L25

4102

4.21.2016

abhi shelat

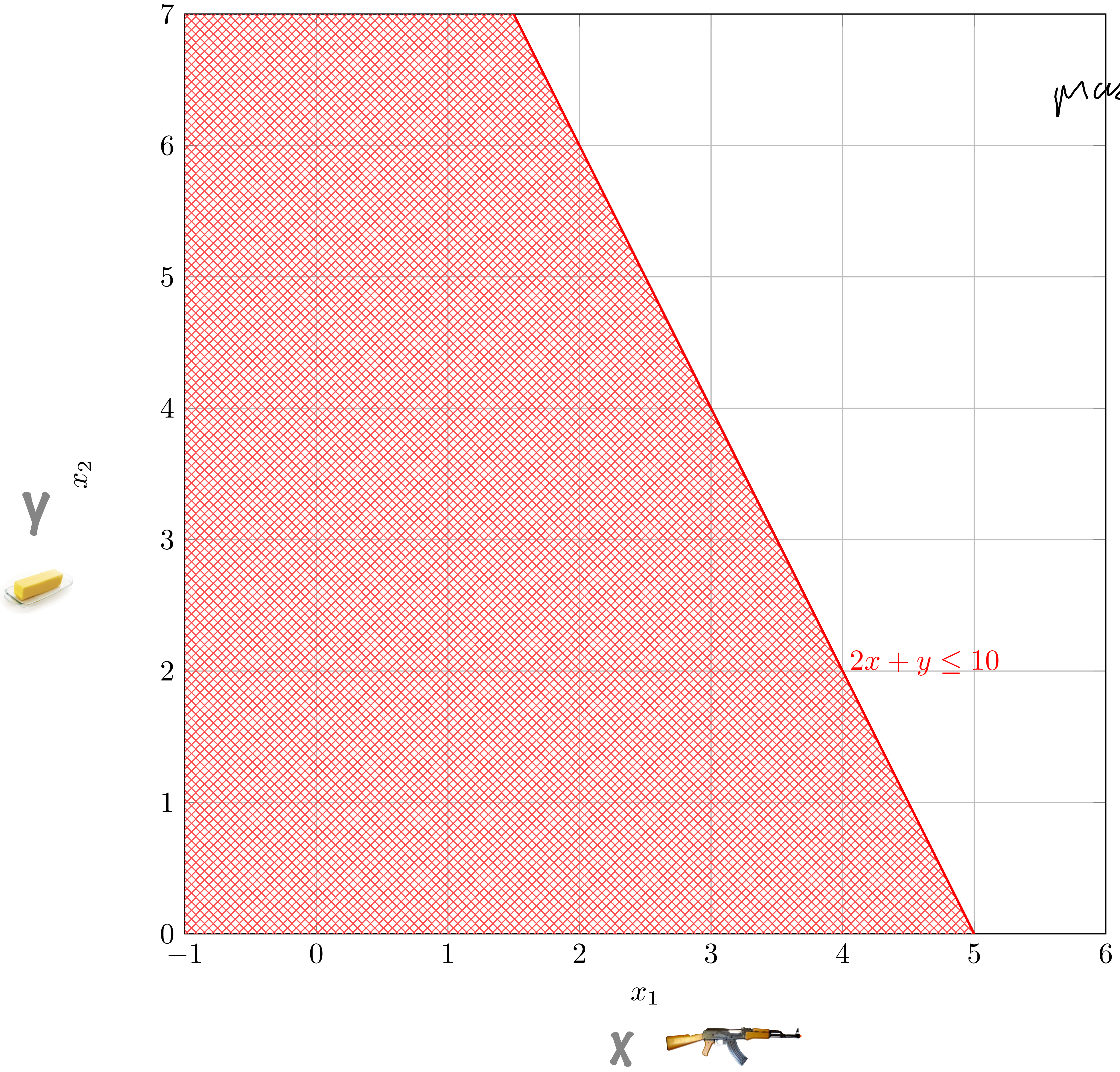
Guns and butter

$$\max x + y$$

$$\begin{aligned} 4x - y &\leq 8 \\ 2x + y &\leq 10 \\ 5x - 2y &\geq -2 \\ x, y &\geq 0 \end{aligned}$$

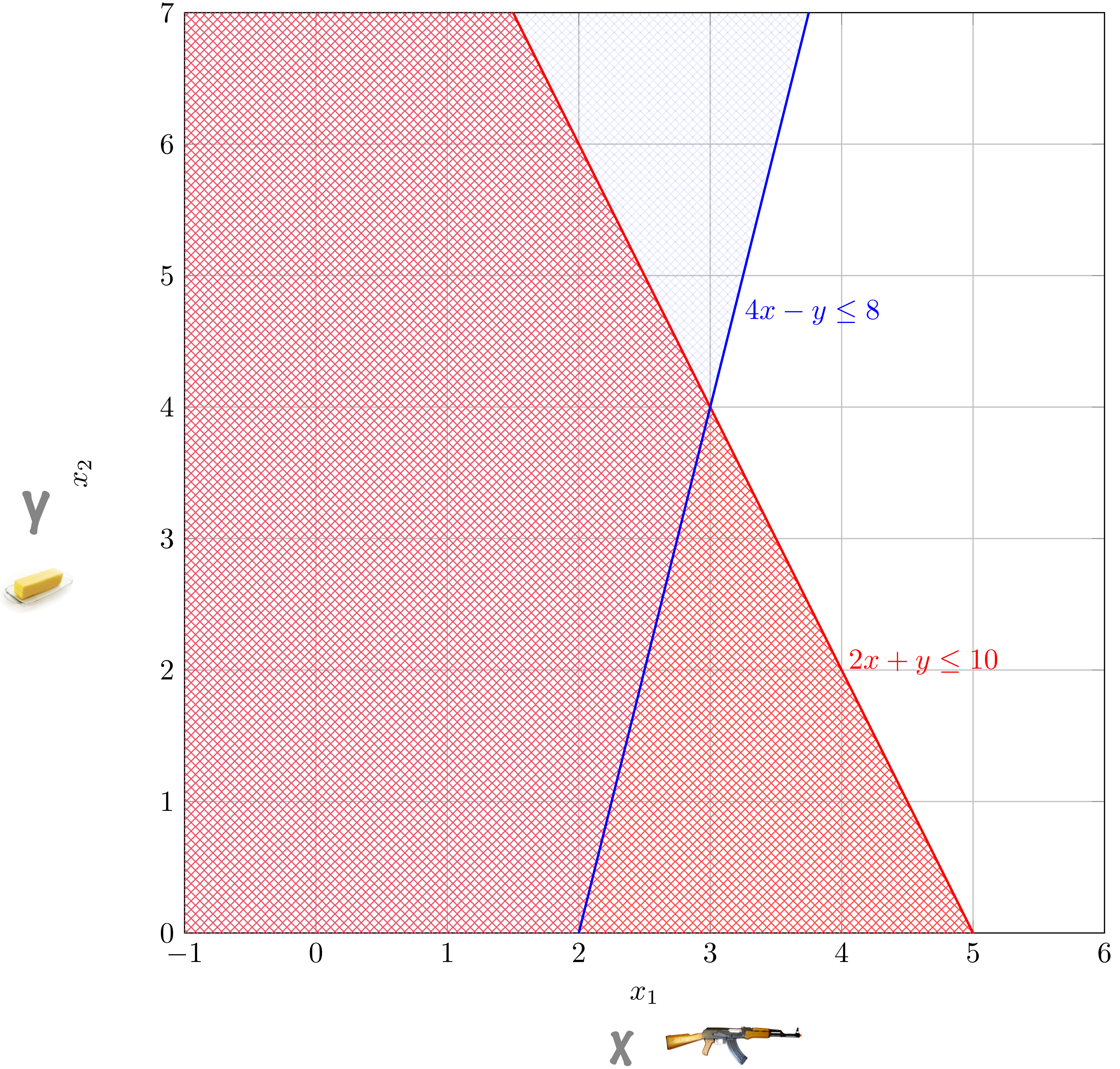
http://i16.photobucket.com/albums/b20/safebuy/ak47/ak47-electric_lg.jpg

http://2.bp.blogspot.com/_NX4zcMNX4VE/Sb8MQffflII/AAAAAAAAAL0/eu4J0dfFhJE/s400/gourmet-butter.jpg

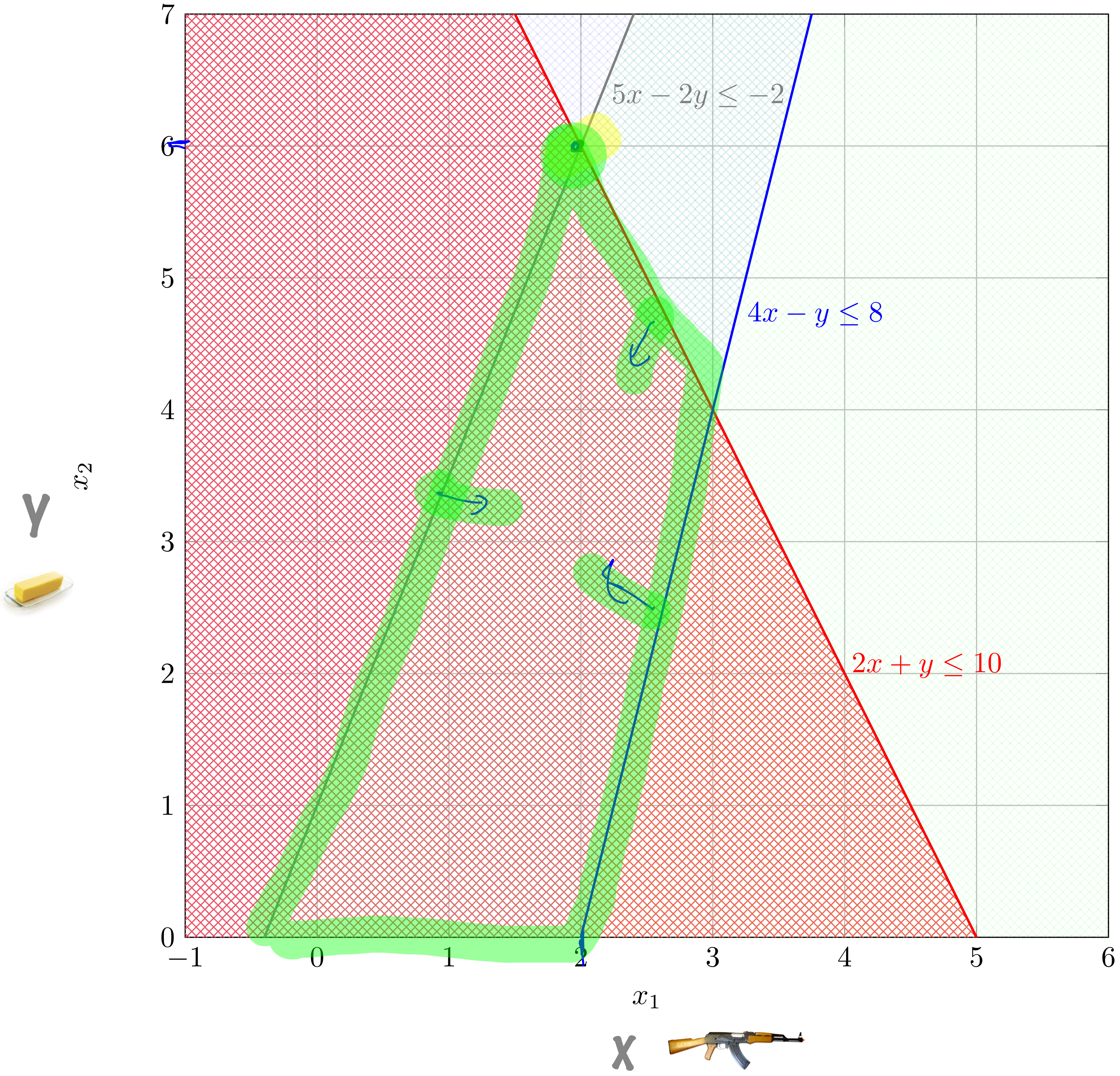


max $2x + 5y$

$$
\begin{aligned}
4x - y &\leq 8 \\
2x + y &\leq 10 \\
5x - 2y &\geq -2 \\
x, y &\geq 0
\end{aligned}
$$



$$
\begin{aligned}
4x - y &\leq 8 \\
2x + y &\leq 10 \\
5x - 2y &\geq -2 \\
x, y &\geq 0
\end{aligned}
$$



$$
\begin{aligned}
4x - y &\leq 8 \\
2x + y &\leq 10 \\
5x - 2y &\geq -2 \\
x, y &\geq 0
\end{aligned}
$$

Certificate of optimality

$$\max x + y$$

$$\begin{aligned} 4x - y &\leq 8 \\ 2x + y &\leq 10 \quad \cdot 7 \\ 5x - 2y &\geq -2 \quad \cdot -1 \\ x, y &\geq 0 \end{aligned}$$

$$\begin{aligned} 14x + 7y &\leq 70 \\ -5x + 2y &\leq 2 \\ \hline 9x + 9y &\leq 72 \end{aligned}$$

$$x + y \leq 8$$

Certificate of optimality

$$\max x + y$$

$$\begin{aligned} 4x - y &\leq 8 \\ 2x + y &\leq 10 \\ 5x - 2y &\geq -2 \\ x, y &\geq 0 \end{aligned}$$

7	$14x + 7y \leq 70$
-1	$-5x + 2y \leq 2$

$9x + 9y \leq 72$

	Brownie	Dumpling	Espresso	Roots
cost	5	2	3	8
cals	400	200	150	500
choc	3	2	0	0
sugar	2	2	4	4
fat	2	4	0	5

requirements: 500 calories, 6 oz choc, 10 oz sugar, 8 oz fat

$\min 5x_1 + 2x_2 + 3x_3 + 8x_4$

$400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500$

___

___

___

	Brownie	Dumpling	Espresso	Roots
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choc	3	2	0	0
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fat	2	4	0	5

requirements: 500 calories, 6 oz choc, 10 oz sugar, 8 oz fat

$$\min 5x_1 + 2x_2 + 3x_3 + 8x_4$$

A x b

$$\begin{bmatrix} 400 & 200 & 150 & 500 \\ 3 & 2 & 0 & 0 \\ 2 & 2 & 4 & 4 \\ 2 & 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \geq \begin{bmatrix} 500 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$\min 5x_1 + 2x_2 + 3x_3 + 8x_4$$

$$\begin{bmatrix} 400 & 200 & 150 & 500 \\ 3 & 2 & 0 & 0 \\ 2 & 2 & 4 & 4 \\ 2 & 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \geq \begin{bmatrix} 500 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$\min 5x_1 + 2x_2 + 3x_3 + 8x_4$$

$$\begin{bmatrix} 400 & 200 & 150 & 500 \\ 3 & 2 & 0 & 0 \\ 2 & 2 & 4 & 4 \\ 2 & 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \geq \begin{bmatrix} 500 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$\leq$

