L3 4102

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@cs4102

5-7pm 512

shelat

your id:

Solve for x: $\boxed{\left(\frac{14}{9}\right)}^{100} = 2^x$

$$\left(2^{\left(\log \frac{14}{9}\right)}\right)^{100} = 2^x$$

$$100 \log_2\left(\frac{14}{9}\right) = x \cdot \log_2 2$$

$$\Rightarrow x = 100 \cdot \log_2\left(\frac{14}{9}\right)$$

$$2^{\log_2\left(\frac{14}{9}\right)} = \left(\frac{14}{9}\right)$$

Karatsuba algorithm

n-digit mult.

$\overbrace{1\ 7}^{a}\ \overbrace{8\ 9}^{b}\ \bigstar\ \overbrace{1\ 4}^{c}\ \overbrace{3\ 2}^{d}$

$$ac \cdot (100)^2 + (ad+bc)\,100 + bd$$

1. ac
2. bd
3. $(a+b)(c+d) = ac + (ad+bc) + bd$
4. ③ − ① − ② $= (ad+bc)$

Karatsuba(ab, cd)

Base case: return b*d if inputs are 1-digit

ac = Karatsuba(a,c)

bd = Karatsuba(b,d)

t = Karatsuba((a+b),(c+d)) $\left(\frac{n}{2}+1\right)$

$3T(n/2) + 2n$

mid = t - ac - bd

$4n$

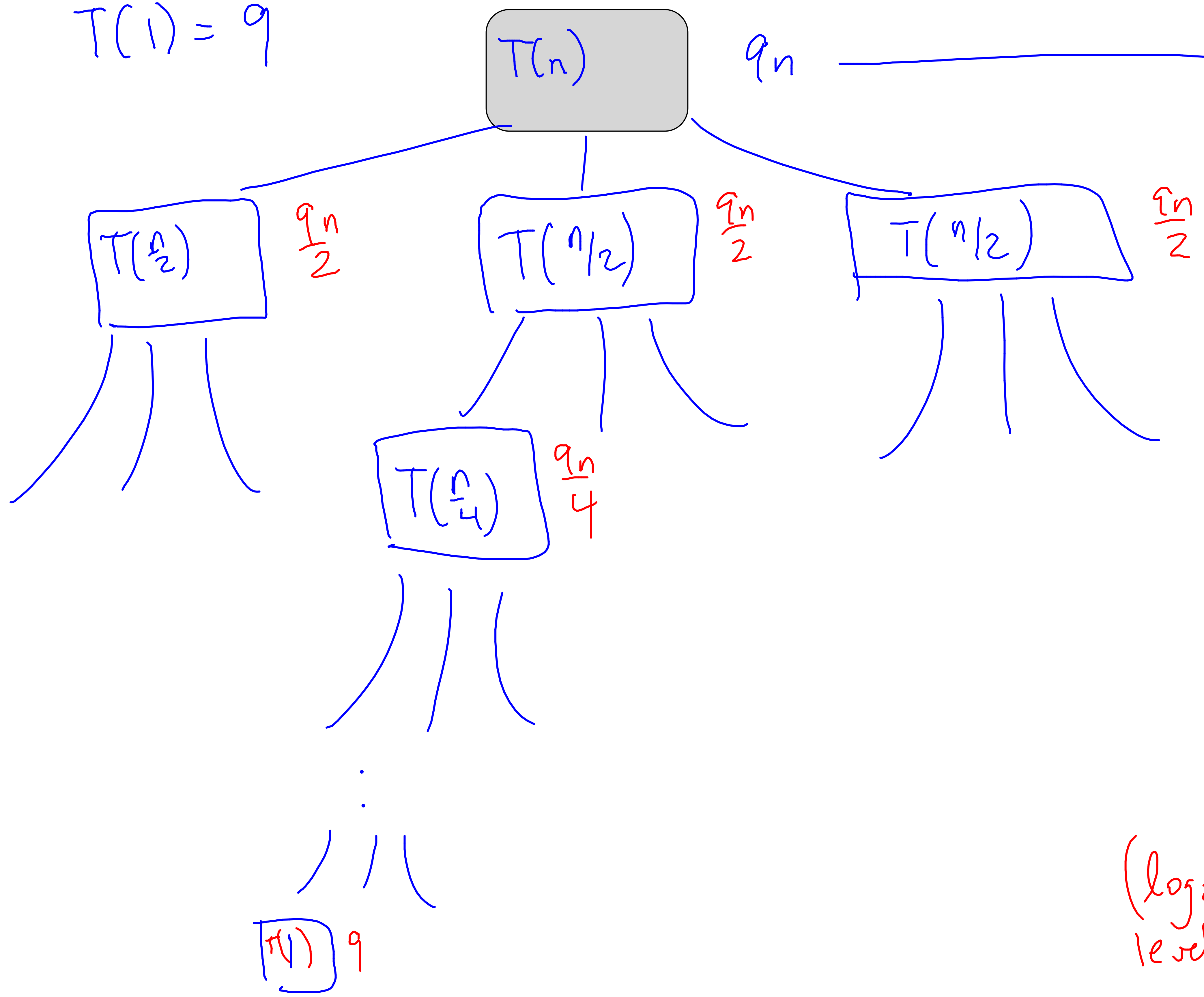
RETURN ac*100^2 + mid*100 + bd

$3n$

$T(n) = 3T\left(\frac{n}{2}\right) + 9n$

$$T(n) = 3T(n/2) + \underline{9n}$$

$T(1) = 9$



Lets add all of these

$\rightarrow 9n$

$\rightarrow 3 \cdot \frac{9n}{2}$

$\rightarrow 9 \cdot \frac{9n}{4}$

$(\log_2 n)$ levels

$\rightarrow 3^{\log_2 n} \cdot \frac{9n}{2^{\log n}}$

calculations:

