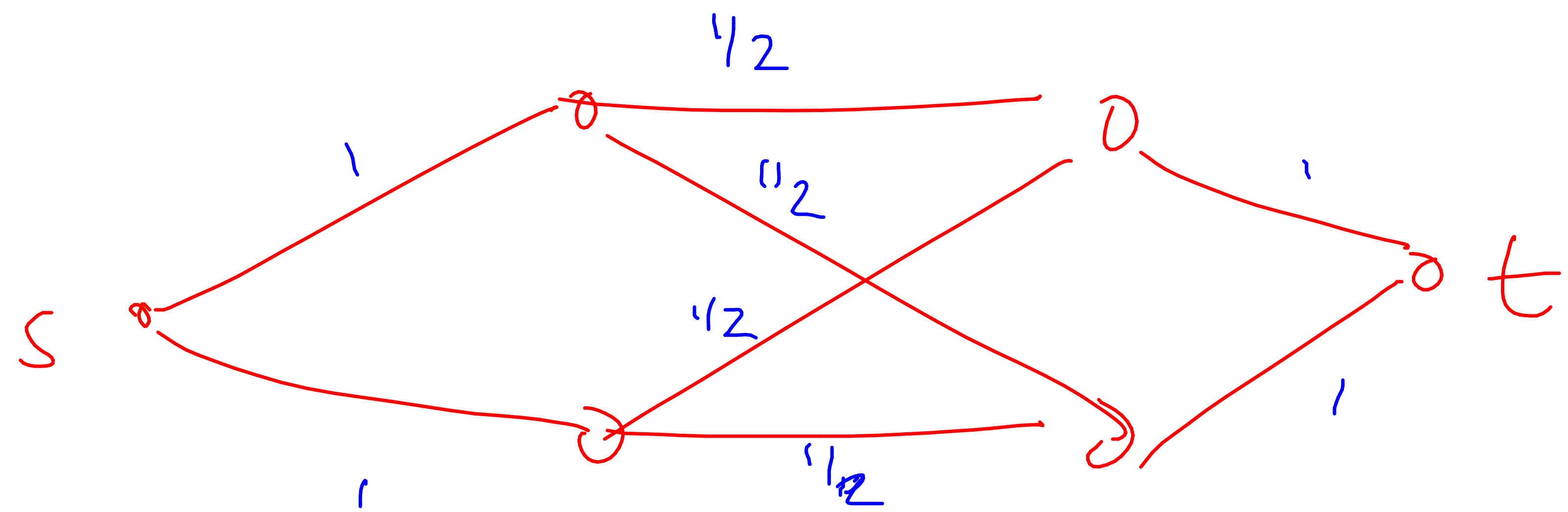
INFLOW(x) = OUTflow(x)

correctness

IF G' HAS A FLOW OF K, THEN then G has a MATCHING of size K.

"Add all edges w/ f(e)=1 to a set M, it follows that |M|=K"

NOT TRIVIAL (wrong)



$|f| = 2$

integrality theorem

IF CAPACITIES ARE ALL INTEGRAL, THEN there exists a MAXFLOW whose flows are integral. Furthermore, FF produces such a flow.

Proof: By induction. FF starts w/ an integral flow.

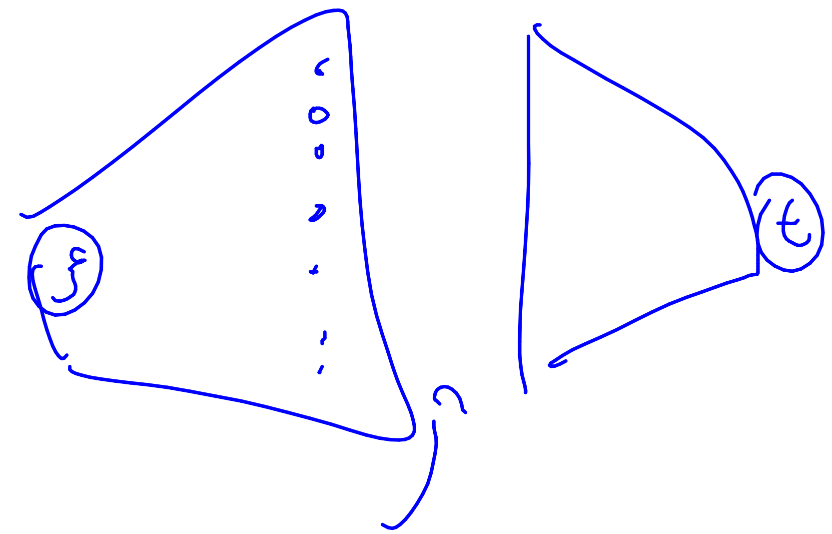
True after i iterations.