$x_1 \geq 0$
$x_2 \geq 0$
$x_3 \geq 0$
$x_4 \geq 0$

```
H-representation
begin
8 4 rational
-500 400 200 150 500
-6    3   2   0    0
-10   2   2   4    4
-8    2   4   0    5
0     1   0   0    0
0     0   1   0    0
0     0   0   1    0
0     0   0   0    1
end
minimize
0 5 2 3 8
```

$$\min 5x_1 + 2x_2 + 3x_3 + 8x_4$$

$$\begin{bmatrix} 400 & 200 & 150 & 500 \\ 3 & 2 & 0 & 0 \\ 2 & 2 & 4 & 4 \\ 2 & 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \geq \begin{bmatrix} 500 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

```
H-representation
begin
8 4 rational
-500 400 200 150 500
-6   3   2   0   0
-10  2   2   4   4
-6   2   4   0   5
0    1   0   0   0
0    0   1   0   0
0    0   0   1   0
0    0   0   0   1
end
minimize
0 5 2 3 8
```

```
*Objective function is
 0 + 5 X[1] + 2 X[2] + 3 X[3] + 8 X[4]
*LP status: a dual pair (x, y) of optimal
begin
  primal_solution
  1 :  0
  2 :  3
  3 :  1
  4 :  0
  dual_solution
  2 :  -1/4
  5 :  -11/4
  3 :  -3/4
  8 :  -5
  optimal_value : 9
end
*number of pivot operations = 4
```

max flow as lp

$$\max \sum_v f(s,v) - \sum_v f(v,s)$$

$f(u,v) \le c(u,v)$ for (u,v) in E

$\sum_u f(u,v) = \sum_w f(v,w) \quad \forall v$

$f(u,v) \ge 0$ for (u,v) in E

MAX $f_1 + f_8 - f_{11}$

$f_1 \le 3$

$f_2 \le 2$

⋮

$f_{11} \le 2$

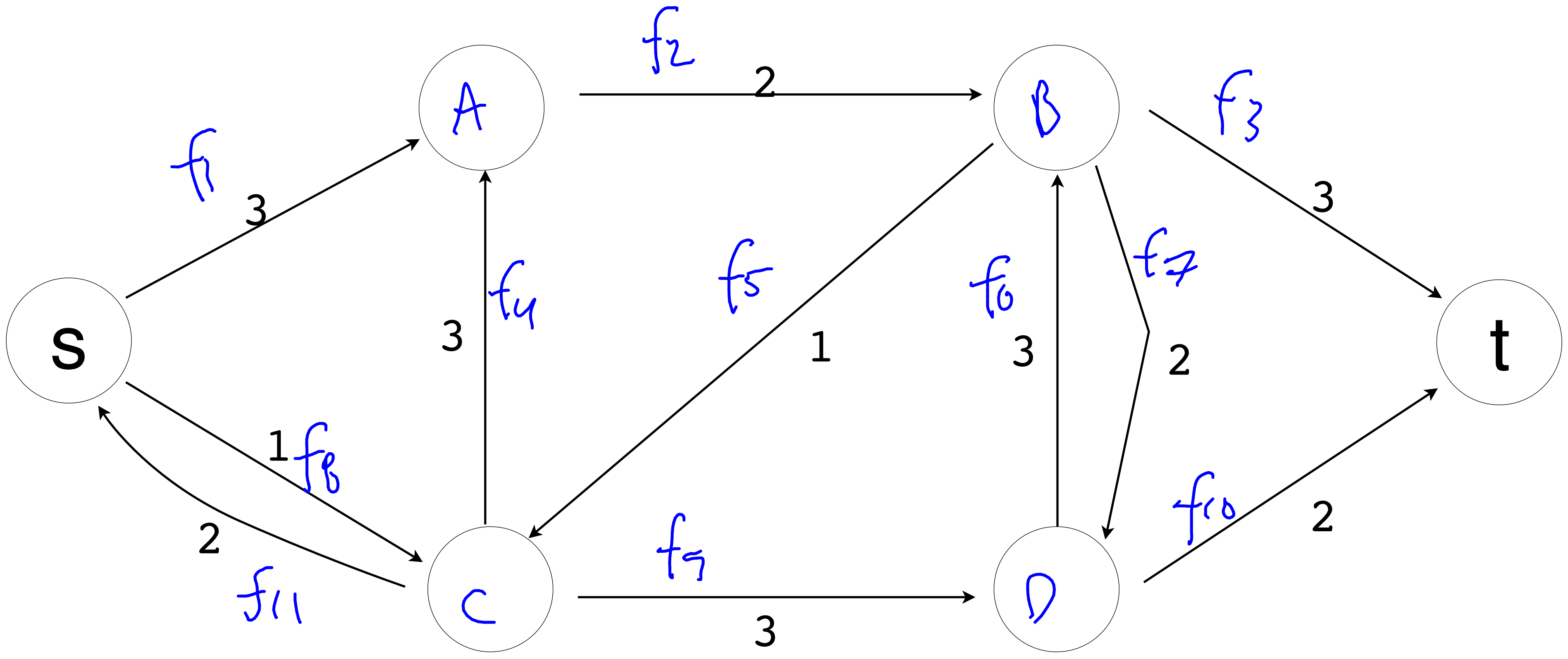
$f_1 + f_4 - f_2 = 0$ (A)

(B)

(C)

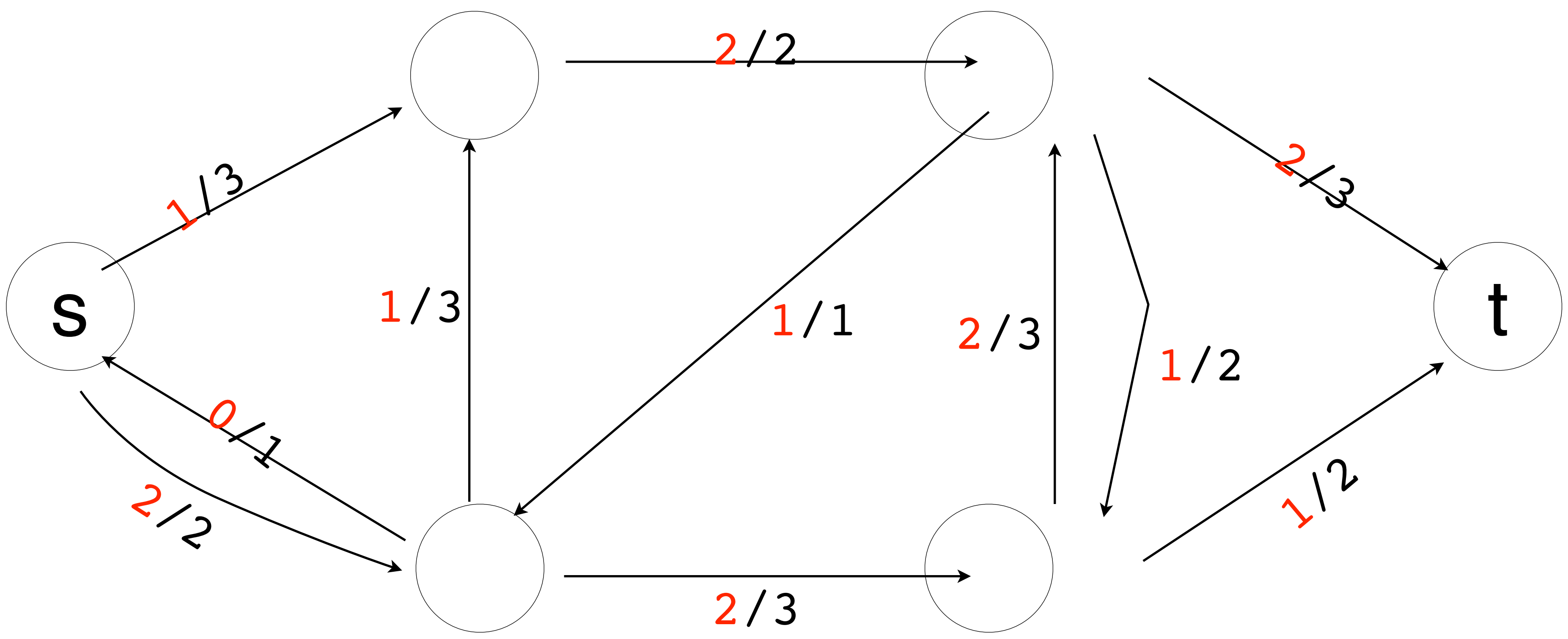
(D)

$f_i \ge 0$



min-cost flow as lp

input: (G, c, s, t) $G = (V, E)$ $c : E \to \mathbb{Z}_+$ $x : E \to \mathbb{Z}_+$ d



min-cost flow as lp

$$\min_e x_e \cdot f(e)$$

$$f(e) \leq c(e)$$

$$f(e) \geq 0$$

$$\sum_u f(u,v) = \sum_w f(v,w)$$

$$\sum_v f(s,v) - \sum_v f(v,s) = d$$

zero-sum games

		COLIN	
		c_1	
ROWENA	r_1	3 -3	-1
	r_2	-2	1

$$\sum_{i,j} G_{ij} r_i c_j$$

zero-sum games

		colin	
rowena		3	-1
		-2	1

rowena announces
her strategy first:

zero-sum games

colin

rowena r_1	3	-1
r_2	-2	1

rowena announces her strategy first:

(r_1, r_2)

$$\min\{\ 3r_1 - 2r_2,\ -r_1 + r_2\ \}$$

z z

zero-sum games

colin

rowena

3	-1
-2	1

rowena announces her strategy first:

$$\max z$$

$$z \leq 3r_1 - 2r_2$$

$$z \leq -r_1 + 2r_2$$

$$r_1 + r_2 = 1$$

$$r_1, r_2 \geq 0$$

zero-sum games

colin

rowena

rowena	3	-1
	-2	1

colin announces his strategy first:

(c_1, c_2)

zero-sum games

c_1 c_2

colin

rowena

	3	-1
	-2	1

colin announces his strategy first:

pick (c_1, c_2) so as to min $\max\{3c_1 - c_2,\ -2c_1 + c_2\}$

zero-sum games

	colin	
rowena	3	-1
	-2	1

colin announces
his strategy first:

$$\min w$$

$$\begin{aligned} -3c_1 + c_2 + w &\geq 0 \\ 2c_1 - c_2 + w &\geq 0 \\ c_1 + c_2 &= 1 \\ c_1, c_2 &\geq 0 \end{aligned}$$

zero-sum games

	colin	
rowena	3	-1
	-2	1

$$\max z$$

$$\begin{aligned} -3r_1 + 2r_2 + z &\leq 0 \\ r_1 - r_2 + z &\leq 0 \\ r_1 + r_2 &= 1 \\ r_1, r_2 &\geq 0 \end{aligned}$$

zero-sum games

	colin	
rowena	3	-1
	-2	1

$$\max z$$

$$\begin{aligned} -3r_1 + 2r_2 + z &\leq 0 \\ r_1 - r_2 + z &\leq 0 \\ r_1 + r_2 &= 1 \\ r_1, r_2 &\geq 0 \end{aligned}$$

4 rows

=

$$\min w$$

$$\begin{aligned} -3c_1 + c_2 + w &\geq 0 \\ 2c_1 - c_2 + w &\geq 0 \\ c_1 + c_2 &= 1 \\ c_1, c_2 &\geq 0 \end{aligned}$$

zero-sum games

2 move

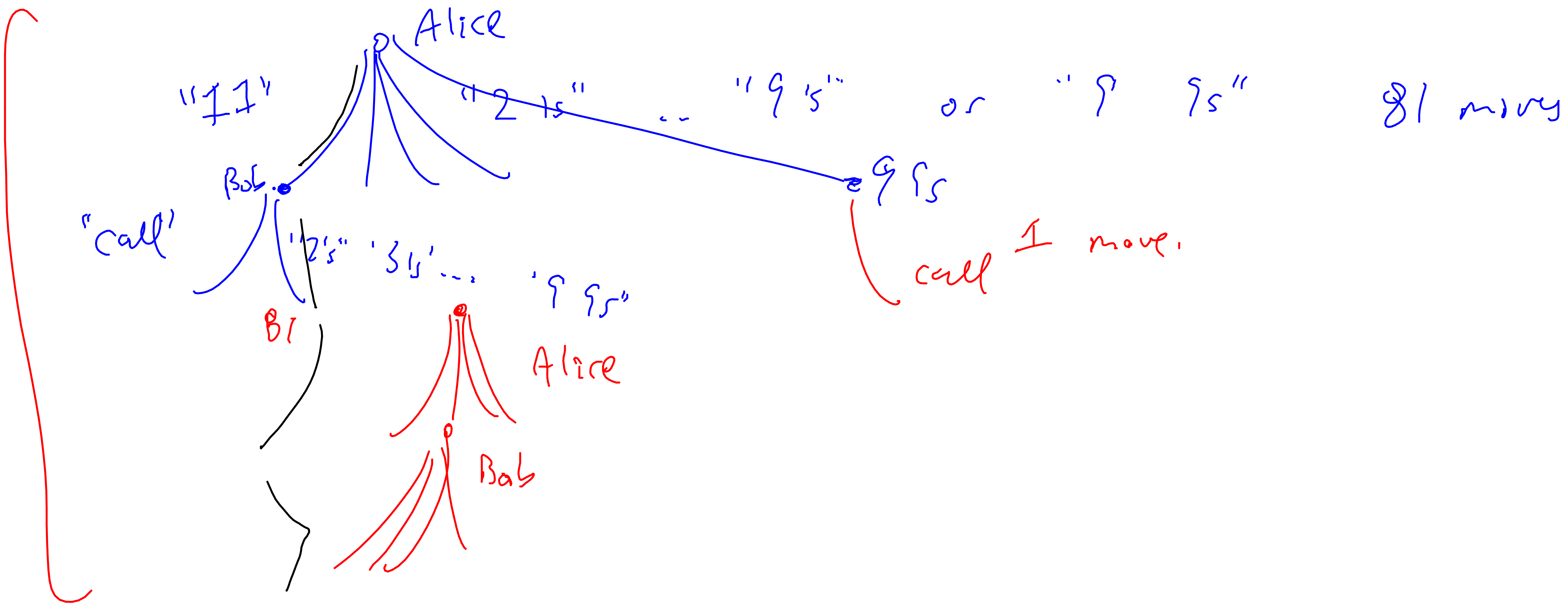
		2/7	5/7	colin
rowena	3/7	3	-1	
	4/7	-2	1	