$$T(n) = 9n + 3\cdot\frac{9n}{2} + 3^2\cdot\frac{9n}{4} + 3^3\cdot\frac{9n}{8} + \cdots + 3^{\log_2 n}\cdot\frac{9n}{2^{\log_2 n}}$$

$$= 9n\cdot\sum_{i=0}^{\log_2 n}\left(\frac{3}{2}\right)^i = 9n\cdot\left[\frac{\left(\frac{3}{2}\right)^{\log_2 n+1}-1}{3/2-1}\right] = 2\cdot 9n\left[\left(\frac{3}{2}\right)^{\log n+1}\right]-18n$$

$$= 2\cdot 9n\left[2^{\log_2\left(\frac{3}{2}\right)}\right]^{\log_2 n+1} - 18n = 18n\left[2^{(\log_2 3-1)}\right]^{(\log_2 n+1)} - 18n$$

$$= 18n\left[\left(2^{\log_2(n)\cdot\log_2(3) - \log_2(n) + \log_2(3) - 1}\right)\right] - 18n$$

$$= 18\cancel{n}\cdot\left[\frac{n^{\log_2 3}\cdot 2^{\log_2 3-1}}{\cancel{n}}\right] - 18n = \left(18\cdot 2^{\log_2 3-1}\right)\cdot n^{\log_2 3} - 18n$$

$$= O(n^{\log_2 3})$$

$$\left(\frac{3}{2}\right)^{\log_2 n + 1} = \left[2^{\log_2\left(\frac{3}{2}\right)}\right]^{(\log_2(n)+1)}$$

$$\left(\frac{3}{2}\right) = \underbrace{2^{\log_2\left(\frac{3}{2}\right)}}$$

$$\left[2^{\log_2(3) - \underbrace{\log_2(2)}^{=1}}\right]$$

$$= \left[2^{\log_2(3) - 1}\right]$$

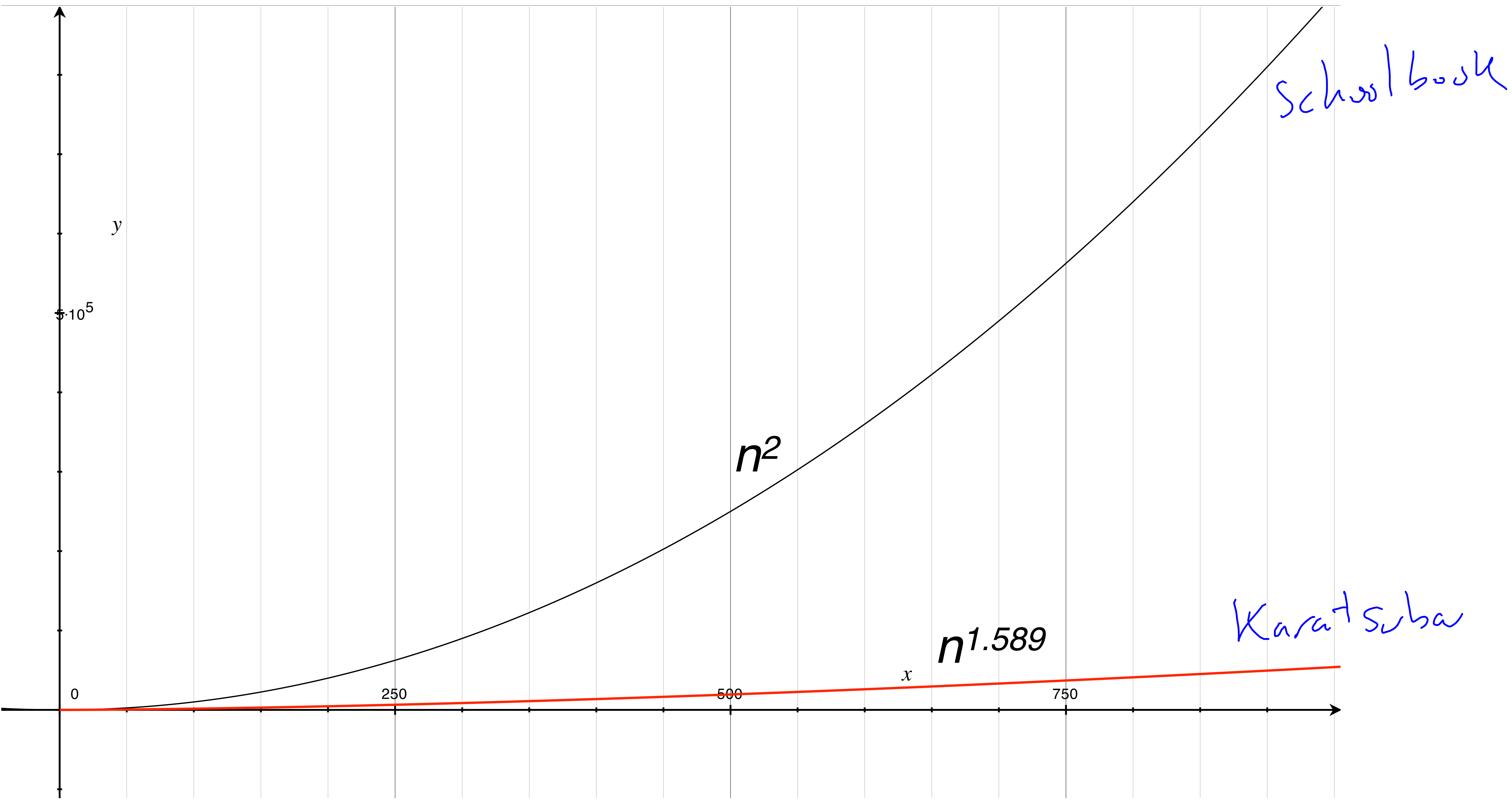
calculations:

$$T(n) = 3T(n/2) + 9n$$

$$O(n^{\log_2(3)})$$

$$T(n) = 3T(n/2) + 9n$$

$$\underline{O(n^{\log_2(3)})} \qquad O(n^{\underline{1.589}})$$


Schoolbook
Karatsuba
n^2
$n^{1.589}$
y
x
$5 \cdot 10^5$
0
250
500
750

1	7	8	9	★	1	4	3	2
a		b			c		d	

$$T(n) = 3T(n/2) + \overset{100n}{\underline{9n}}$$

$$T(n) = 4T(n/2) + \underline{3n}$$

simpler proof technique?