On the $(i+1)^{st}$ iteration, push $\min_{e \in p} c(e)$, which is also integral. Why?? B/c capacities are integral.

$\Rightarrow$ flow @ $(i+1)$ integral

$\Rightarrow$ Final answer is integral. ☒

correctness



G' HAS A FLOW OF K, THEN G HAS K-MATCHING.

① Our algorithm uses FF, so flow for G' is integral.

Now define $M = \{ e \mid f(e) = 1 \text{ and } e = (x, y) \text{ s.t. } x \in L, y \in R \}$

Prove that M is a matching.

- All flows are integral & capacity $c(e) = 1$. so $f(e) = 0$ or 1 for $e \in E$.

  Thus for all $v \in L$ ($v \in R$), v is incident to at most 1 edge in M.

  By the flow constraint. By MIN. cuts, $|M| = K$

running time

① $\Theta(L+R)$

② $\Theta(E^2 V)$ or $\Theta(E \cdot V) = \Theta(E \cdot L)$

$\Theta(E|f|)$

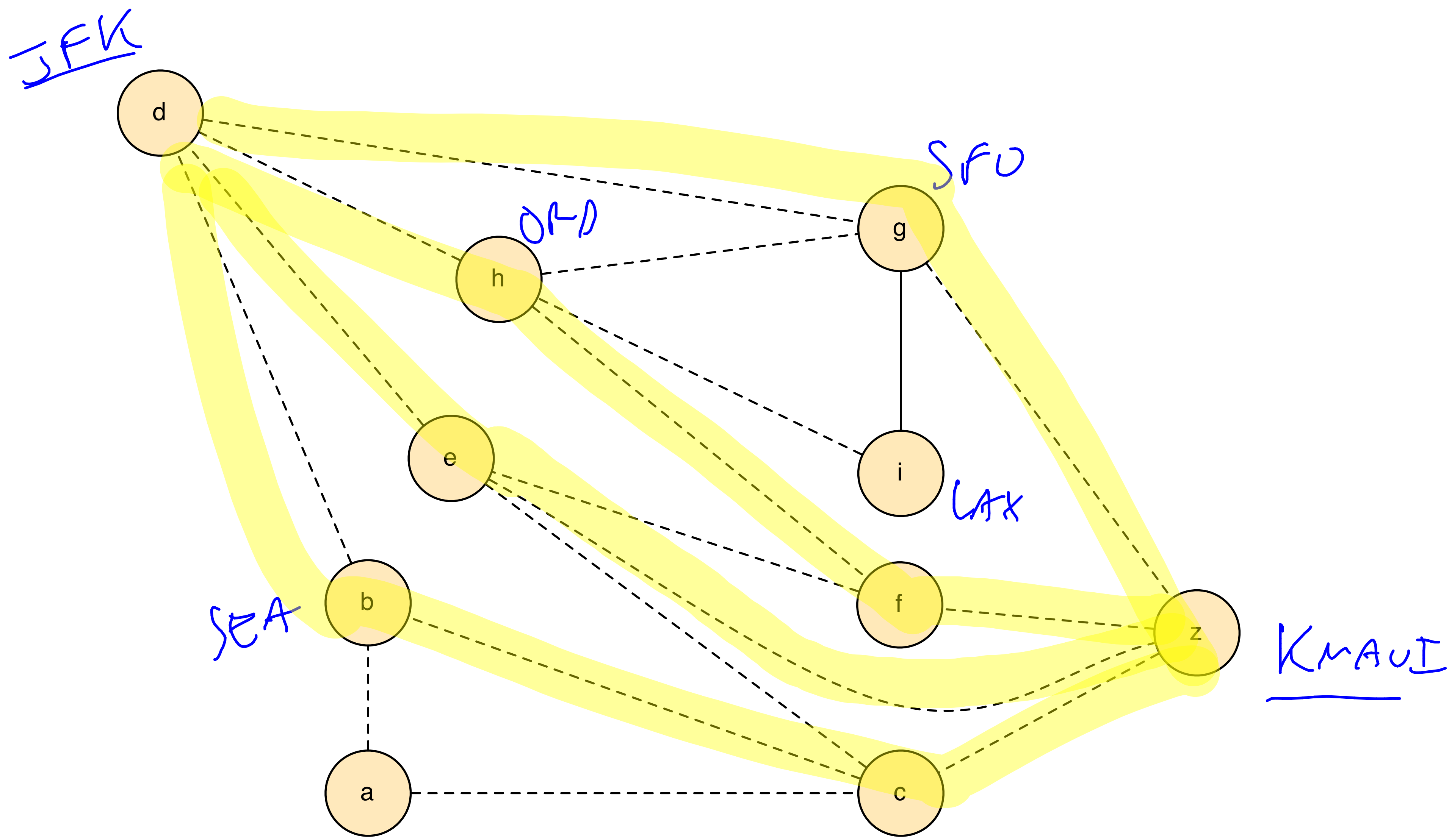
$\to \min(L, R)$

maxflow $|f| \le L$

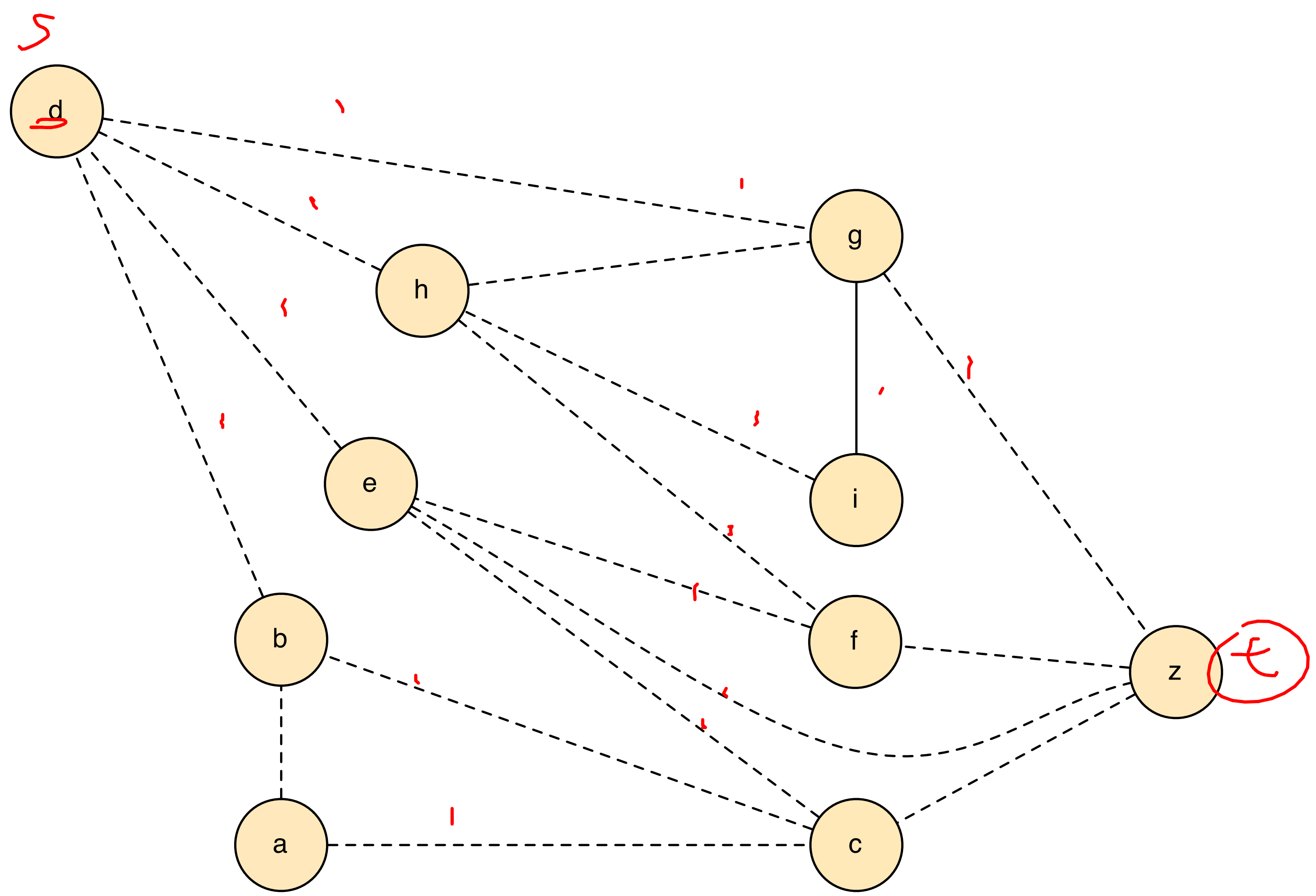
③ $\Theta(L)$

$\Theta(EL)$

edge-disjoint paths



algorithm



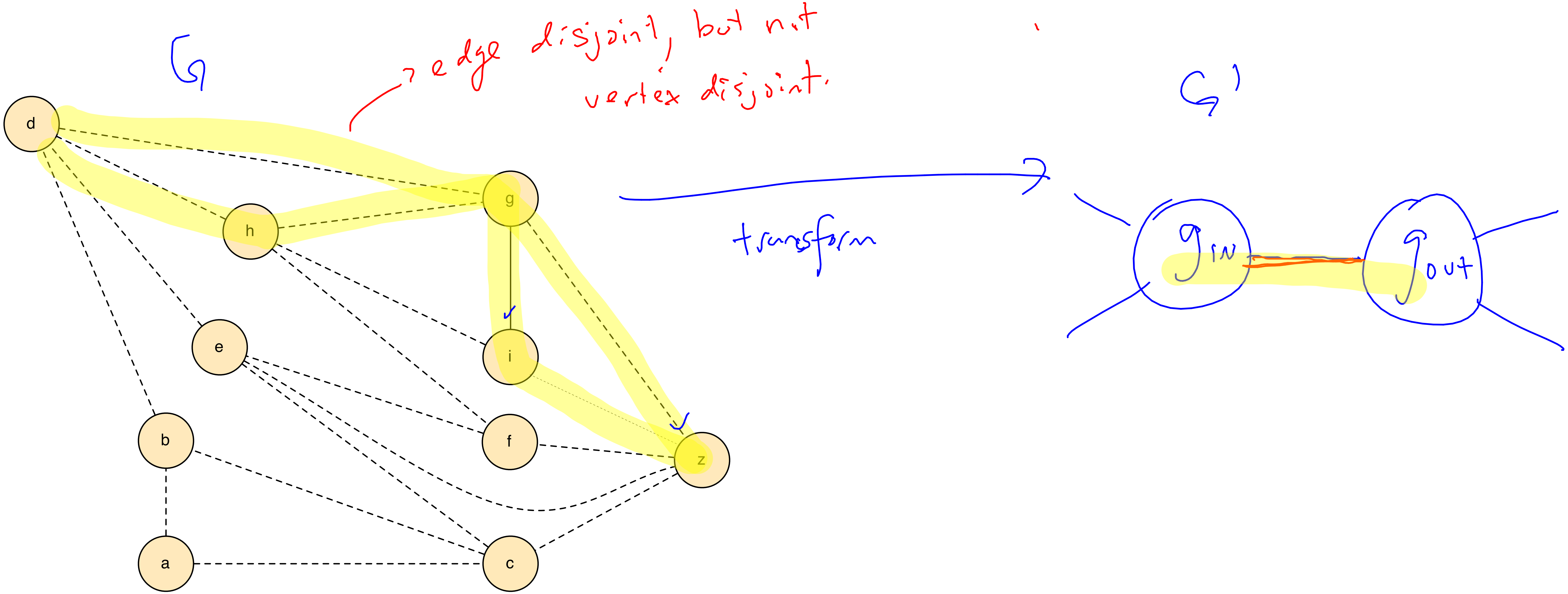
analysis

IF G HAS K DISJOINT PATHS, THEN

analysis

G' HAS A FLOW OF K, THEN

