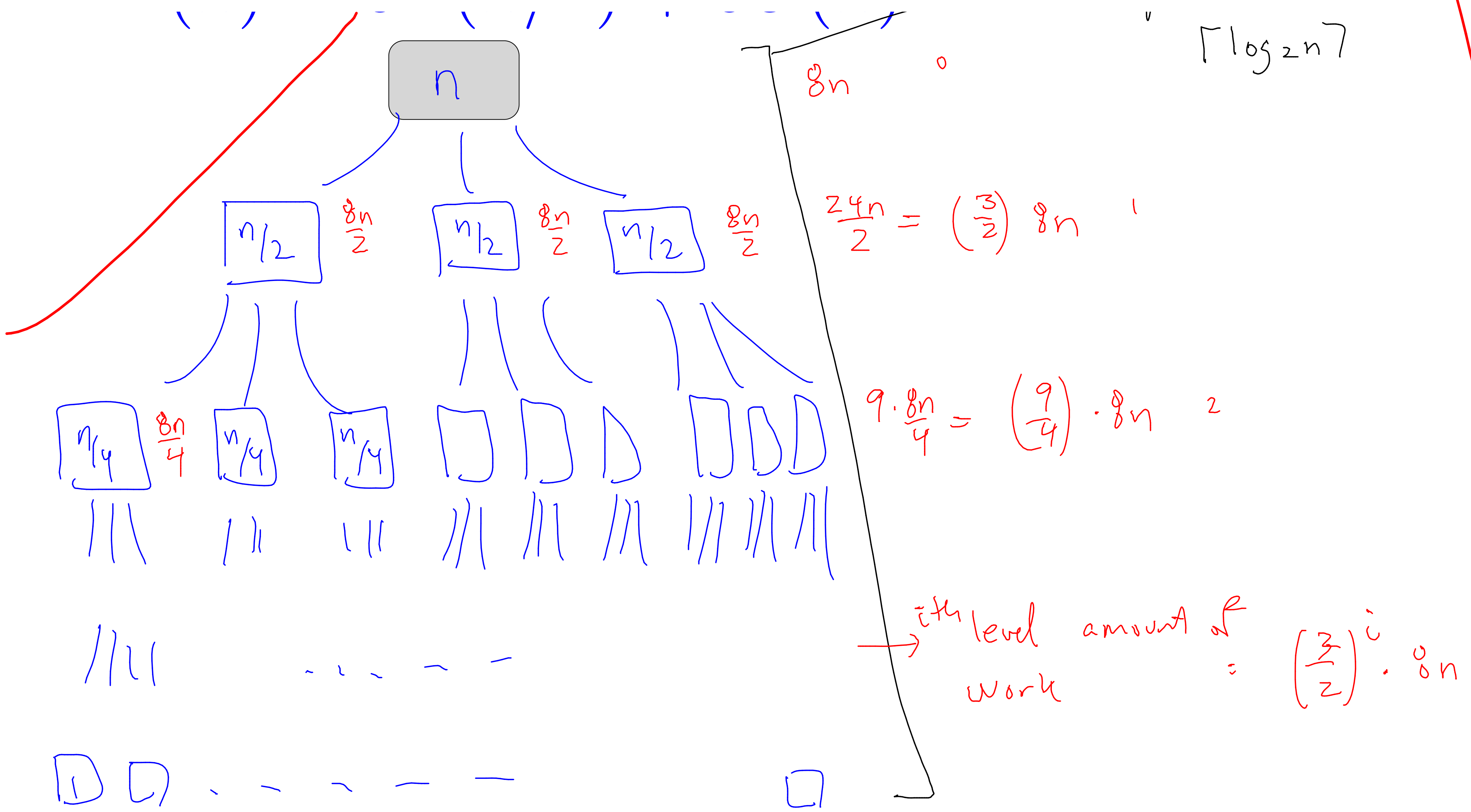
L4

shelat
4102
feb 2 2016

$$T(n) = 3T(n/2) + 9n$$

$$O(n^{\log_2(3)})$$

$O(n^{1.6})$



$$T(n) = 3T(n/2) + 9n$$

(w/ appropriate base case)

Prove $T(n) < n^{\log_2(3)} - 18n$.