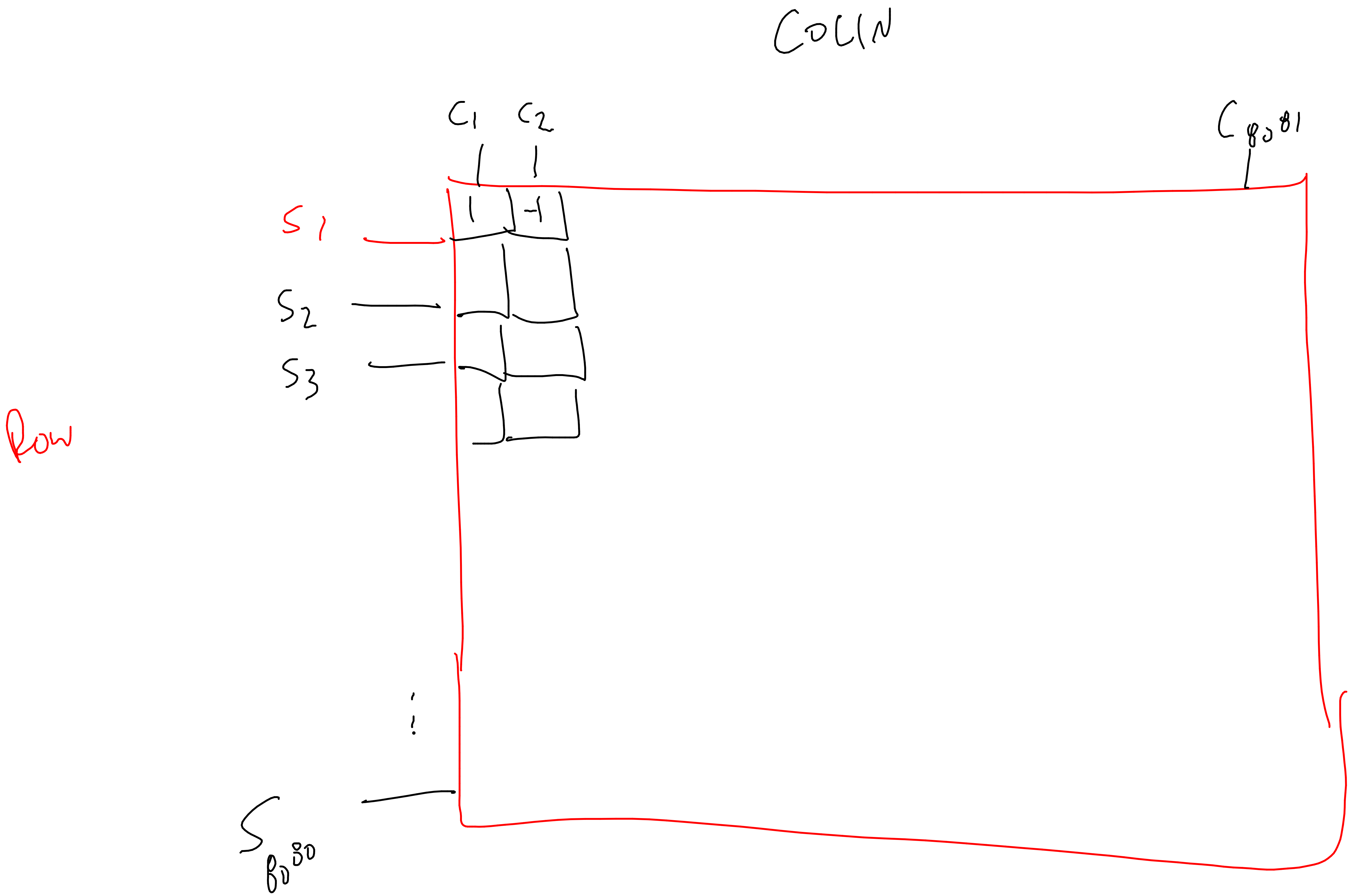
value of the game is : 1/7

Alice

Bob

game tree. enormous!!!..




COLIN
C1
C2
C8081
S1
S2
S3
S8080
Row
1
-1

$$\max_x \min_y \sum_{i,j} G_{ij} x_i y_j = \min_y \max_x \sum_{i,j} G_{ij} x_i y_j$$

https://www.inf.ethz.ch/personal/fukudak/cdd_home/

Welcome to the cdd and cddplus Homepage

Last update: May 15, 2015

Currently, the C-library version cddlib of cdd packages is the only one being updated, while standalone codes cdd and cddplus are still useful. To know what cdd, cddplus and cddlib are, please read

- cddplus readme
- cddlib readme

Manuals (html version):

- cdd/cdd+ manual
- cddlib manual

Get source codes:

- cdd/cddpuls directory click here
- cdd package cdd-061a.tar.gz
- cddplus package cdd+-077a.tar.gz (to be compled with g++ 4.1. With more recent g++, try patch) NEW . With g++ 3.1, use cdd+-077.tar.gz
- cddlib package cddlib-094h.tar.gz NEW

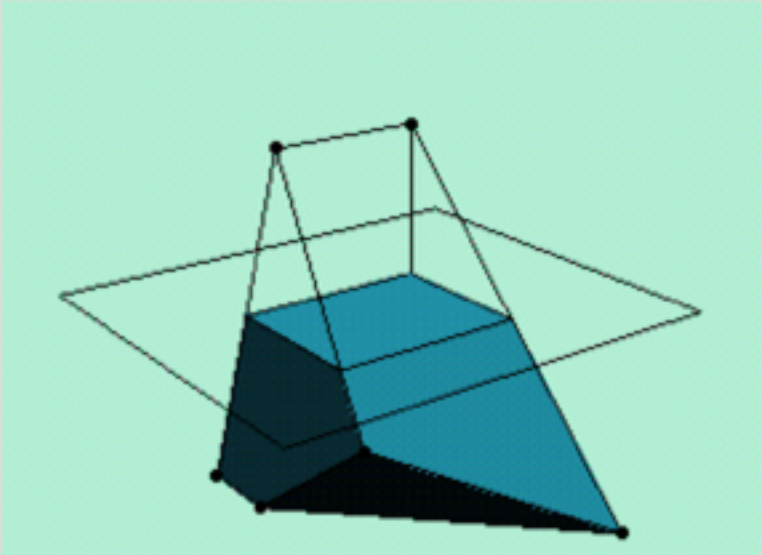
To know the implementation:

- ``The double description revisited'' gzipped ps file

To learn the fundamental concepts of Convex Hull, Vornonoi, Delaunay, etc.:

- ``Polyhedral Computation FAQ'' (still experimental) html version or pdf file

Links to cdd/cdd+/cddlib users and more. NEW



how to “evaluate” an lp

$$\max c^T \vec{x}$$

$$A\vec{x} \leq \bar{b}$$

$$\vec{x} \geq 0$$

simplex

init: at the origin, $(0,0,\ldots,0)$

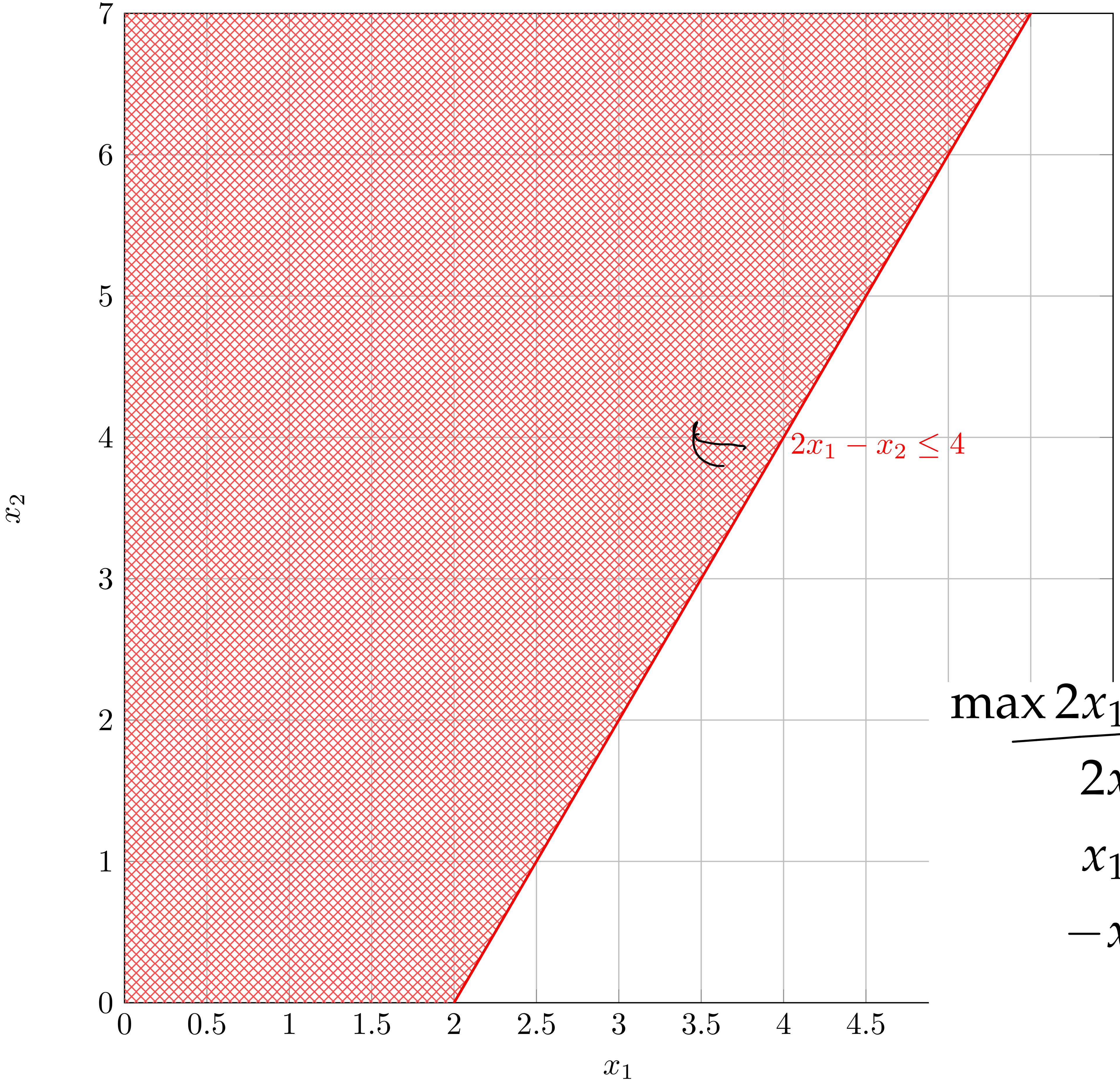
while a neighbor x' has a higher value of $c^T \cdot x' > c^T x$

do: move to that neighbor (if it has not been visited)

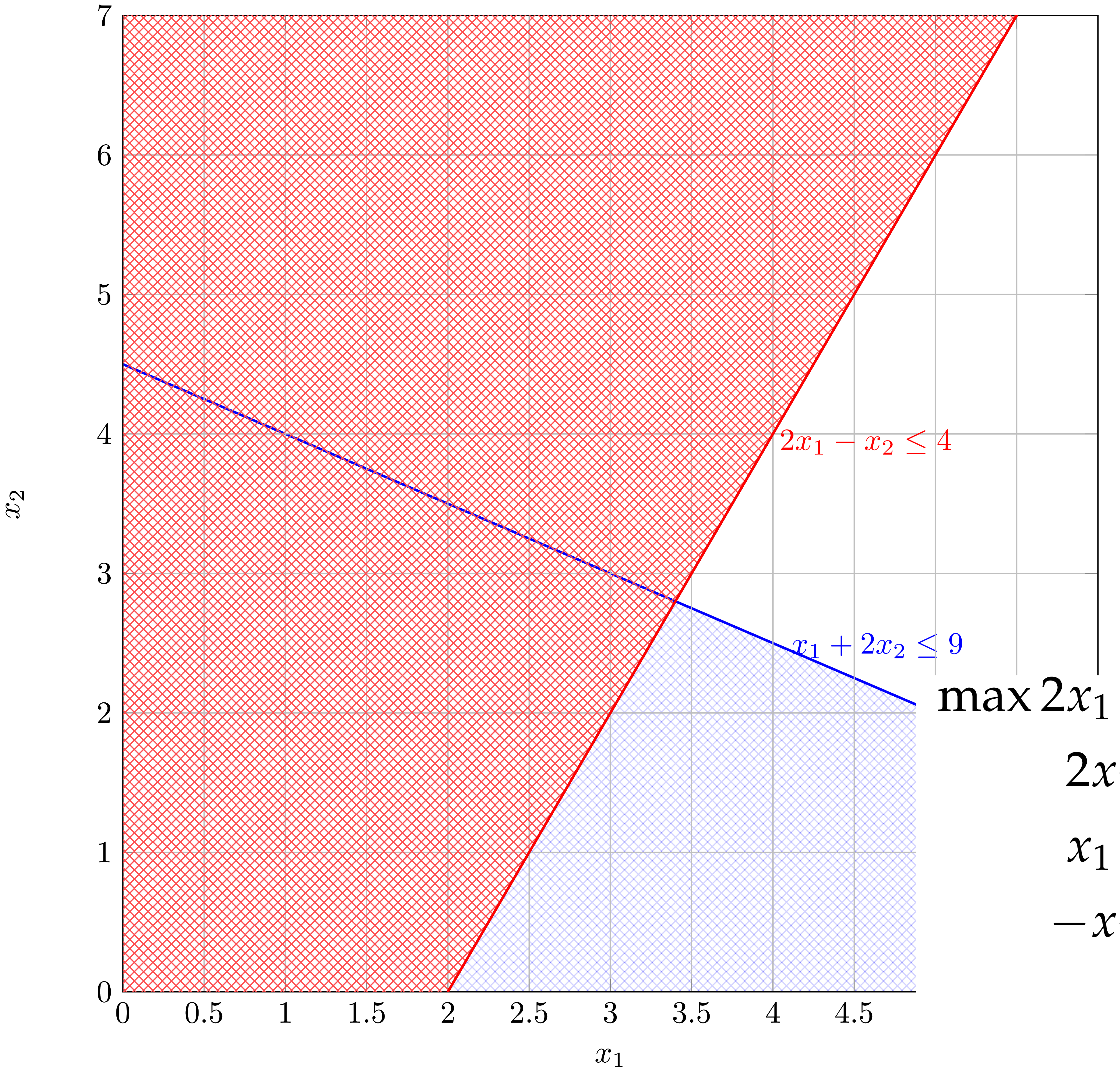
greedy ALG!!