1

induction redux

classic goal: prove that some property P(k) is true for all k

$\forall k, P(k)$ holds

❶ one long proof...

classic goal: prove that some property P(k) is true for all k

$$\forall k, P(k) \text{ holds}$$

❶ Induction

classic
base case: $P(1)$

classic
inductive
step:

$$\left.\begin{array}{c} P(1) \\ \cdots \\ P(k) \end{array}\right\} \text{ implies } P(k+1) \text{ true}$$

base
cases.

❷ induction redux asymptotic style

base case: $P(n^*)$

inductive step: $\left.\begin{matrix} P(n^*) \\ \dots \\ P(k) \end{matrix}\right\}$ implies $P(k+1)$ true

simpler proof (guess +chk)

$$T(n) = 3T(n/2) + 9n$$

Prove: $T(n) < 3000 \cdot n^{1.6}$ $\Rightarrow O(n^{1.6})$

Notice that for small $n = 1, 2,$ $T(n) < 3000 \cdot n^{1.6}$.

Suppose that $T(n) < 3000 \cdot n^{1.6}$ for all $n < n_0$.

Consider $T(n_0+1)$, we would like to show $T(n_0+1) < 3000(n_0+1)^{1.6}$

$$T(n_0+1) = 3T\left(\frac{n_0+1}{2}\right) + 9(n_0+1)$$

$$< 3\left(\frac{n_0+1}{2}\right)^{1.6} \cdot 3000 + 9(n_0+1)$$

$$< (0.997)(3000)(n_0+1)^{1.6} + 9(n_0+1)$$

is negative b/c $9(n_0+1)^{1.6} > 9(n_0+1)$

$$< 3000(n_0+1)^{1.6} + \boxed{9(n_0+1) - (0.003)(3000)(n_0+1)^{1.6}}$$

$$< 3000(n_0+1)^{1.6}$$

simpler proof

mergesort

goal: Sort the array

technique:

6	4	9	12	2	5	8	7

mergesort

6	4	9	12	2	5	8	7

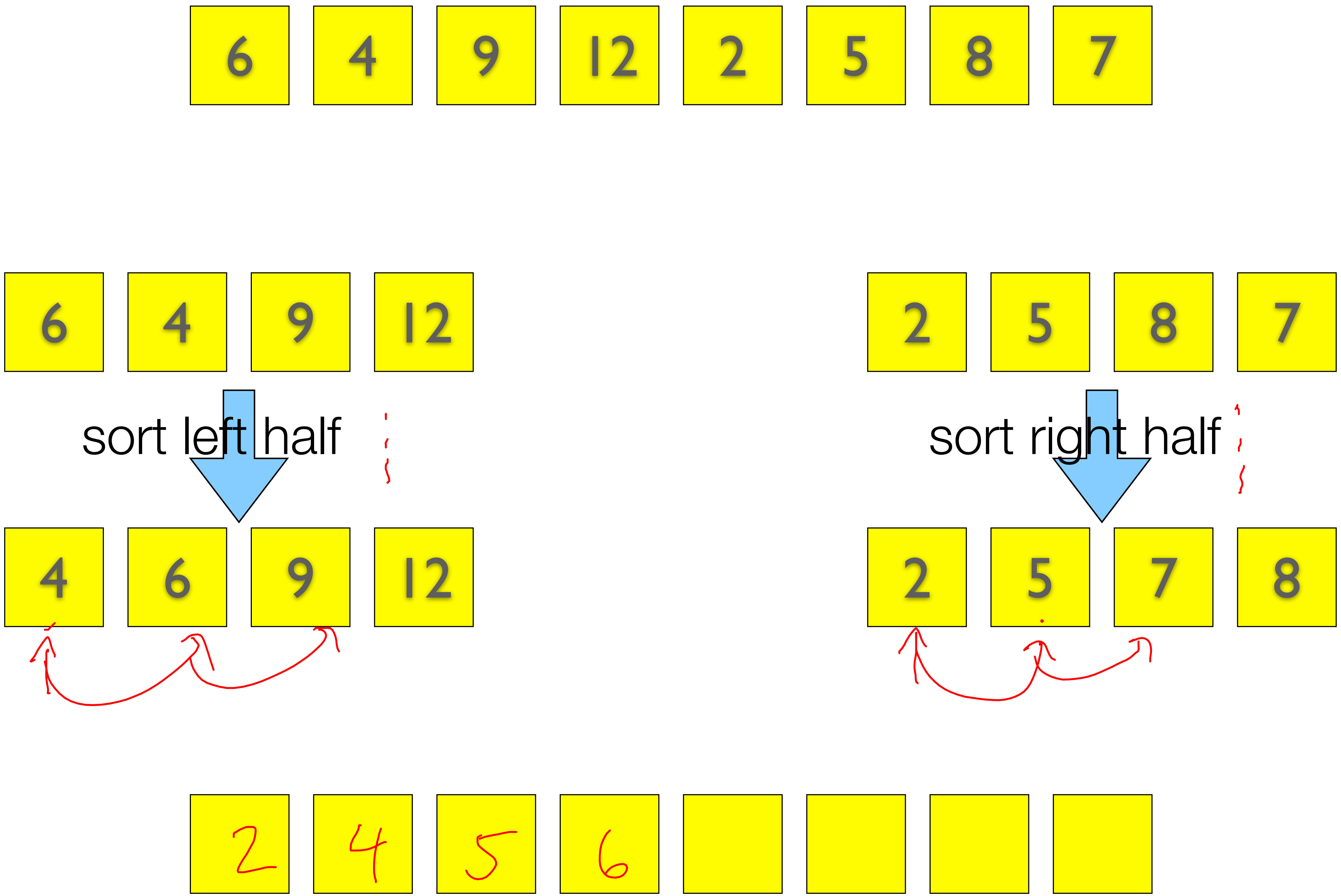
left

6	4	9	12

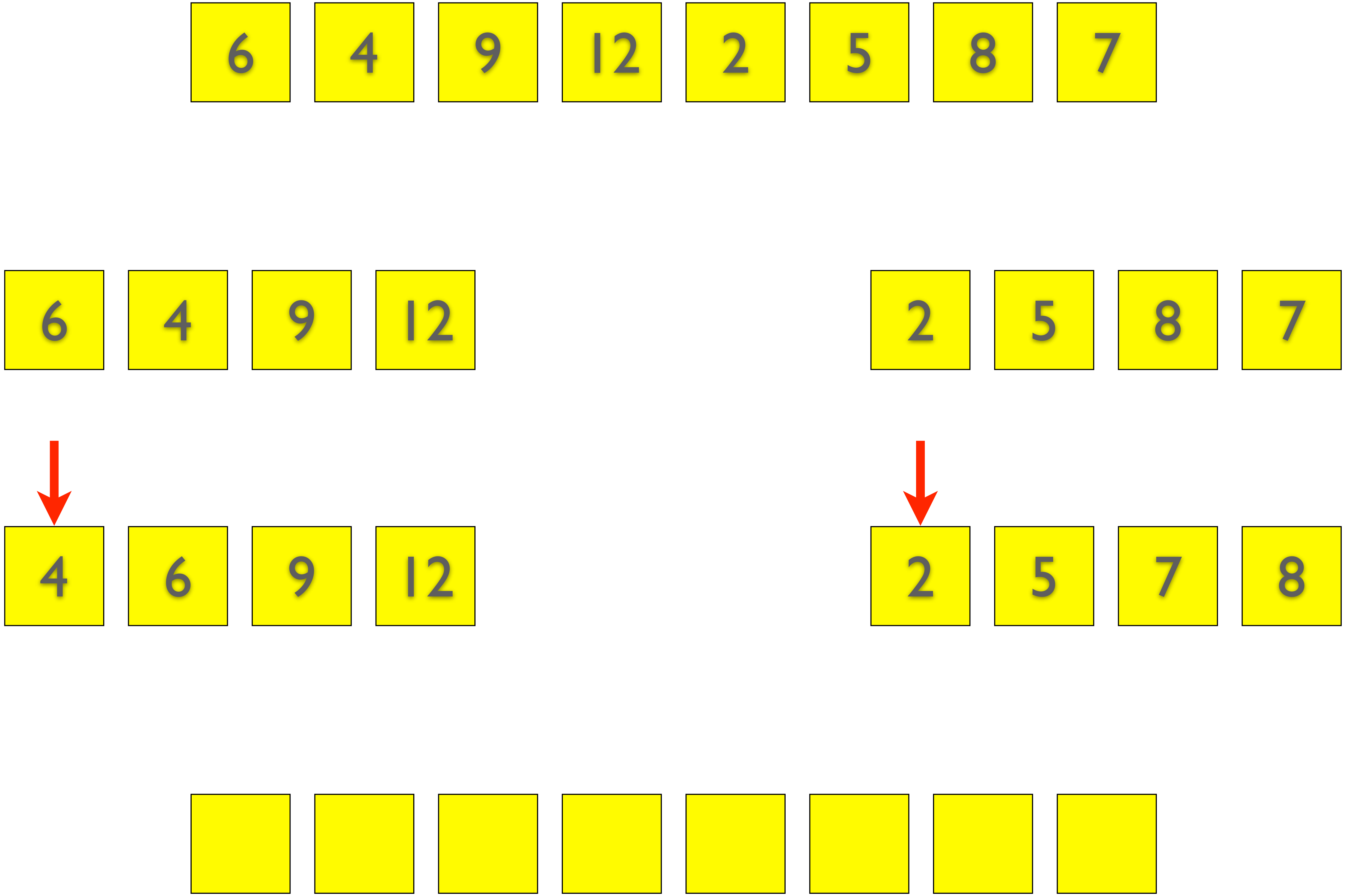
right