By inspection, the statement holds for small n. Suppose that it holds for all $n < n_0$. Consider

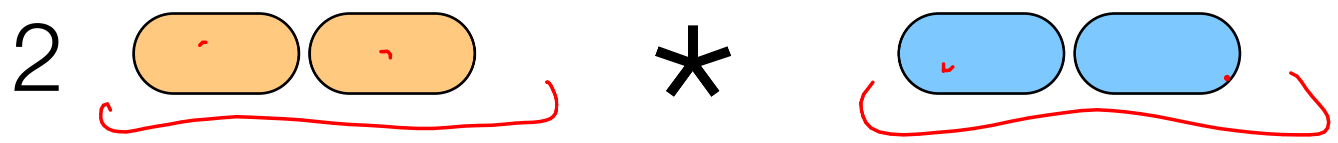
$$T(n_0+1) = 3T\left(\frac{n_0+1}{2}\right) + 9(n_0+1)$$

$$< 3\left[\left(\frac{n_0+1}{2}\right)^{\log_2(3)} - 18\left(\frac{n_0+1}{2}\right)\right] + 9(n_0+1)$$

$$= \frac{3}{3}(n_0+1)^{\log_2(3)} - 27(n_0+1) + 9(n_0+1)$$

$$= (n_0+1)^{\log_2(3)} - 18(n_0+1)$$

☒

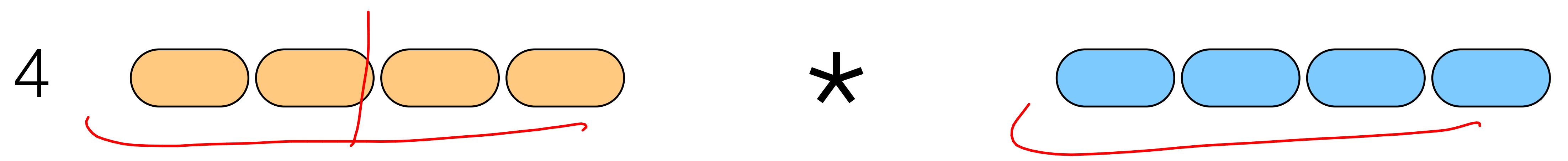


Base case for Karatsuba was $n=2$

4 mult, 3 additions —

I inspected the assembly language.

$T(2) = 10.$ 10 operations.



3 operation on $T(2) = 3 \cdot T(2) + \square$

we combined the steps. $= 3 \cdot 10 + \boxed{32}$

$T(4) = 62$

Digits	# operations
2	10
4	62
8	250
16	878
32	2890
64	9182
128	28570

Digits	# operations	$14n^{\log_2 3} - 16n$
2	10	10
4	62	62
8	250	250
16	878	878
32	2890	2890
64	9182	9182
128	28570	28570

$$T(n) = 3T(n/2) + 8n$$ (guess +chk)

Prove $T(n) \le 14n^{\log_2 3} - 16n$.

$$T(n) = 3T(n/2) + 8n$$ (guess +chk)

Lets prove that $T(n) \leq 14n^{\log_2 3} - 16n$

$$T(n) = 3T(n/2) + 8n$$ (guess +chk)

Lets prove that $T(n) \leq 14n^{\log_2 3} - 16n$

By inspection, indeed, $T(n) \leq 14n^{\log_2 3} - 16n$ when n<1024. $= n_0$.