works in practice

(exp. time in the worst case)



$$\max 2x_1 + 5x_2$$
$$2x_1 - x_2 \leq 4$$
$$x_1 + 2x_2 \leq 9$$
$$-x_1 + x_2 \leq 3$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$



$$\max 2x_1 + 5x_2$$

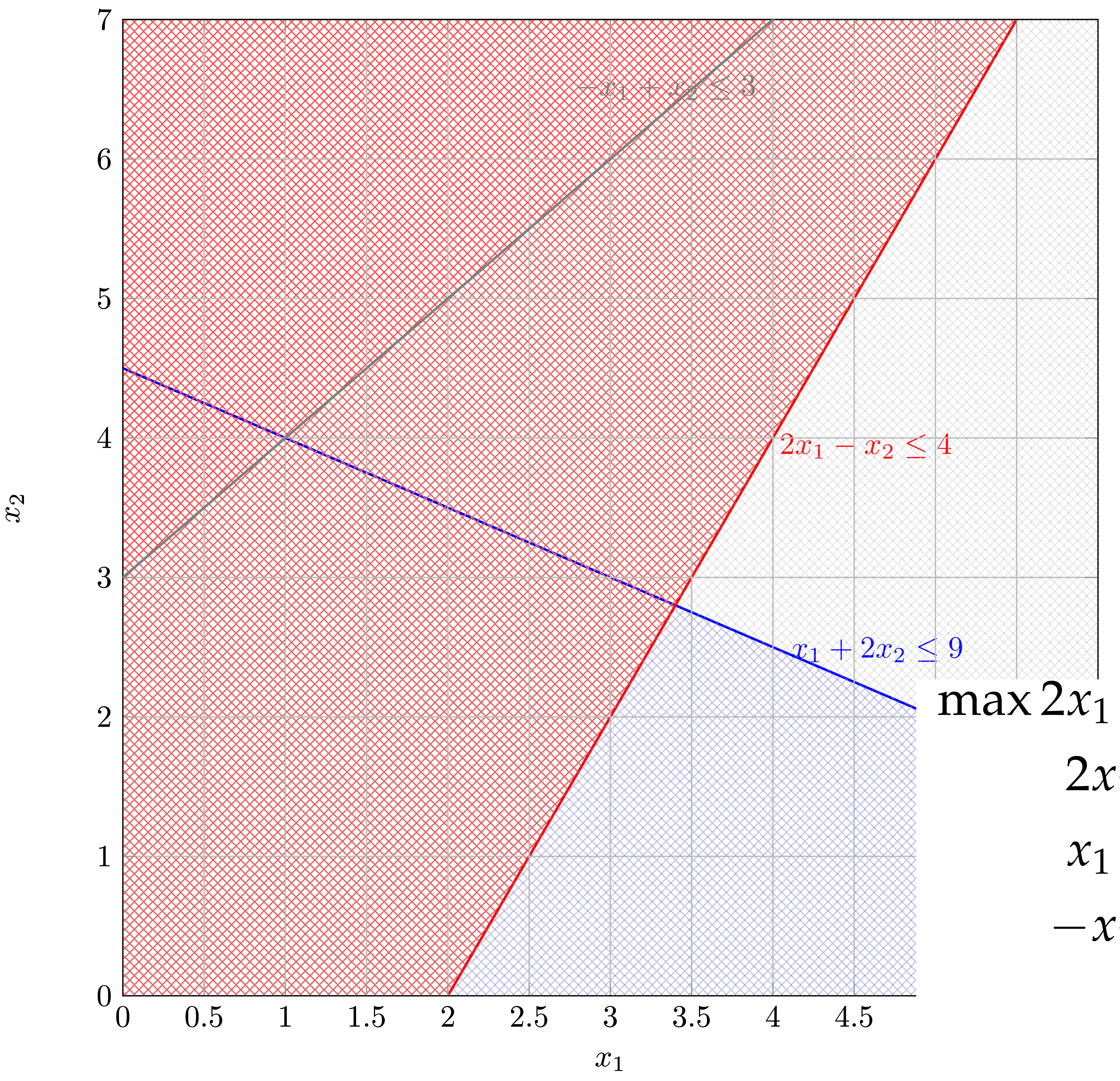
$$2x_1 - x_2 \leq 4$$

$$x_1 + 2x_2 \leq 9$$

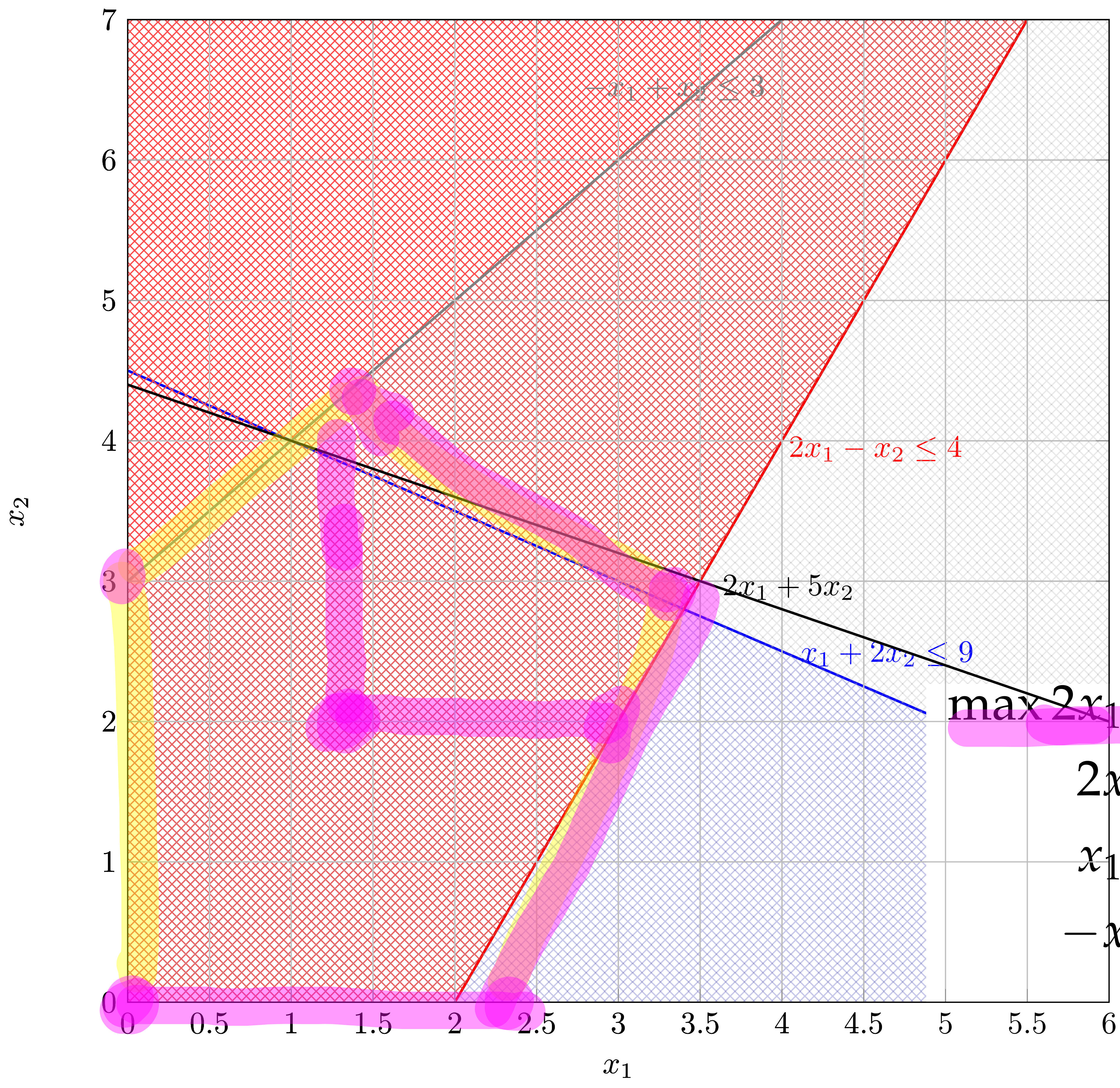
$$-x_1 + x_2 \leq 3$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$



$$
\begin{aligned}
\max 2x_1 + 5x_2 & \\
2x_1 - x_2 &\leq 4 \\
x_1 + 2x_2 &\leq 9 \\
-x_1 + x_2 &\leq 3 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}
$$



$$\max 2x_1 + 5x_2$$
$$2x_1 - x_2 \le 4$$
$$x_1 + 2x_2 \le 9$$
$$-x_1 + x_2 \le 3$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

n constraints in 2d,

$\Rightarrow O(n^2)$ vertices

---

n constraints $\frac{n}{2}$ variables.

$\Rightarrow O\binom{n}{\frac{n}{2}} \sim$ exp in n

Pivot

$$\max 2x_1 + 5x_2$$
$$2x_1 - x_2 \le 4$$
$$x_1 + 2x_2 \le 9$$
$$-x_1 + x_2 \le 3$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

$x_1 = 0$

$x_2 = 0$

x_2 can go to ∞

$x_2 \le 4.5$

$x_2 \le 3.$

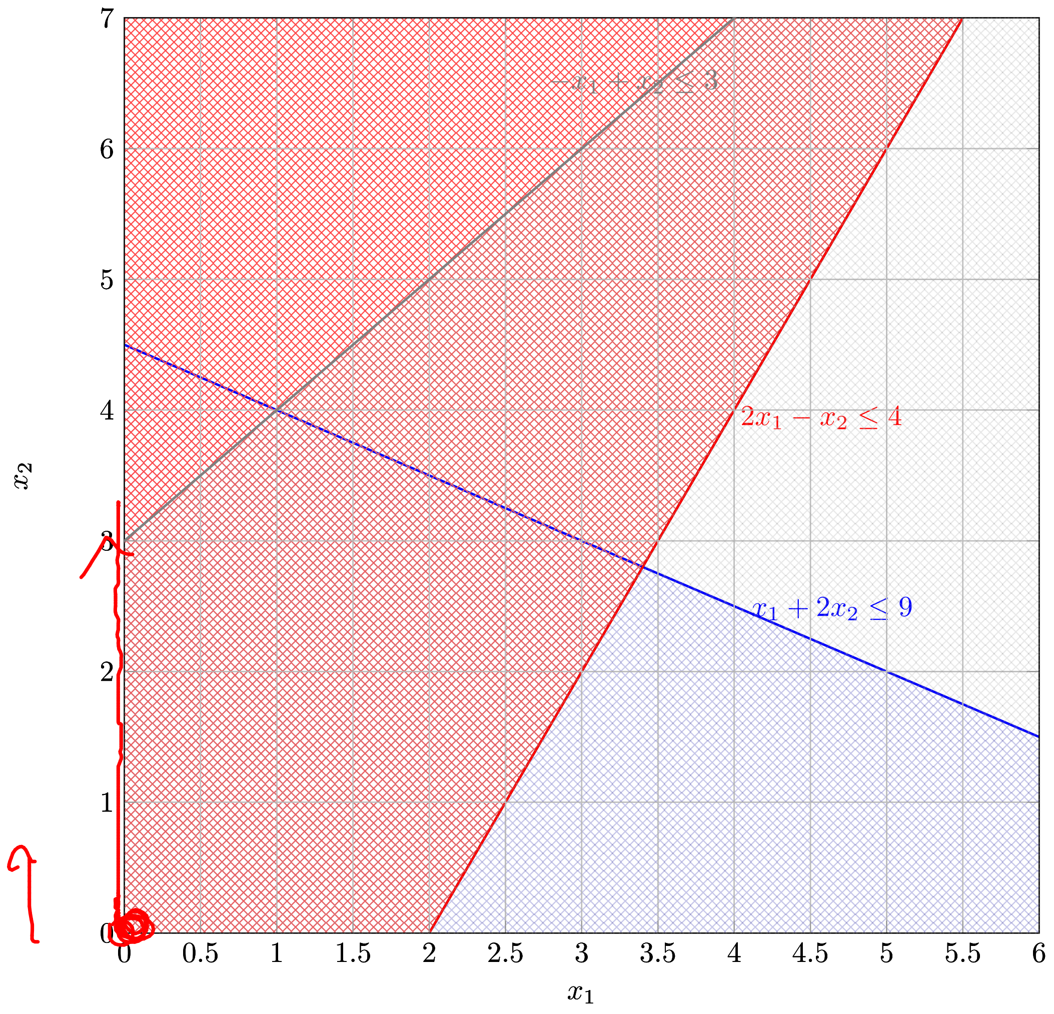
Change of variables.