2	5	8	7

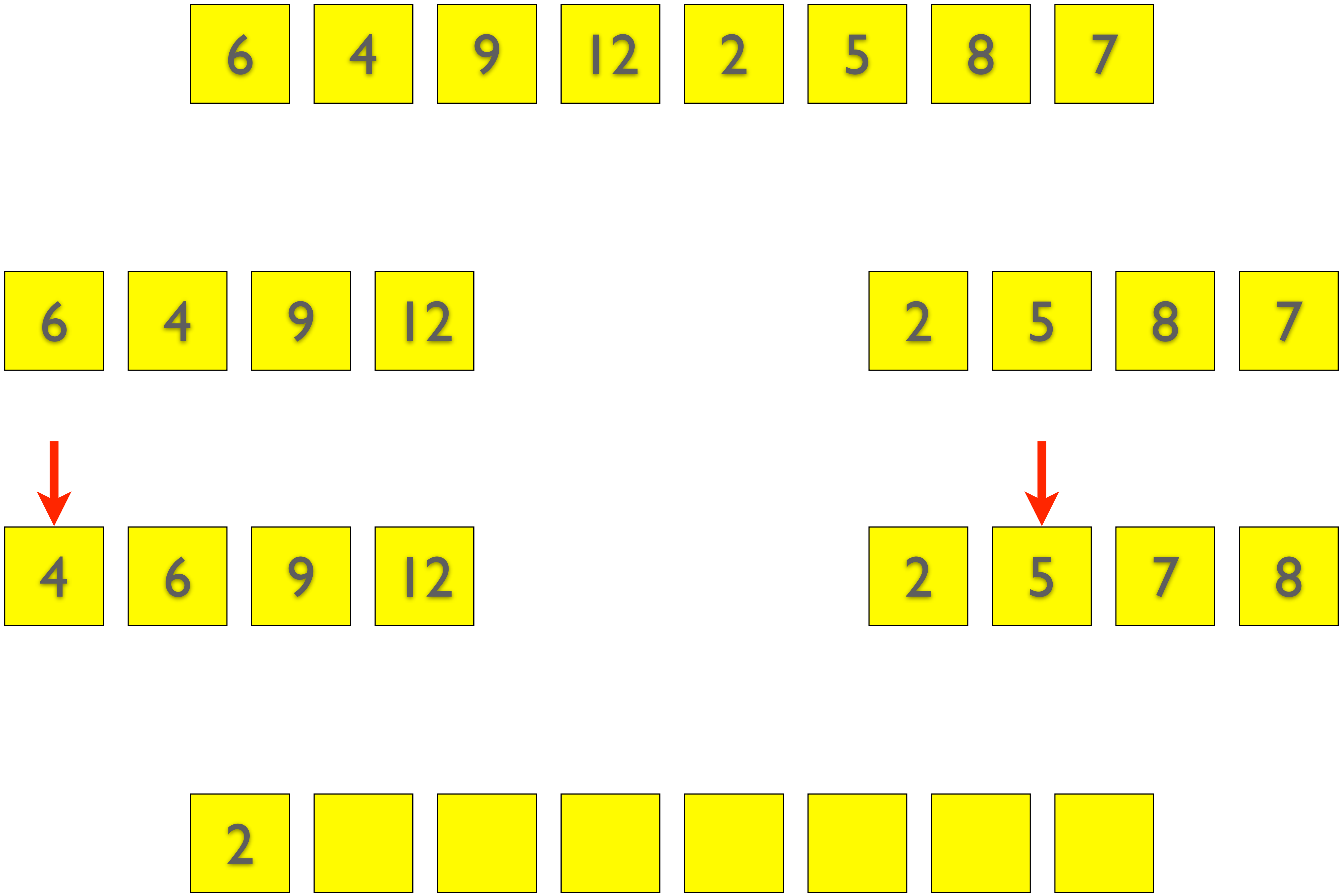
mergesort


6 4 9 12 2 5 8 7
6 4 9 12
2 5 8 7
sort left half
sort right half
4 6 9 12
2 5 7 8
2 4 5 6