vertex-disjoint paths



Baseball elimination

	W	L	Left	A	P	N	M
				Against			
ATL	83	71	8	-	1	6	1
PHL	80	79	3	1	-	0	2
NY	78	78	6	6	0	-	0
MONT	77	82	3	1	2	0	-

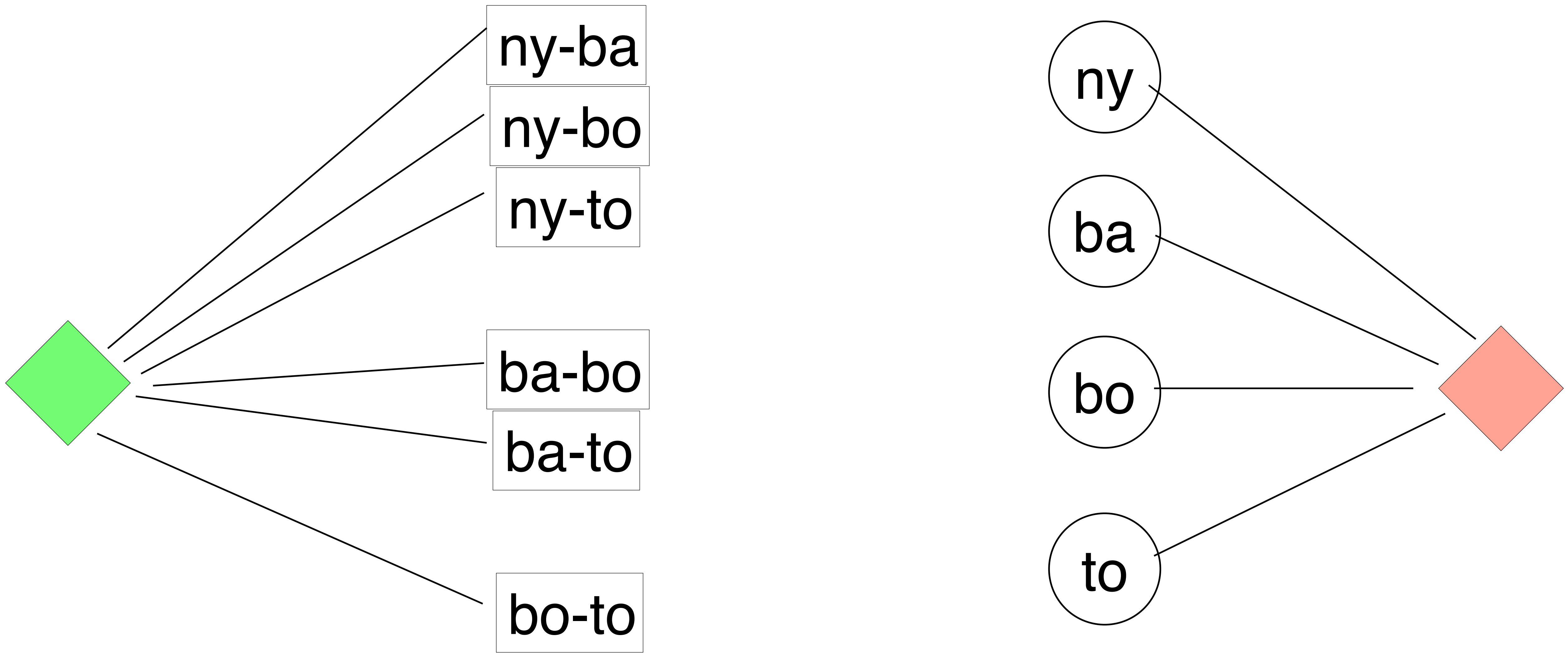
80 games

Baseball elimination

	W	L	Left	Against: N	B	Bo	T	D
NY	75	59	28		3	8	7	3
BAL	71	63	28	3		2	7	4
BOS	69	66	27	8	2			
TOR	63	72	27	7	7			
DET	49	86	27	3	4			