$y_1 = x_1$

$y_2 = 3 + x_1 - x_2$

$x_2 = 3 + y_1 - y_2$

$$\max 2y_1 + 5(3 + y_1 - y_2)$$
$$2y_1 - (3 + y_1 - y_2) \le 4$$
$$y_1 + 2(3 + y_1 - y_2) \le 9$$
$$-y_1 + (3 + y_1 - y_2) \le 3$$
$$y_1 \ge 0$$
$$y_2 \ge 0$$



Pivot

$$\max 2x_1 + 5x_2$$

$$\begin{aligned} 2x_1 - x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 9 \\ -x_1 + x_2 &\leq 3 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

$$y_1 = x_1, y_2 = 3 + x_1 - x_2$$

pivot

$$\max 7y_1 - 5y_2 + 15$$
$$y_1 + y_2 \leq 7$$
$$3y_1 - 2y_2 \leq 3$$
$$y_2 \geq 0$$
$$y_1 \geq 0$$
$$-y_1 + y_2 \leq 3$$

(0, 0)

tightest constraint on y_1

$z_1 = 3 - 3y_1 + 2y_2$

$z_2 = y_2$

$$
\begin{aligned}
\max 7y_1 - 5y_2 + 15 \\
y_1 + y_2 &\leq 7 \\
3y_1 - 2y_2 &\leq 3 \\
y_2 &\geq 0 \\
y_1 &\geq 0 \\
-y_1 + y_2 &\leq 3
\end{aligned}
$$

$$z_1 = 3 - 3y_1 + 2y_2, z_2 = y_2$$

$$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2 \quad \rightarrow \quad \underline{\underline{22}}$$

$$-\frac{1}{3}z_2 + \frac{5}{3}z_2 \leq 6$$

$$z_1 \geq 0$$

$$z_2 \geq 0$$

$$\frac{1}{3}z_1 - \frac{2}{3}z_2 \leq 1$$

$$\frac{1}{3}z_1 + \frac{1}{3}z_2 \leq 4$$

$z_1 = 0$

$z_2 = 0$

$$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$$

$$-\frac{1}{3}z_2 + \frac{5}{3}z_2 \leq 6$$

$$z_1 \geq 0$$

$$z_2 \geq 0$$

$$\frac{1}{3}z_1 - \frac{2}{3}z_2 \leq 1$$

$$\frac{1}{3}z_1 + \frac{1}{3}z_2 \leq 4$$

$$z_1 = 3 - 3y_1 + 2y_2, z_2 = y_2$$

$$y_1 = x_1, y_2 = 3 + x_1 - x_2$$

Optimality

c

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4$$

$$x_1 + 2x_2 \leq 9 \qquad \tfrac{7}{3}$$

$$-x_1 + x_2 \leq 3 \qquad \tfrac{1}{3}$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$\Rightarrow \quad \tfrac{7}{3}x_1 + \tfrac{14}{3}x_2 \leq \tfrac{63}{3}$$

$$-\tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 \leq 1$$

$$\tfrac{6}{3}x_1 + \tfrac{15}{3}x_2 \leq 22$$

$$2x_1 + 5x_2 \leq 22$$

simplex

simple fact: origin is optimal if and only if c is negative.

problems

initial vertex

no solution?

run time

WE HAVE BEEN SOLVING PROBLEM A BY SOLVING SMALLER VERSIONS OF PROBLEM A

GENERAL IDEA:

SOLVE PROBLEM A BY SOLVING PROBLEM B

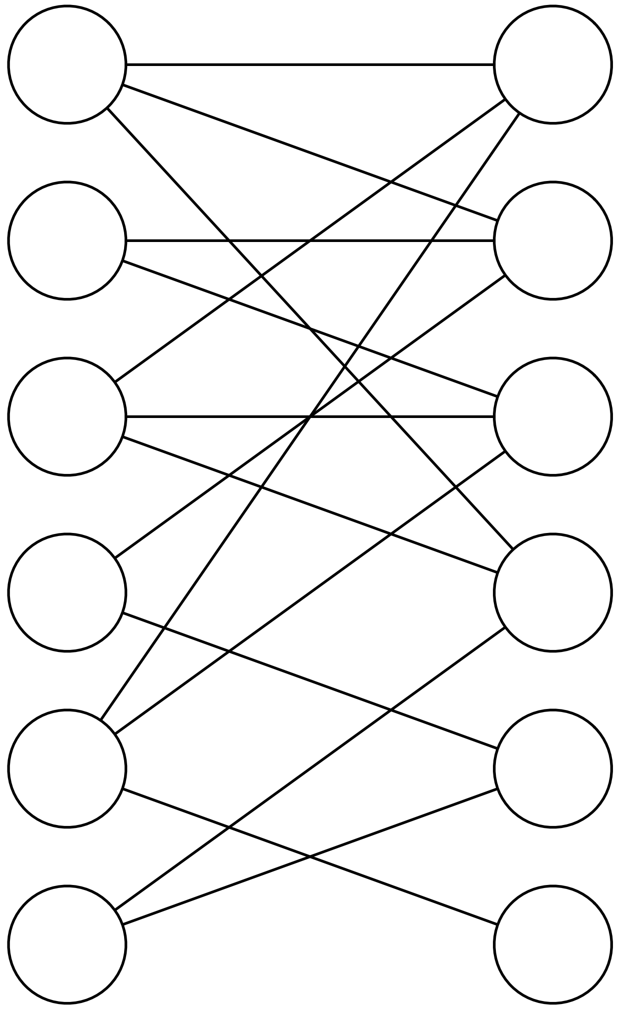
Bipartite Matching Algorithm

BP(L,R,E)

1. MAKE NEW G'
FROM INPUT G.

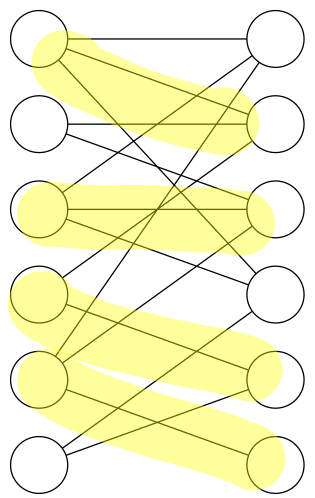
2. RUN FF ON G'

3. OUTPUT ALL MIDDLE EDGES
WITH FLOW F(E)=1.



Bipartite Matching Instance

(L,R,E)



transform

G'

Bipartite Matching Instance

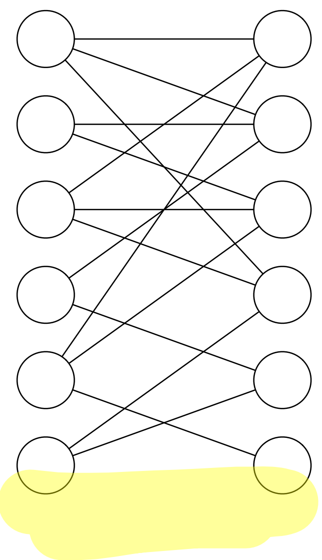
BP Algorithm

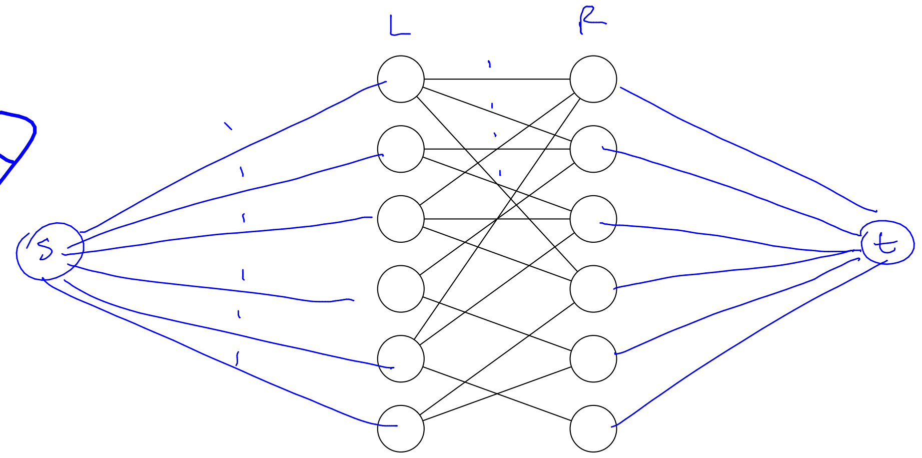
Max flow Instance

G'

G' has a MAXFLOW k

if LRE has a solution





matching

MAXFlow G'

f

matching of size k

size k

IF G HAS A MATCHING OF SIZE K, THEN $\Longrightarrow$ G' has a MAXFLOW of K.

Proof: (Outline) Let M^* be the matching of size K for G,

Construct flow f to be

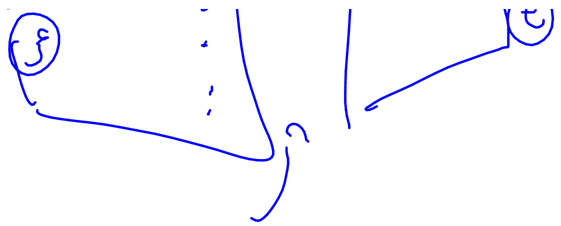
$f(e) = 1$ if $e \in M^*$, and if $e = (x, y)$ then $f(s, x) = 1$, $f(y, t) = 1$

$\Rightarrow$ flow f satisfying ① capacity constraint

② flow constraint

INFLOW(x) = OUTflow(x)

$|f| = K.$



G' $\Longleftarrow$

' HAS A FLOW OF K, THEN G HAS K-MATCHING.

① Our algorithm uses FF, so flow for G' is integral.

Now define $M: \{ e \mid f(e)=1$ and $e=(x,y)$ s.t. $x \in L$ $y \in R \}$

Prove that M is a matching.

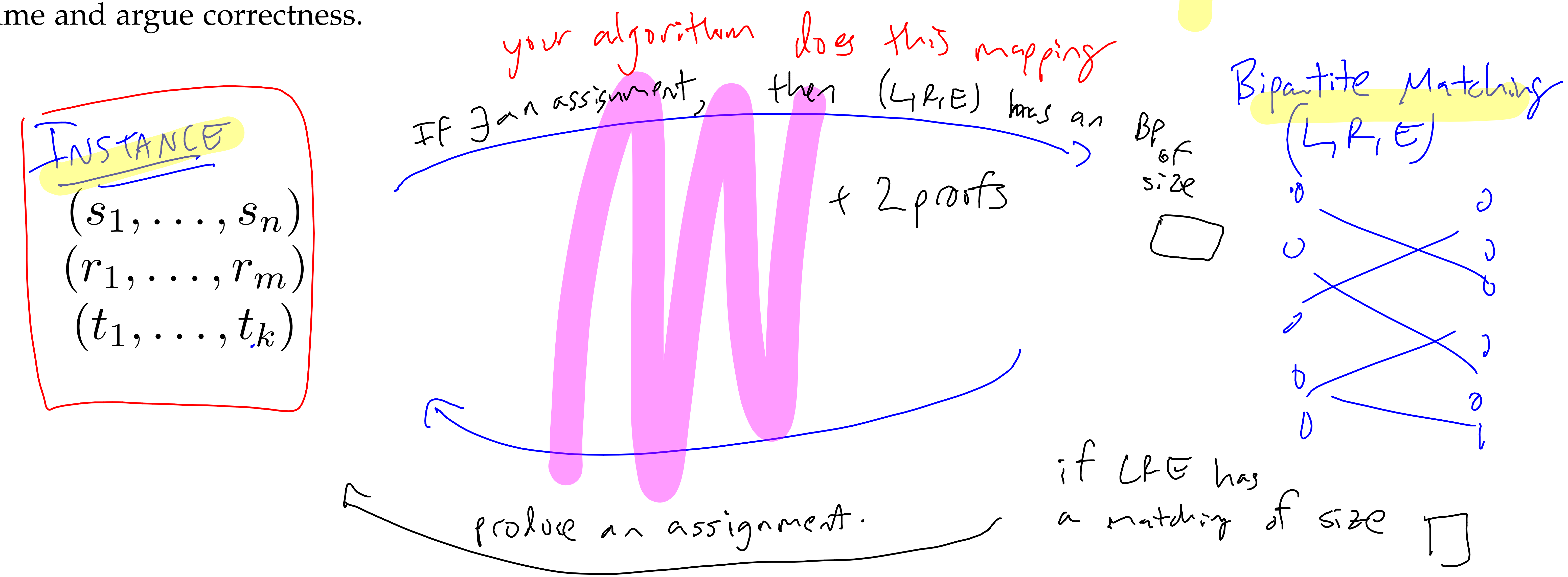
- All flows are integral & capacity $c(e)=1$. so $f(e) = 0$ or 1 for $e \in E$.

Thus for all $v \in L$, ($v \in R$) v is incident to at most 1 edge in M.

By the flow constraint. By (MIN.) cuts, $|M| = K$

PROBLEM 2 *Classrooms*

Before the start of the Spring semester, the Registrar must assign each class to a time and a classroom. The classroom must be larger than the class it holds to properly seat all the students. Suppose there are n classes such that class i has s_i students enrolled. The university has m rooms, and room j can hold r_j students. Finally, there are non-overlapping time slots $t_1, \ldots, t_k$ for the classes. For example t_1 is "MW9-10.15" and t_2 is "MW10.30-11.45" and so on. Given all this data, namely, given $(s_1, \ldots, s_n), (r_1, \ldots, r_m), (t_1, \ldots, t_k)$, design an efficient algorithm that assigns classes to times and classrooms. Analyze the running time and argue correctness.