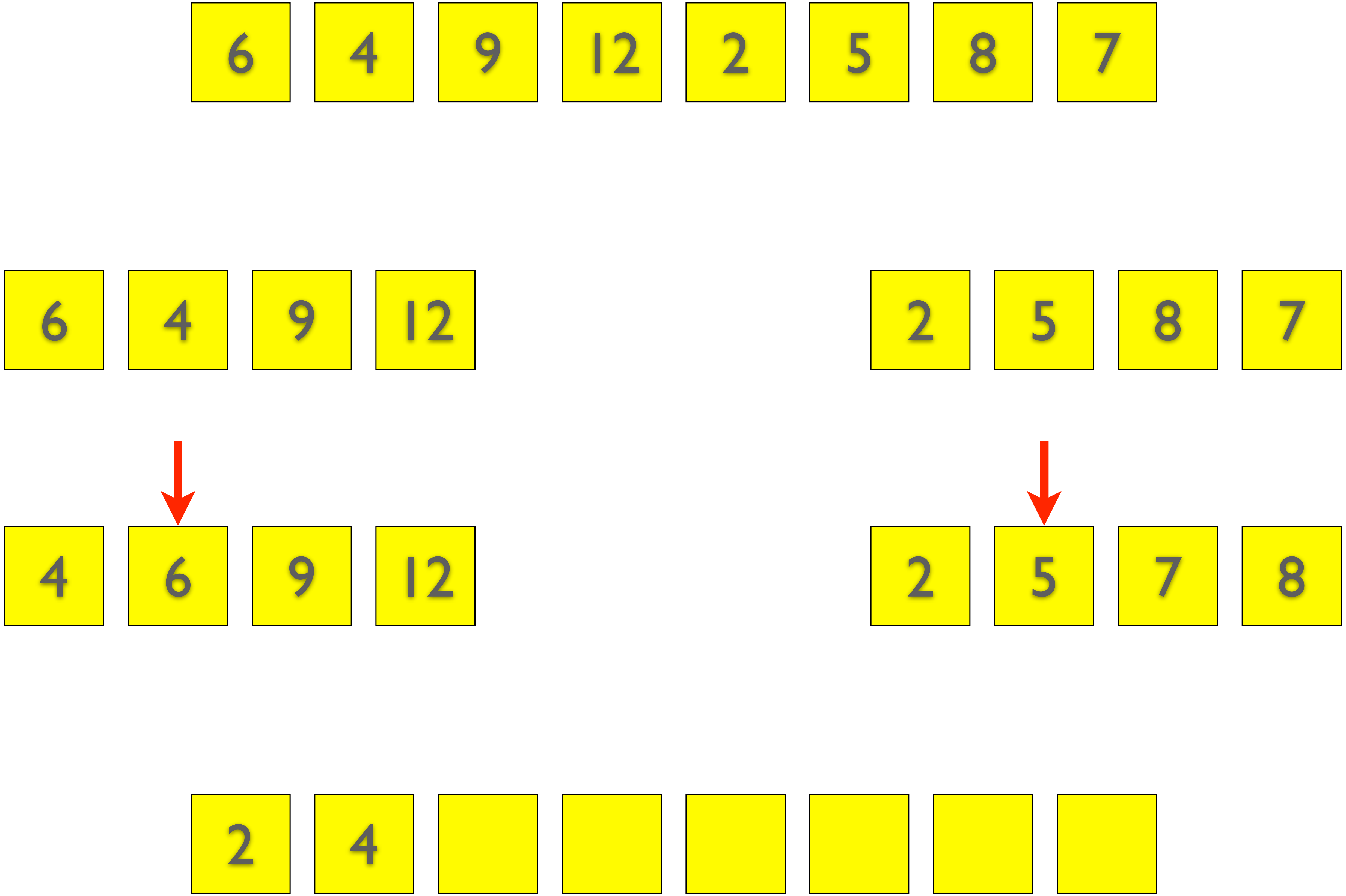
mergesort



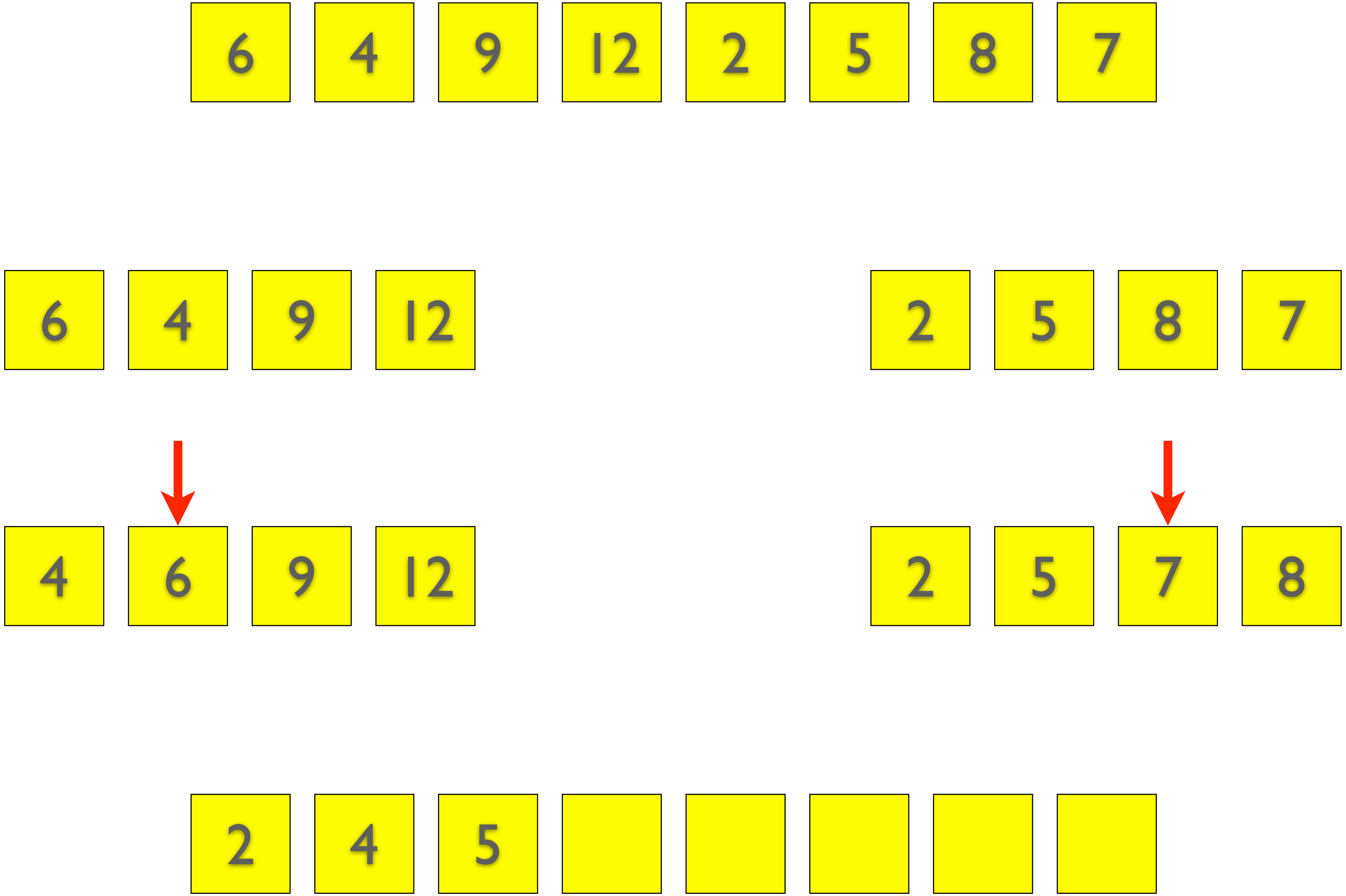
mergesort



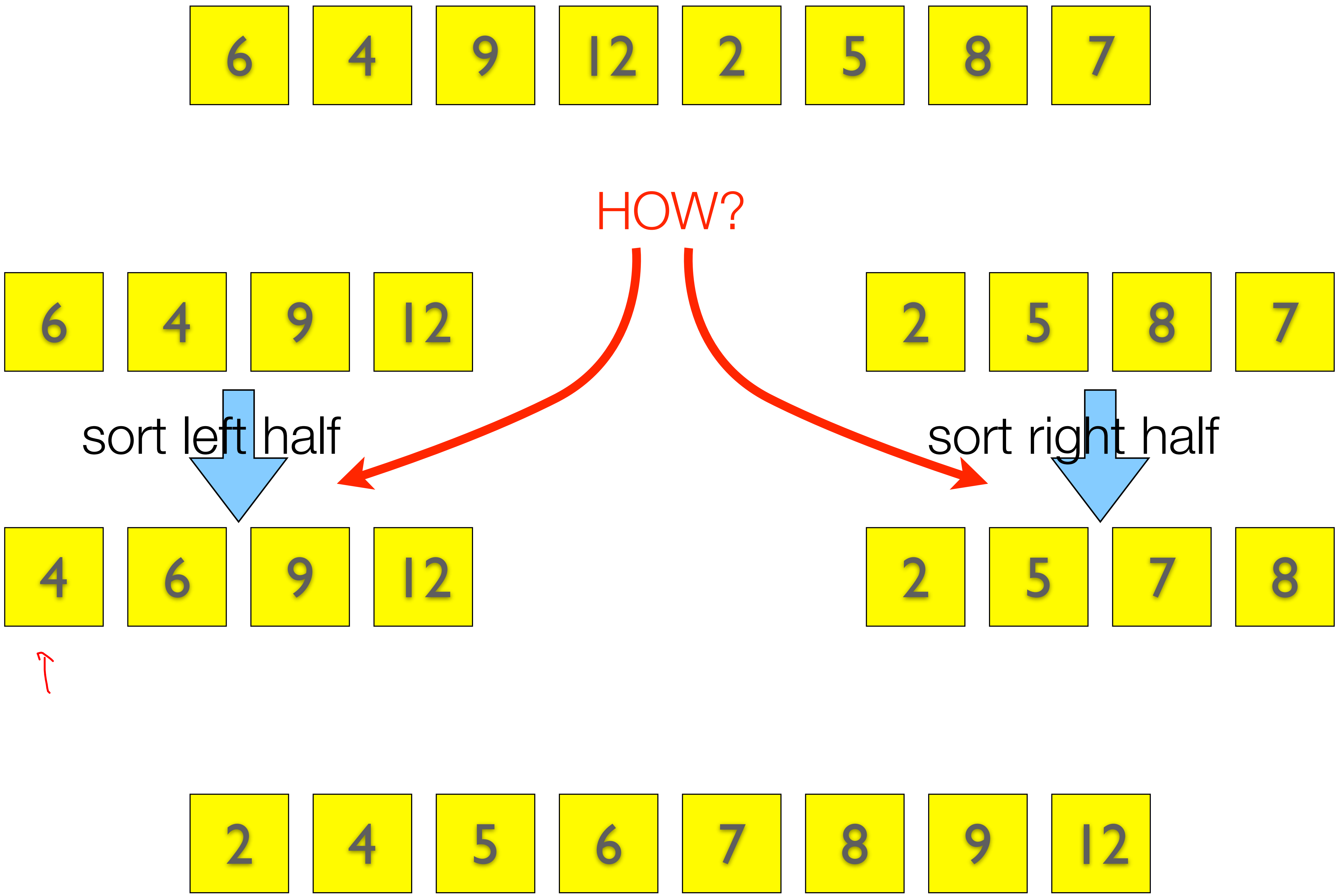
mergesort



mergesort



mergesort



6 4 9 12 2 5 8 7
HOW?
6 4 9 12
2 5 8 7
sort left half
sort right half
4 6 9 12
2 5 7 8
2 4 5 6 7 8 9 12

mergesort(`A,start,end`)

❶

❷

❸

❹

❺

mergesort(`A,start,end`) $T(n)$

1. `if start < end`

2. midpt
   $q \leftarrow \lfloor(\text{start} + \text{end})/2\rfloor$ 1

3. `mergesort(A,start,q)` left
   `mergesort(A,q+1,end)` right $\rightarrow 2T(\frac{n}{2})$

4. `merge(A,start,q,end)` merge $\rightarrow$ n steps

5. `else ...`
   base case

$$T(n) = 2T(\tfrac{n}{2}) + 2n$$

mergesort(`A,start,end`)

1. `if start < end`

2. $q \leftarrow \lfloor (\text{start} + \text{end})/2 \rfloor$

3. ```
       mergesort(A,start,q)
       mergesort(A,q+1,end)
   ```

4. ```