Consider $T(n_0+1) = 3T\left(\frac{n_0+1}{2}\right) + 8(n_0+1)$

$$< 3\left[14\left(\frac{n_0+1}{2}\right)^{\log_2 3} - 16\left(\frac{n_0+1}{2}\right)\right] + 8(n_0+1)$$

$$T(n) = 3T(n/2) + 8n$$ (guess +chk)

Lets prove that $T(n) \leq 14n^{\log_2 3} - 16n$

By inspection, indeed, $T(n) \leq 14n^{\log_2 3} - 16n$ when n<1024.

A1: Lets assume that $T(n) \leq 14n^{\log_2 3} - 16n$ when $n < n_0$

$$T(n) = 3T(n/2) + 8n$$ (guess +chk)

Lets prove that $T(n) \leq 14n^{\log_2 3} - 16n$

By inspection, indeed, $T(n) \leq 14n^{\log_2 3} - 16n$ when n<1024.

A1: Lets assume that $T(n) \leq 14n^{\log_2 3} - 16n$ when $n < n_0$

Consider the case of $T(n_0 + 1)$

$$T(n) = 3T(n/2) + 8n$$ (guess +chk)

Lets prove that $T(n) \leq 14n^{\log_2 3} - 16n$

By inspection, indeed, $T(n) \leq 14n^{\log_2 3} - 16n$ when n<1024.

A1: Lets assume that $T(n) \leq 14n^{\log_2 3} - 16n$ when $n < n_0$

Consider the case of $T(n_0 + 1)$

$$T(n_0 + 1) = 3T((n_0 + 1)/2) + 8(n_0 + 1)$$ By definition

$$T(n) = 3T(n/2) + 8n$$ (guess +chk)

Lets prove that $T(n) \le 14n^{\log_2 3} - 16n$

By inspection, indeed, $T(n) \le 14n^{\log_2 3} - 16n$ when n<1024.

A1: Lets assume that $T(n) \le 14n^{\log_2 3} - 16n$ when $n < n_0$

Consider the case of $T(n_0 + 1)$

$$T(n_0 + 1) = 3T((n_0 + 1)/2) + 8(n_0 + 1)$$ By definition

But since $(n_0 + 1)/2 < n_0$ and A1, it follows that

$$T(n) = 3T(n/2) + 8n$$ (guess +chk)

Lets prove that $T(n) \leq 14n^{\log_2 3} - 16n$

By inspection, indeed, $T(n) \leq 14n^{\log_2 3} - 16n$ when n<1024.

A1: Lets assume that $T(n) \leq 14n^{\log_2 3} - 16n$ when $n < n_0$

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But since $(n_0 + 1)/2 < n_0$ and A1, it follows that

$$T(n_0 + 1) \leq 3\left[14\left(\frac{n_0 + 1}{2}\right)^{\log_2 3} - 16\left(\frac{n_0 + 1}{2}\right)\right] + 8(n_0 + 1)$$

$$\leq \frac{3}{3}\left[14(n_0+1)^{\log_2 3} - 24(n_0+1)\right) + 8(n_0+1)$$

$$T(n_0+1) < 3\left[14\left(\frac{n_0+1}{2}\right)^{\log_2 3} - 16\left(\frac{n_0+1}{2}\right)\right] + 8(n_0+1)$$

$$T(n_0+1) \lesssim 3\left[14\left(\frac{n_0+1}{2}\right)^{\log_2 3} - 16\left(\frac{n_0+1}{2}\right)\right] + 8(n_0+1)$$

$$\lesssim 14(n_0+1)^{\log_2 3} - 24(n_0+1) + 8(n_0+1)$$

$$T(n_0+1) \le 3\left[14\left(\frac{n_0+1}{2}\right)^{\log_2 3} - 16\left(\frac{n_0+1}{2}\right)\right] + 8(n_0+1)$$

$$\le 14(n_0+1)^{\log_2 3} - 24(n_0+1) + 8(n_0+1)$$

$$\le 14(n_0+1)^{\log_2 3} - 16(n_0+1)$$

$$T(n_0+1) \leq 3\left[14\left(\frac{n_0+1}{2}\right)^{\log_2 3} - 16\left(\frac{n_0+1}{2}\right)\right] + 8(n_0+1)$$

$$\leq 14(n_0+1)^{\log_2 3} - 24(n_0+1) + 8(n_0+1)$$

$$\leq 14(n_0+1)^{\log_2 3} - 16(n_0+1)$$

This expression matches our Assumption A1.