$$\frac{\sum \text{wins} + \text{games}}{4} = 76.25$$

$\Rightarrow 77$



	W	L	Left	N	B	Bo	T	D
NY	75	59	28		3	8	7	3
BAL	71	63	28	3		2	7	4
BOS	69	66	27	8	2			
TOR	63	72	27	7	7			
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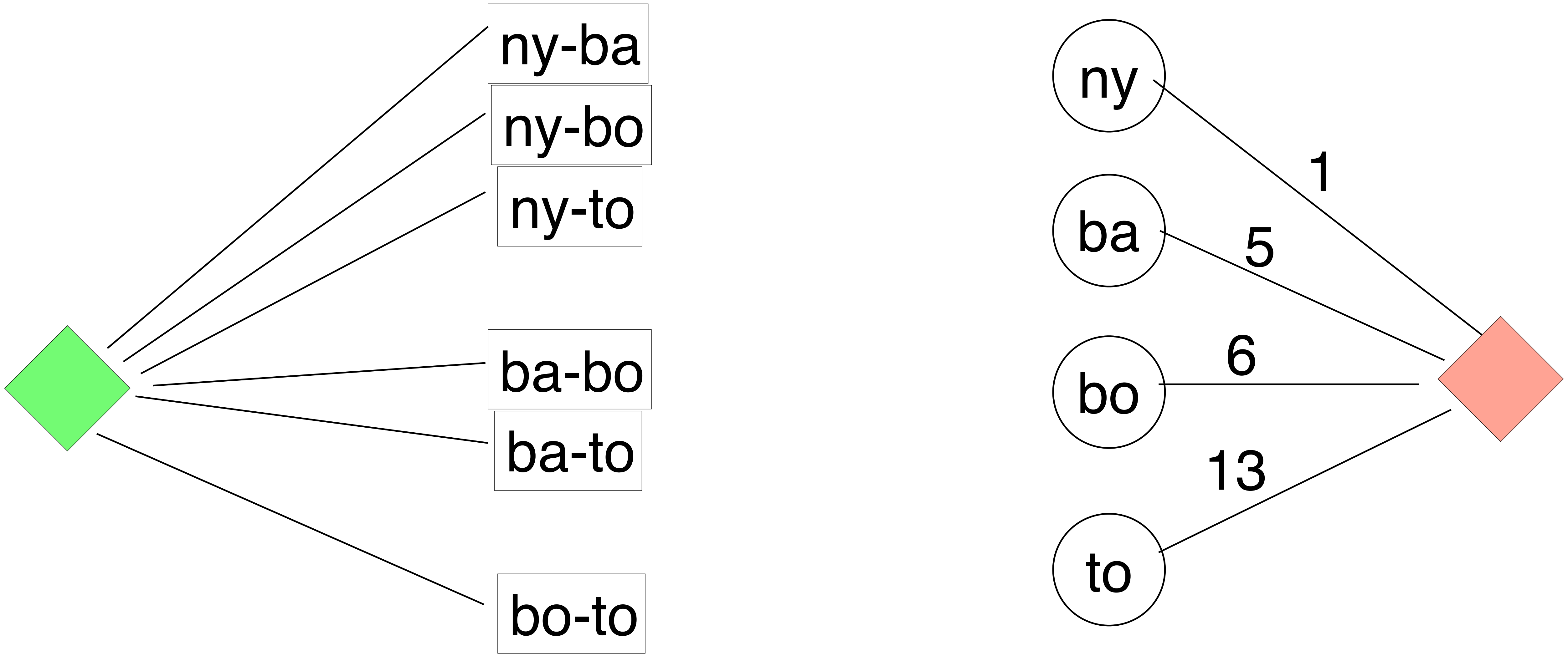