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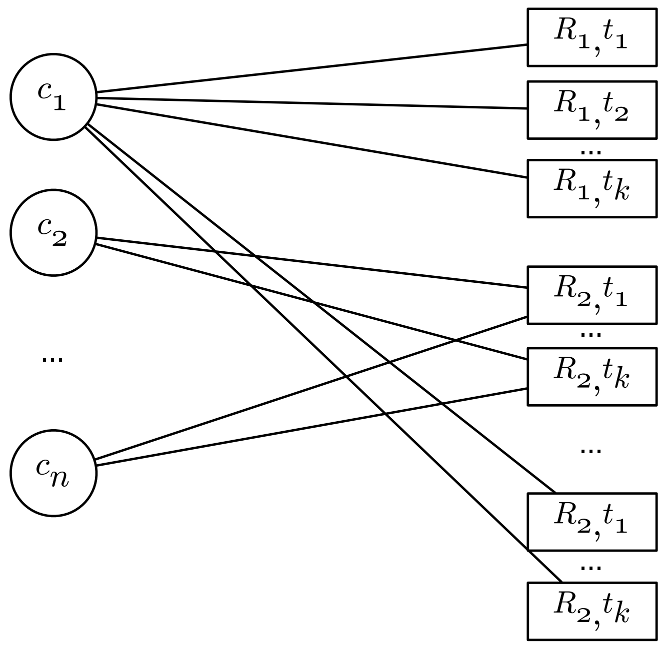
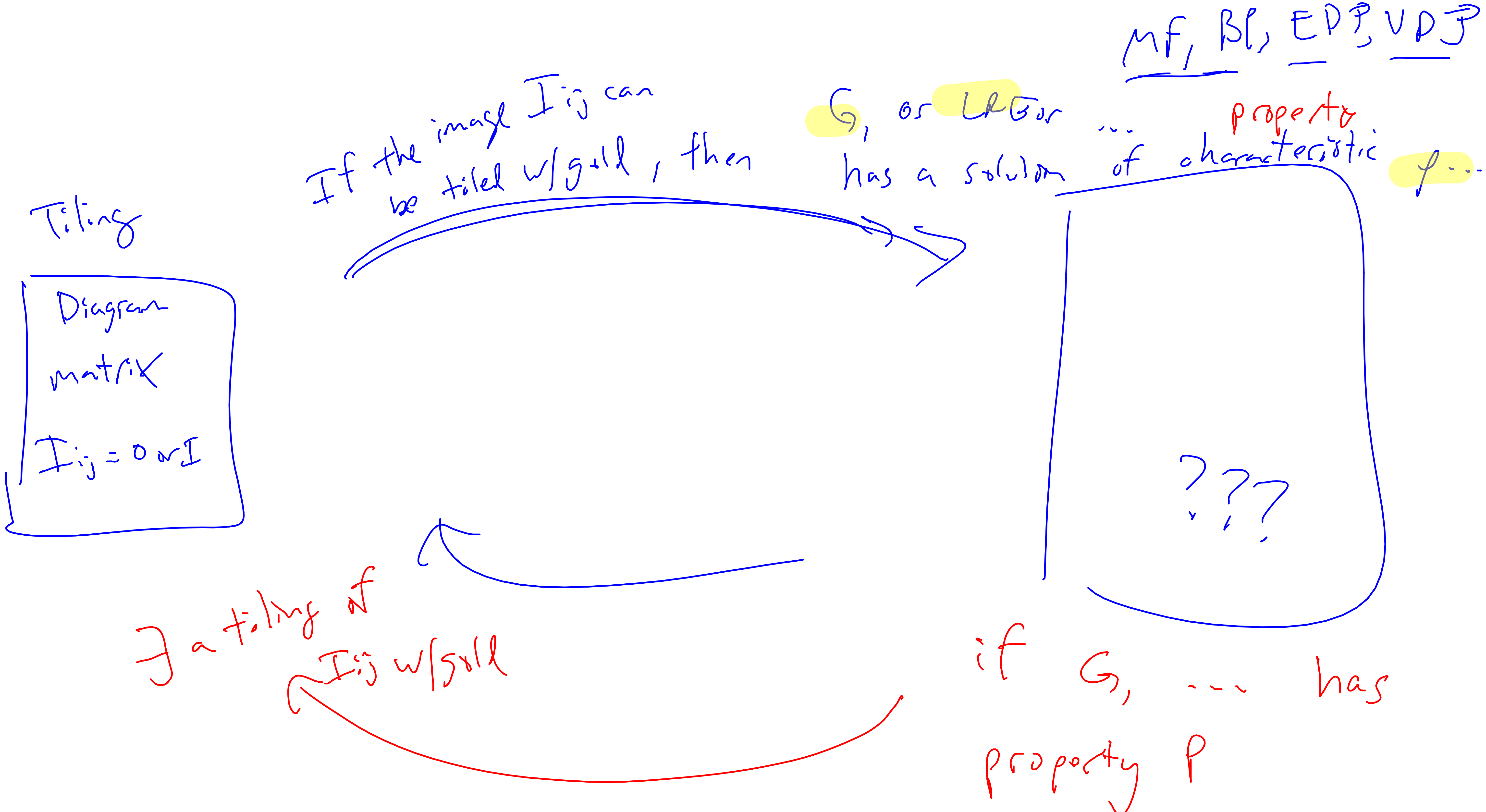
$$(s_1, \ldots, s_n)$$
$$(r_1, \ldots, r_m)$$
$$(t_1, \ldots, t_k)$$



Tiling



Tiling
Diagram
matrix
I_{ij} = 0 or I
If the image I_{ij} can be tiled w/ g-ll, then
G, or LRG or ... has a solution of characteristic property p...
MF, BL, EDP, VDP
???
if G, ... has property P
∃ a tiling of I_{ij} w/ s-ll

Reduction

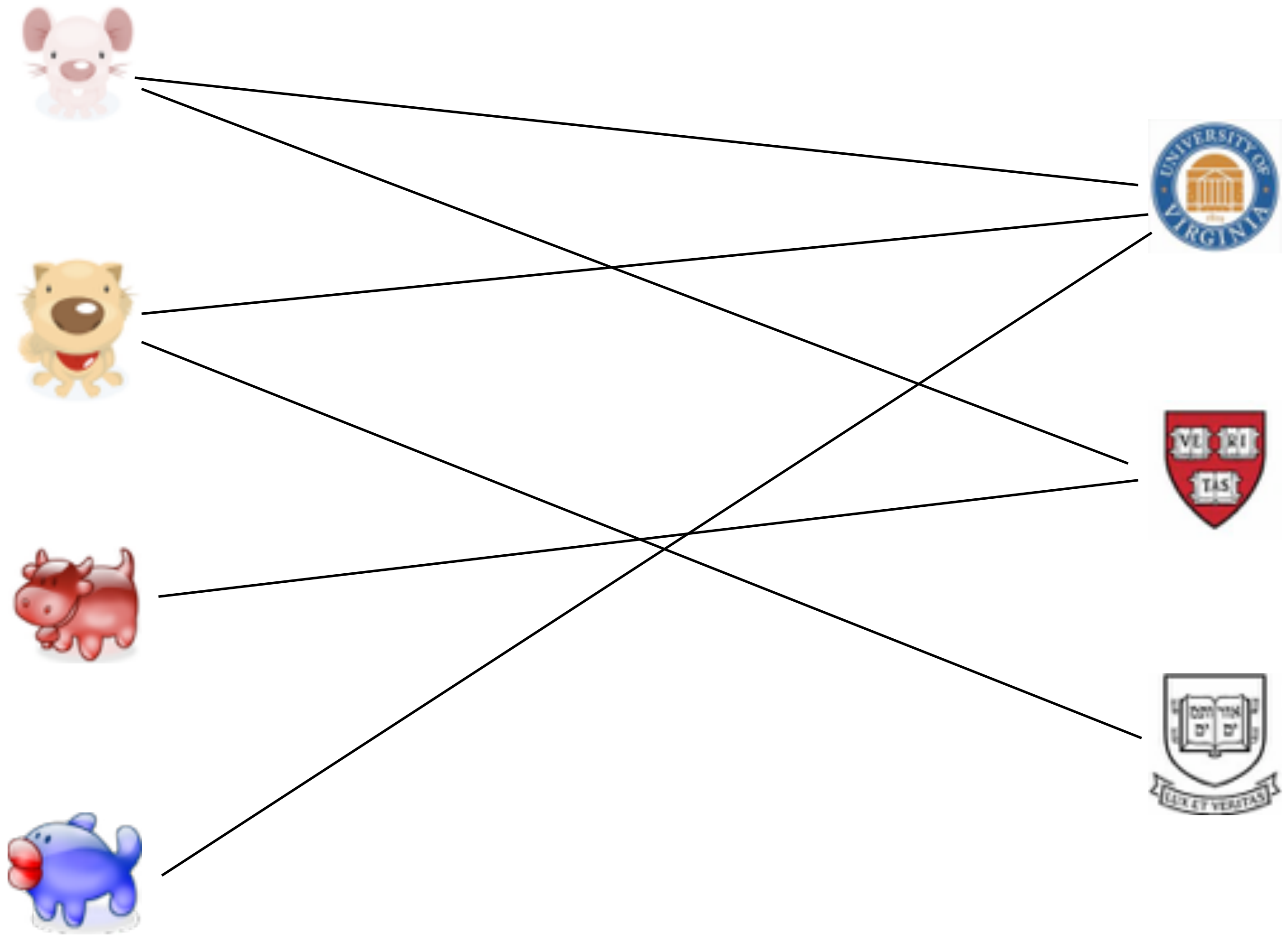
$$\text{PROBLEM}_a \leq_{f(n)} \text{PROBLEM}_b$$