       merge(A,start,q,end)
   ```

5. `else ...`

<u>MERGE($A[1\,..\,n], m$):</u>

$i \leftarrow 1;\ j \leftarrow m+1$
for $k \leftarrow 1$ to n
  if $j > n$
    $B[k] \leftarrow A[i];\ i \leftarrow i+1$
  else if $i > m$
    $B[k] \leftarrow A[j];\ j \leftarrow j+1$
  else if $A[i] < A[j]$
    $B[k] \leftarrow A[i];\ i \leftarrow i+1$
  else
    $B[k] \leftarrow A[j];\ j \leftarrow j+1$
for $k \leftarrow 1$ to n
  $A[k] \leftarrow B[k]$

jeff erickson

mergesort(`A,start,end`)

running time?

1. `if start < end`
2. $q \leftarrow \lfloor (\text{start} + \text{end})/2 \rfloor$
3. `mergesort(A,start,q)`
   `mergesort(A,q+1,end)`
4. `merge(A,start,q,end)`
5. `else ...`

$$T(n) = 2T(n/2) + 2n = O(n \log n)$$

show: Prove $T(n) \le n \log n$ $\quad T(1) = 0,\ T(2) = 2.$

Observe that the statement holds for small n.

Assume that $T(n) \le n \log n$ for all $n < n_0$.