	W	L	Left	N	B	Bo	T	D
NY	75	59	28		3	8	7	3
BAL	71	63	28	3		2	7	4
BOS	69	66	27	8	2			
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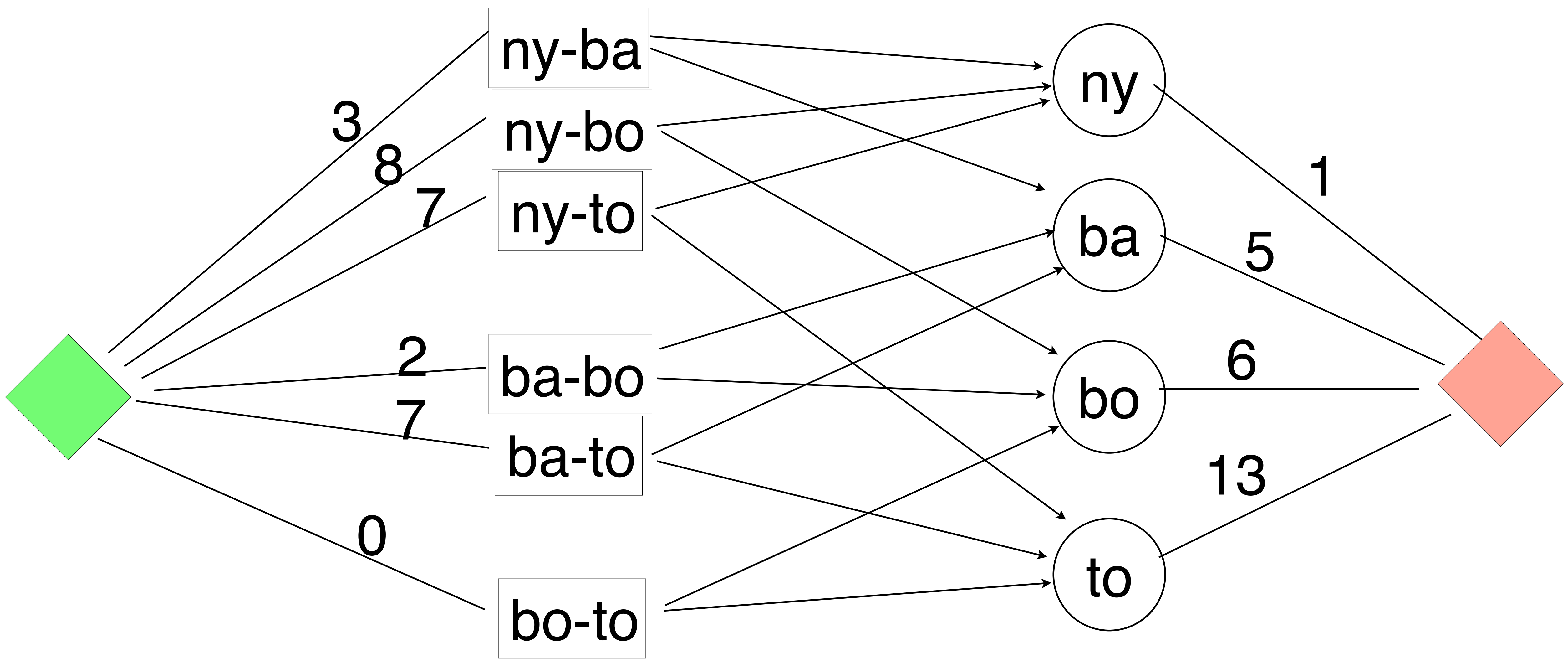

	W	L	Left	N	B	Bo	T	D
NY	75	59	28		3	8	7	3
BAL	71	63	28	3		2	7	4
BOS	69	66	27	8	2			
TOR	63	72	27	7	7			
DET	49	86	27	3	4			