$$\text{PROBLEM}_a \leq_{f(n)} \text{PROBLEM}_b$$

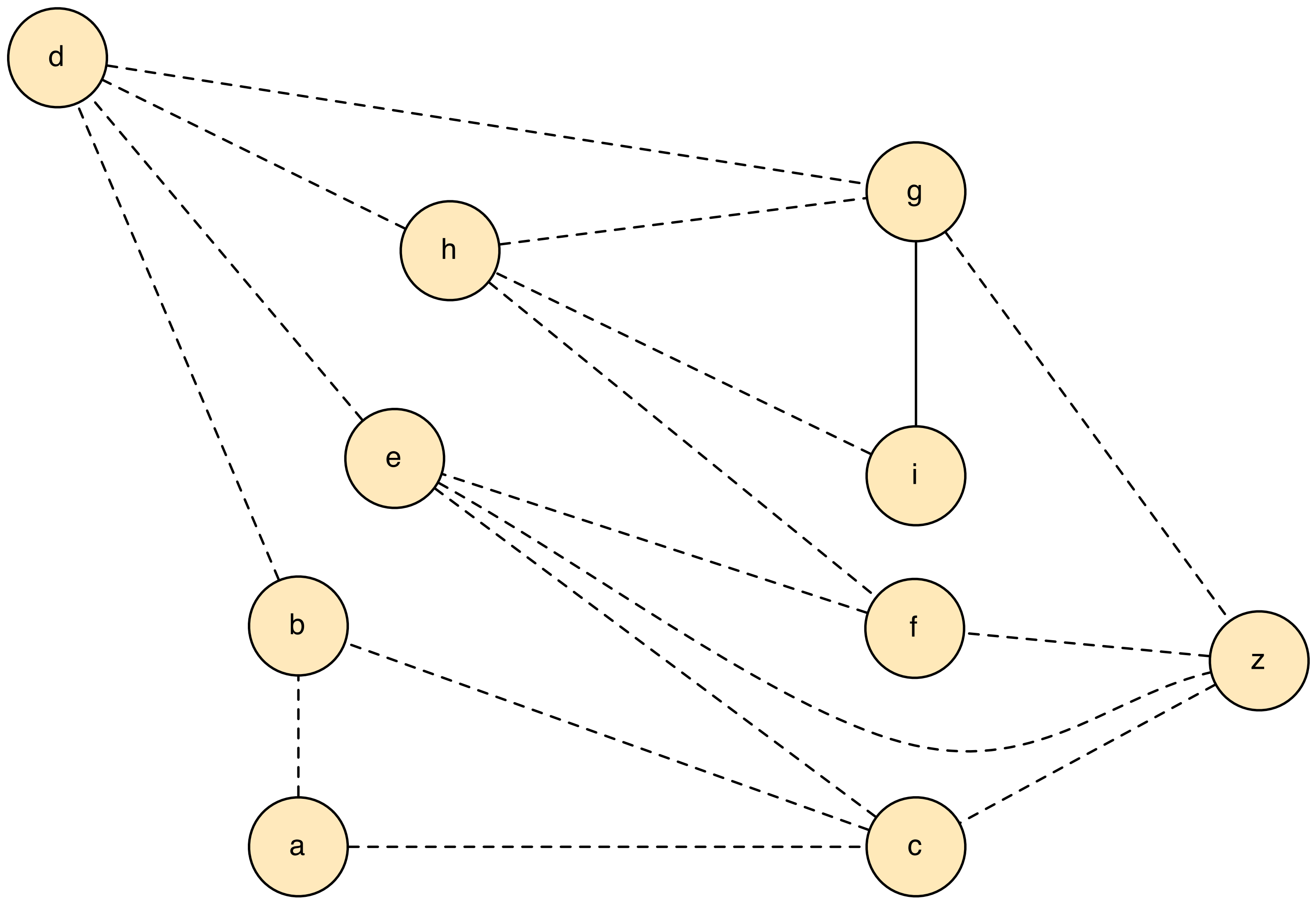
∃c、d

$$T(\text{PROBLEM}_a(n)) \leq f(n) + cT(\text{PROBLEM}_b(dn))$$

Maximum bipartite



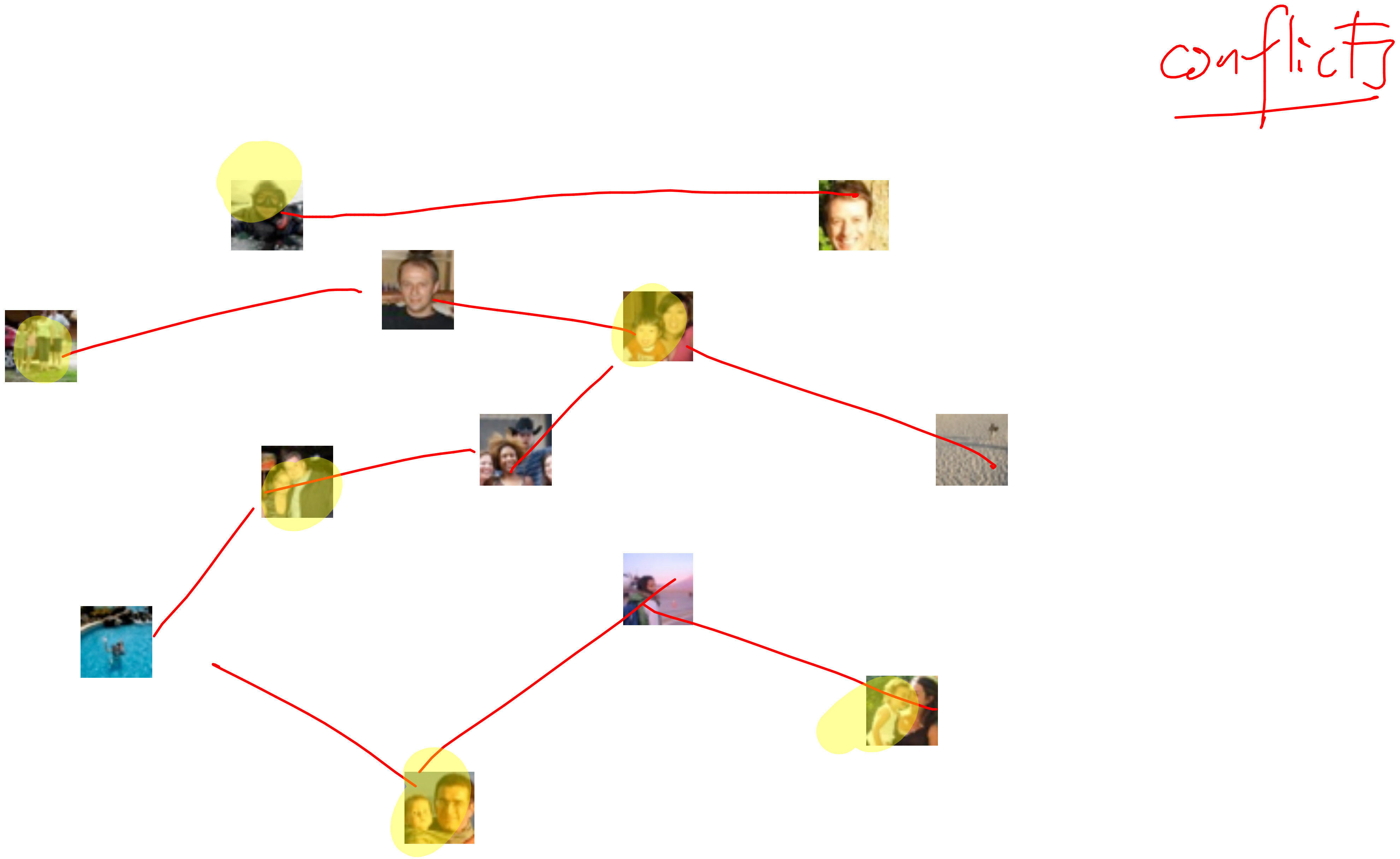
edge-disjoint paths



$\text{MAXBIPARTITE} <_{E+V} \text{MAXFLOW}$

$\text{MAXEDGEDISJ} <_{E+V} \text{MAXFLOW}$

party problem

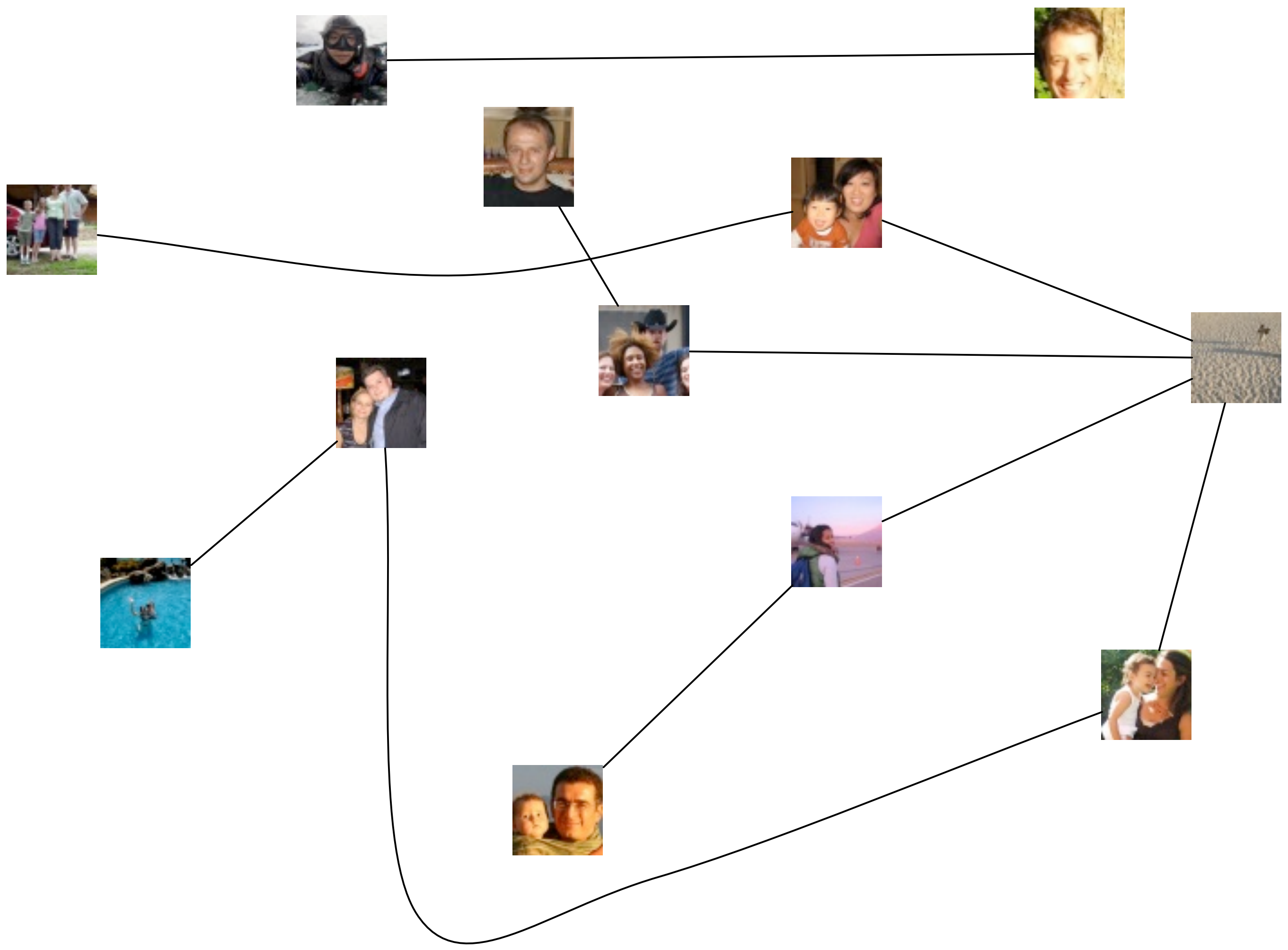


independent set

independent set

a set $S \subseteq V$ is an independent set if
no two nodes in S are joined by an edge.

example

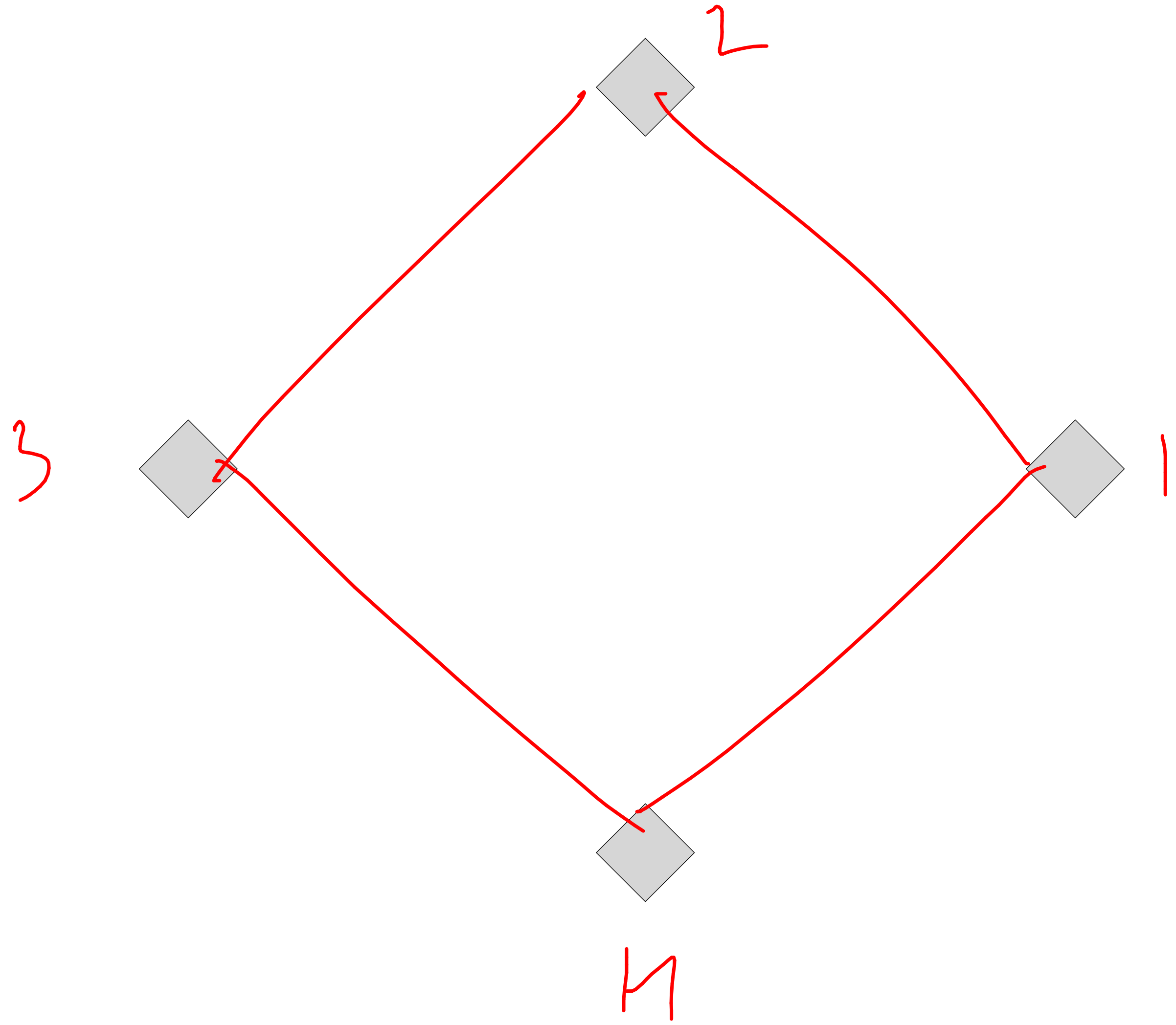


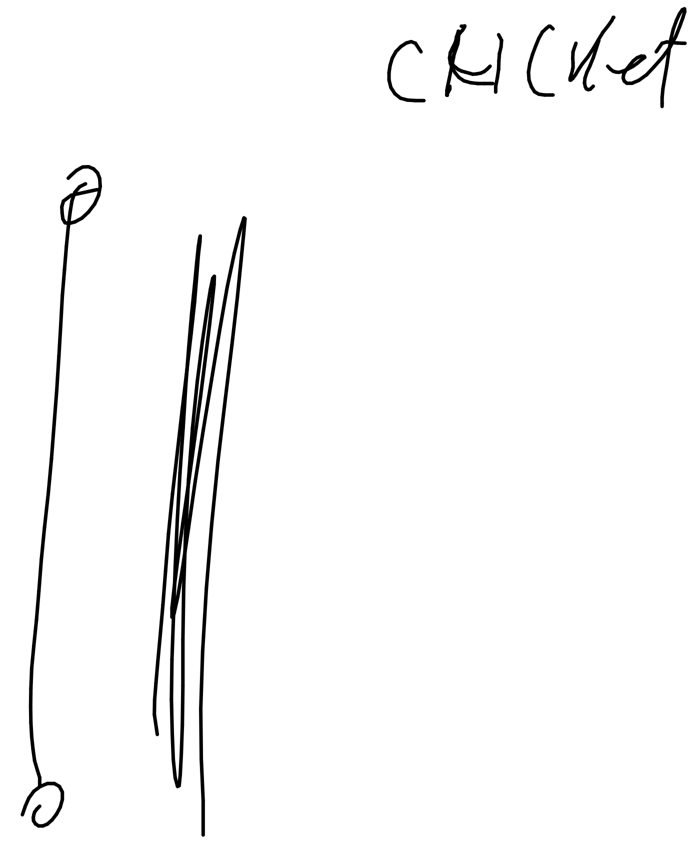
goal:

given a graph G, MAX INDEPENDENT SET,

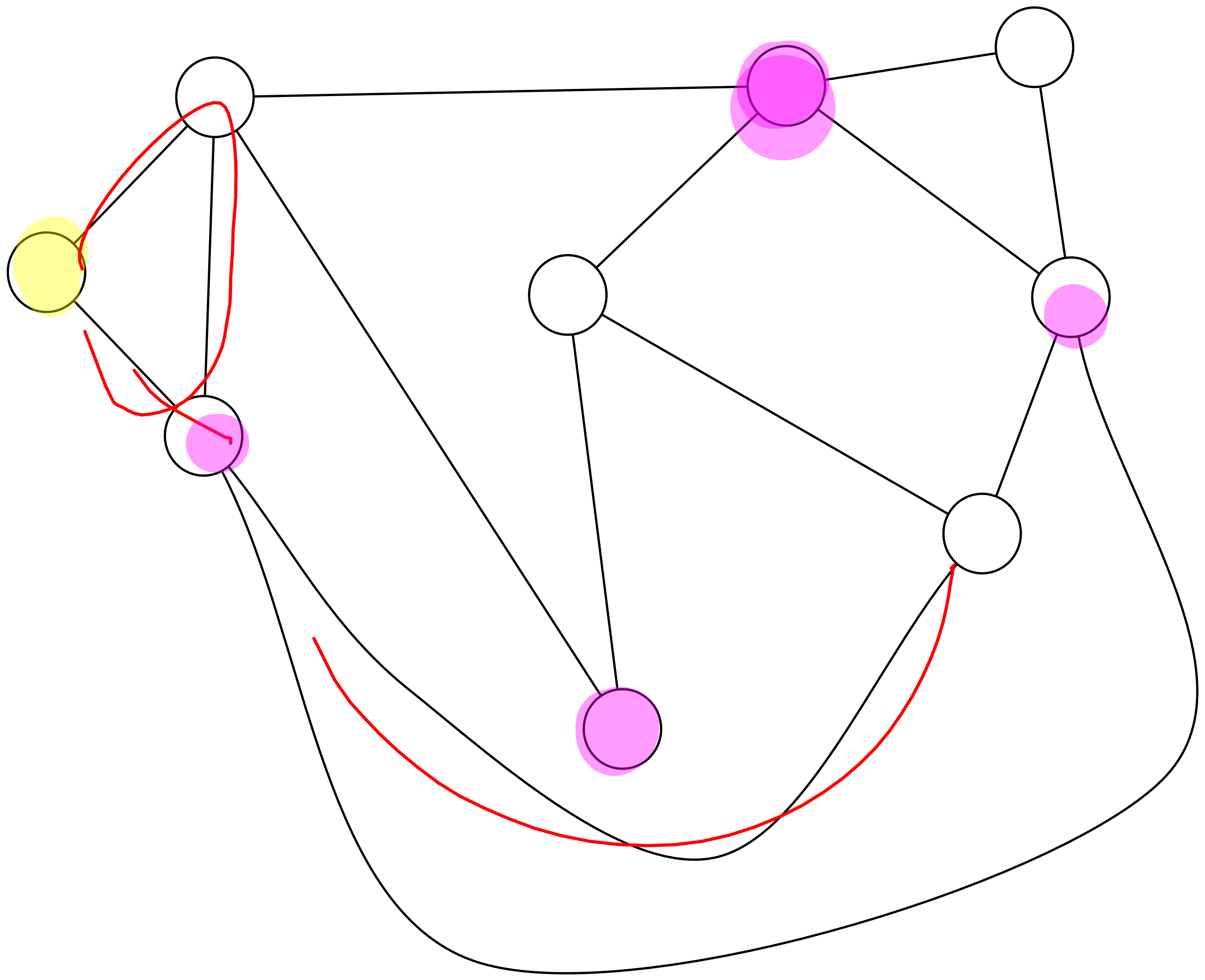
"most people you can sit @ your wedding table"