Consider $T(n_0+1) = 2T\left(\frac{n_0+1}{2}\right) + (n_0+1)$

$$\le 2\left(\frac{n_0+1}{2}\right)\log_2\left(\frac{n_0+1}{2}\right) + (n_0+1)$$

$$\le (n_0+1)\log(n_0+1) - (\log_2 2)(n_0+1) + (n_0+1)$$

$$\le (n_0+1)(\log_2(n_0+1)) \quad \boxed{\times}$$

$$T(n) = 2T(n/2) + n$$

prove:

hypothesis:

base case:

inductive step:

$$T(n) = 2T(n/2) + n$$

prove: $T(n) = O(n \log n)$

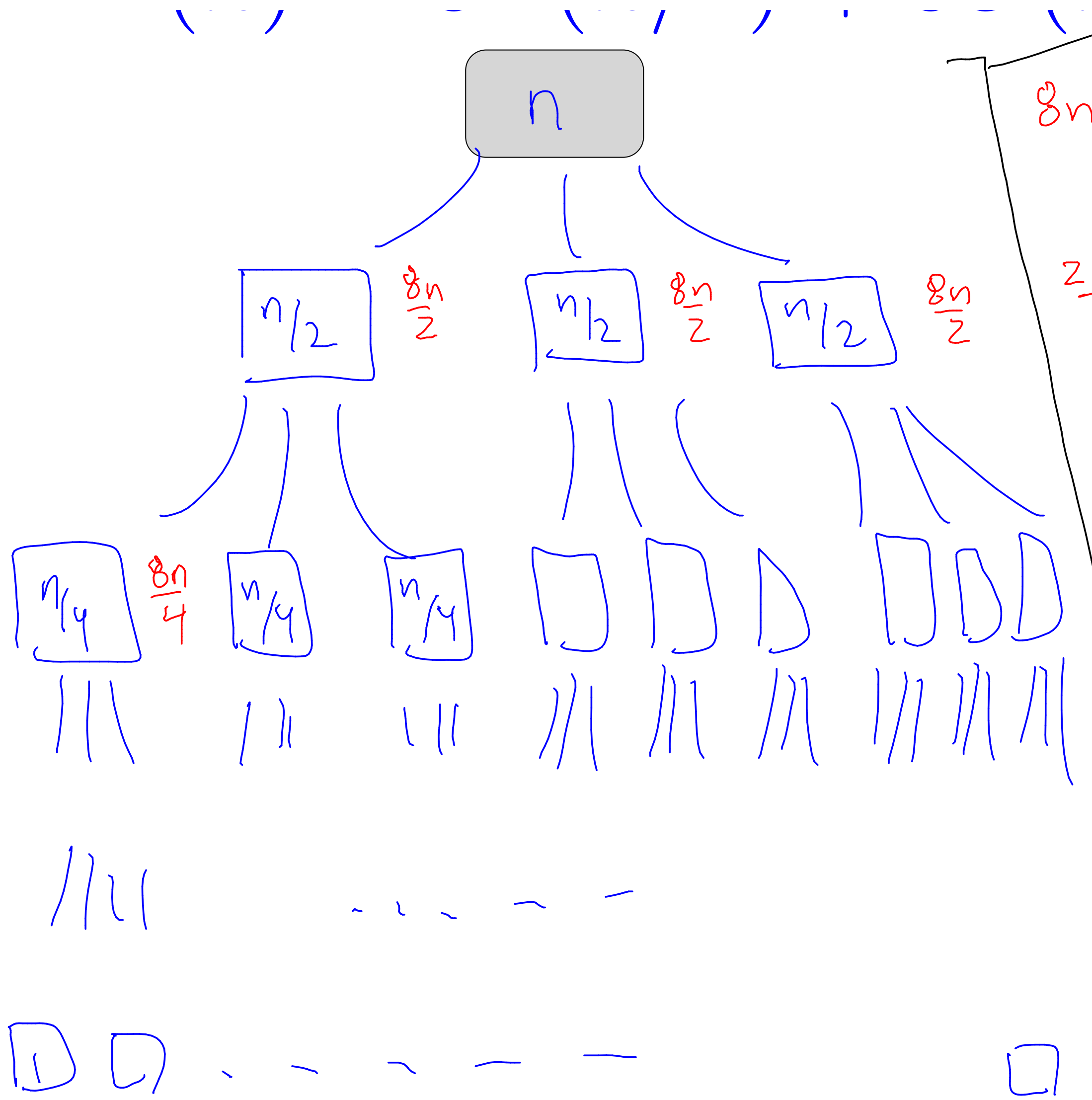
property: $T(n) < cn \log n$ for c>1

base case:

inductive step:

$$T(n) = 3T(n/2) + 9n$$

$O(n^{1.589})$ $\qquad$ $O(n^{\log_2(3)})$



$\lceil \log_2 n \rceil$

$8n$ $\quad 0$

$\frac{24n}{2} = \left(\frac{3}{2}\right) 8n$ $\quad 1$

$9 \cdot \frac{8n}{4} = \left(\frac{9}{4}\right) \cdot 8n$ $\quad 2$

$\rightarrow$ i^{th} level amount of work $= \left(\frac{3}{2}\right)^i \cdot 8n$

$$T(n) = 3T(n/2) + 9n$$ (guess +chk) $= O\left(n^{\log_2(3)}\right)$

Prove $T(n) < n^{\log_2 3} - 27n$

1 classic

one long proof...

goal: prove that some property P(k) is true for all k

$\forall k, P(k)$ holds

$$T(n) = 3T(n/2) + 9n$$ (guess +chk)

show: $T(k) = O(n^{\log 3})$

property:

base case: (handled by constants d' and d")

inductive step

$$T(n) = 8T(n/2) + \Theta(n^2)$$ (guess +chk)