baseball





baseball



a vertex cover of a graph is a set of nodes S such that

for each edge $e=(x,y)$ either

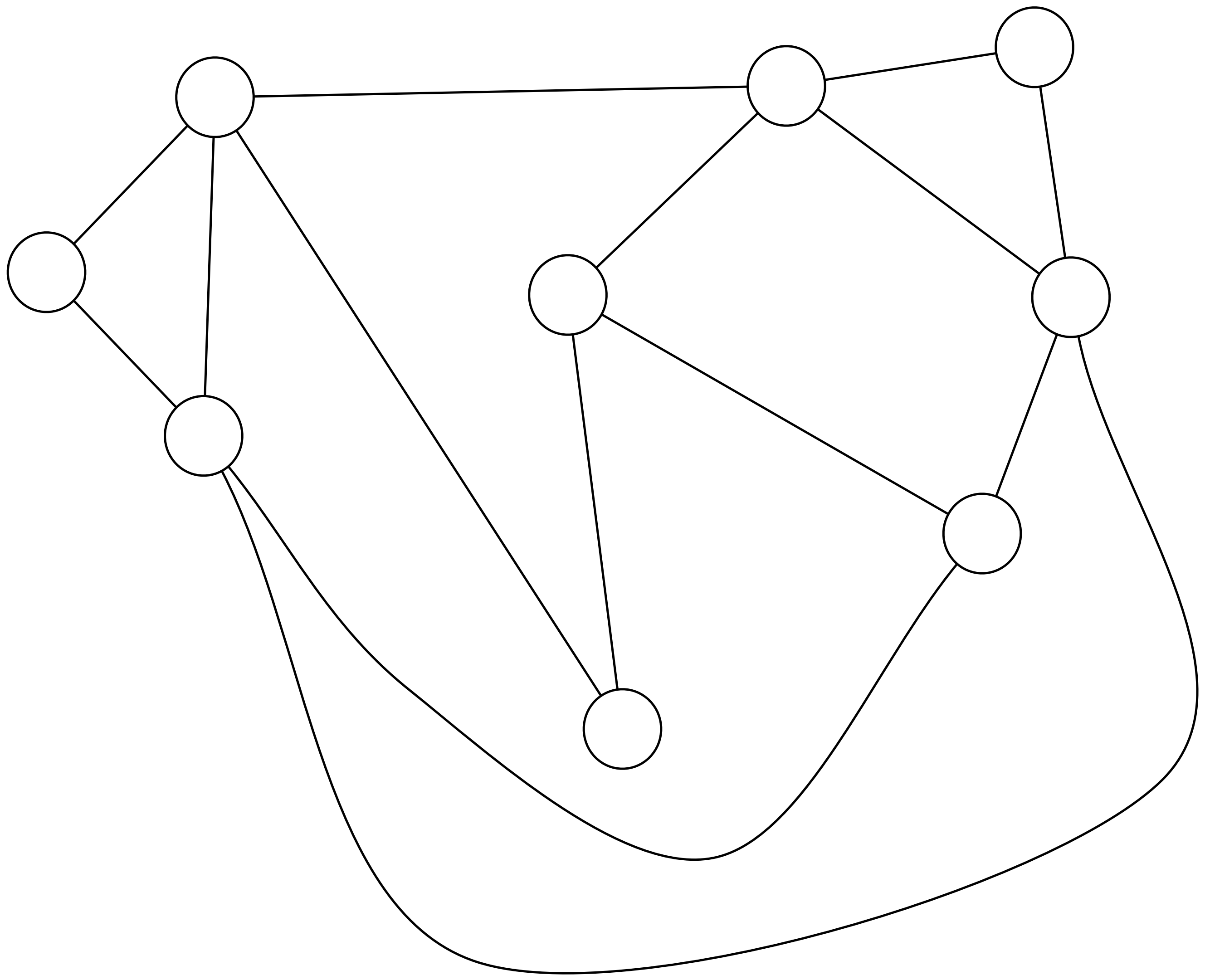
$x \in S$ or $y \in S$.

a vertex cover of a graph is a set $C \subseteq V$

such that $\forall \, (x, y) \in E$

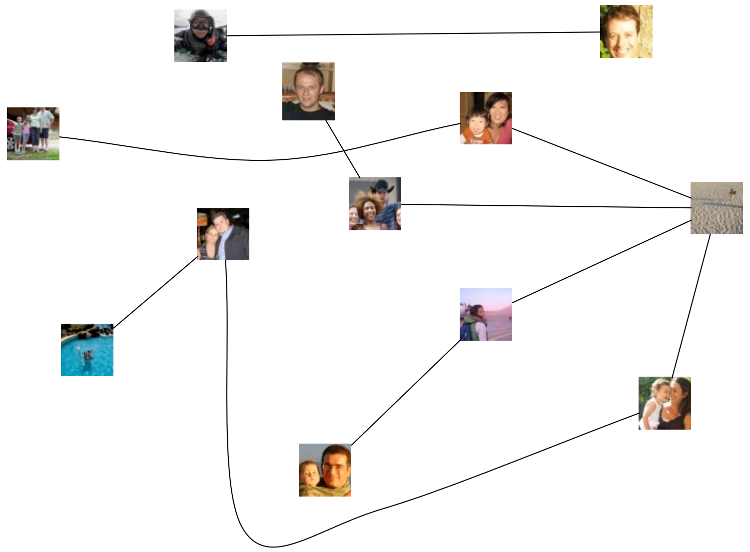
either $x \in C$ or $y \in C$

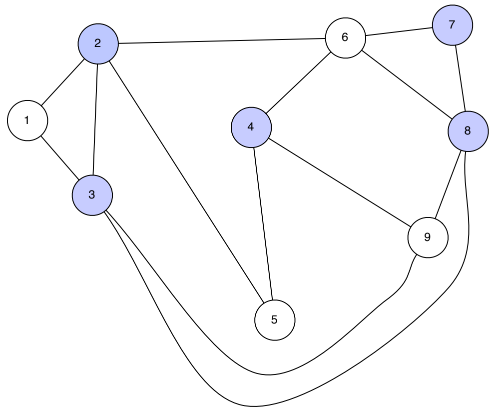
example



goal:

given a graph G,

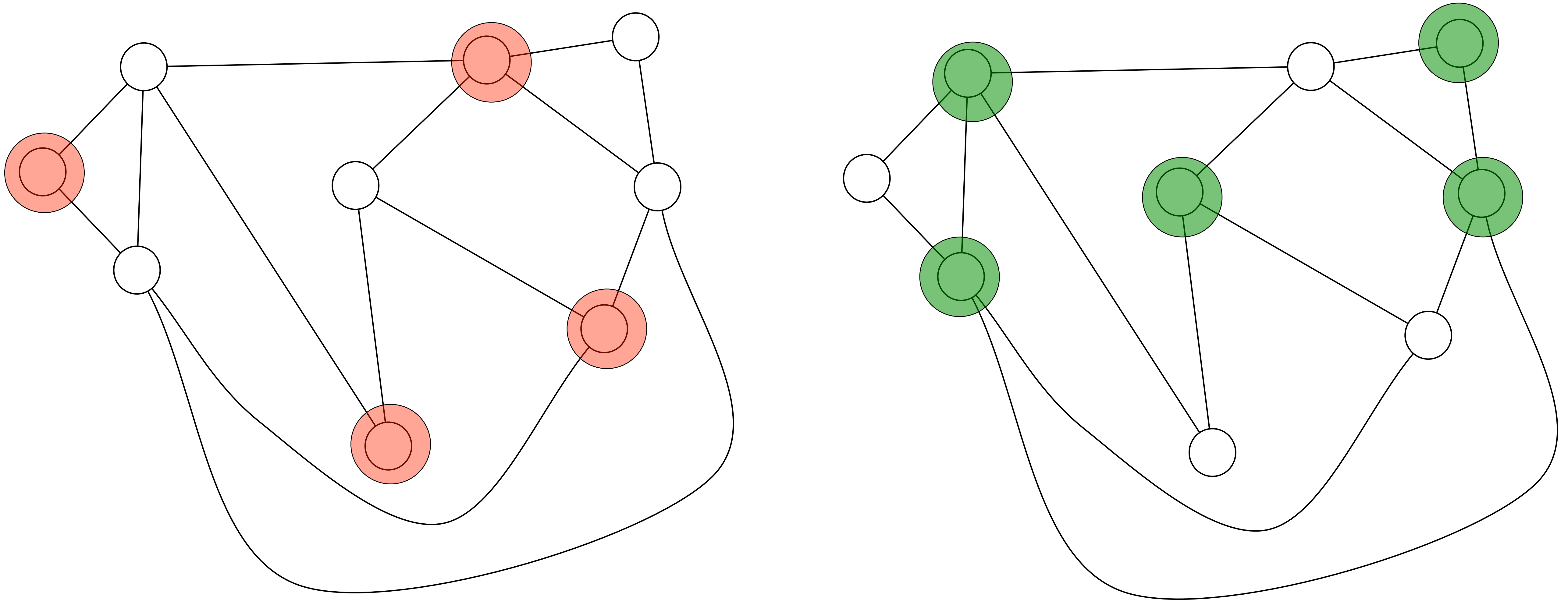




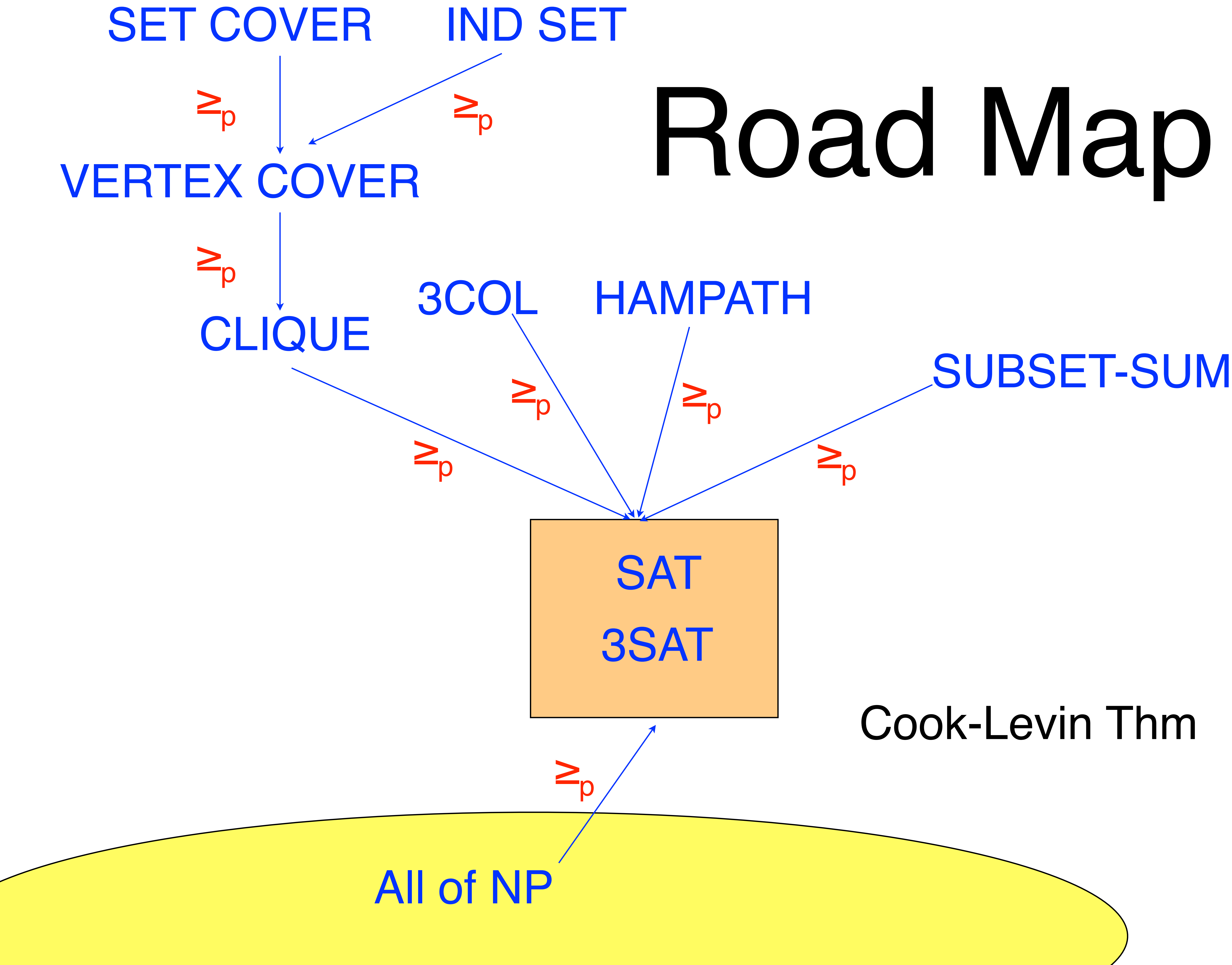
$$\text{MAXINDSET} \leq_{O(V)} \text{MINVERTEXCOVER}$$

A solution to VC can be used to solve INDSET.

thm: set S is an independent set of G iff V-S is a vertex cover.



Road Map


SET COVER
IND SET
≥p
≥p
VERTEX COVER
≥p
CLIQUE
3COL
HAMPATH
SUBSET-SUM
≥p
≥p
≥p
≥p
SAT
3SAT
Cook-Levin Thm
≥p
All of NP