A1: Lets assume that $T(n) \leq 14n^{\log_2 3} - 16n$ when $n < n_0$

$$T(n_0+1) < 3\left[14\left(\frac{n_0+1}{2}\right)^{\log_2 3} - 16\left(\frac{n_0+1}{2}\right)\right] + 8(n_0+1)$$

$$< 14(n_0+1)^{\log_2 3} - 24(n_0+1) + 8(n_0+1)$$

$$< 14(n_0+1)^{\log_2 3} - 16(n_0+1)$$

This expression matches our Assumption A1.

A1: Lets assume that $T(n) \leq 14n^{\log_2 3} - 16n$ when $n < n_0$

This establishes that $T(n) = O(n^{\log_2 3})$

$$T(n_0+1) < 3\left[14\left(\frac{n_0+1}{2}\right)^{\log_2 3} - 16\left(\frac{n_0+1}{2}\right)\right] + 8(n_0+1)$$

$$< 14(n_0+1)^{\log_2 3} - 24(n_0+1) + 8(n_0+1)$$

$$< 14(n_0+1)^{\log_2 3} - 16(n_0+1)$$

This expression matches our Assumption A1.

A1: Lets assume that $T(n) \leq 14n^{\log_2 3} - 16n$ when $n < n_0$

Thus, we can conclude the proof via induction.

This establishes that $T(n) = O(n^{\log_2 3})$

Induction summary

1 $T(n) \leq 14n^{\log_2 3} - 16n$ IS TRUE for one case.

2 $T(n) \leq 14n^{\log_2 3} - 16n$ Suppose TRUE for $n < n_0$

3 Showed that 1,2 imply that

$$T(n_0 + 1) \leq 14(n_0 + 1)^{\log_2 3} - 16(n_0 + 1)$$

4 (Induction)

What happens if
we skip the -16n?

$$T(n) = 3T(n/2) + 8O(n)$$ (guess +chk)

Assume that this term is 8n

Lets prove that $T(n) \leq 14n^{\log_2 3} - 16n$

By inspection, indeed, $T(n) \leq 14n^{\log_2 3} - 16n$ when n<1024.

A1: Lets assume that $T(n) \leq 14n^{\log_2 3} - 16n$ when $n < n_0$

$$T(n) = 3T(n/2) + 8O(n)$$ (guess +chk)

Assume that this term is 8n

Lets prove that $T(n) \le 14n^{\log_2 3} - 16n$

By inspection, indeed, $T(n) \le 14n^{\log_2 3} - 16n$ when n<1024.

A1: Lets assume that $T(n) \le 14n^{\log_2 3} - 16n$ when $n < n_0$

Consider the case of $T(n_0 + 1)$

$$T(n_0 + 1) = 3T((n_0 + 1)/2) + 8(n_0 + 1)$$

By definition

But since $(n_0 + 1)/2 < n_0$ and A1, it follows that

$$T(n_0 + 1) < 3\left[14\left(\frac{n_0 + 1}{2}\right)^{\log_2 3}\right] + 8(n_0 + 1)$$

$$< 14(n_0+1)^{\log_2 3} + 8(n_0+1)$$

$$T(n) = O(n^{\log_2 3})$$

$$\underline{14n^{\log_2 3} - 16n} = O(n^{\log_2 3})$$

$$\leq 100 n^{\log_2 3}$$

$$T(n_0+1) < 3\left[14\left(\frac{n_0+1}{2}\right)^{\log_2 3}\right] + 8(n_0+1)$$

$$< 14(n_0+1)^{\log_2 3} + 8(n_0+1)$$

This expression DOES NOT matches our Assumption A1.
So the induction STOPS!

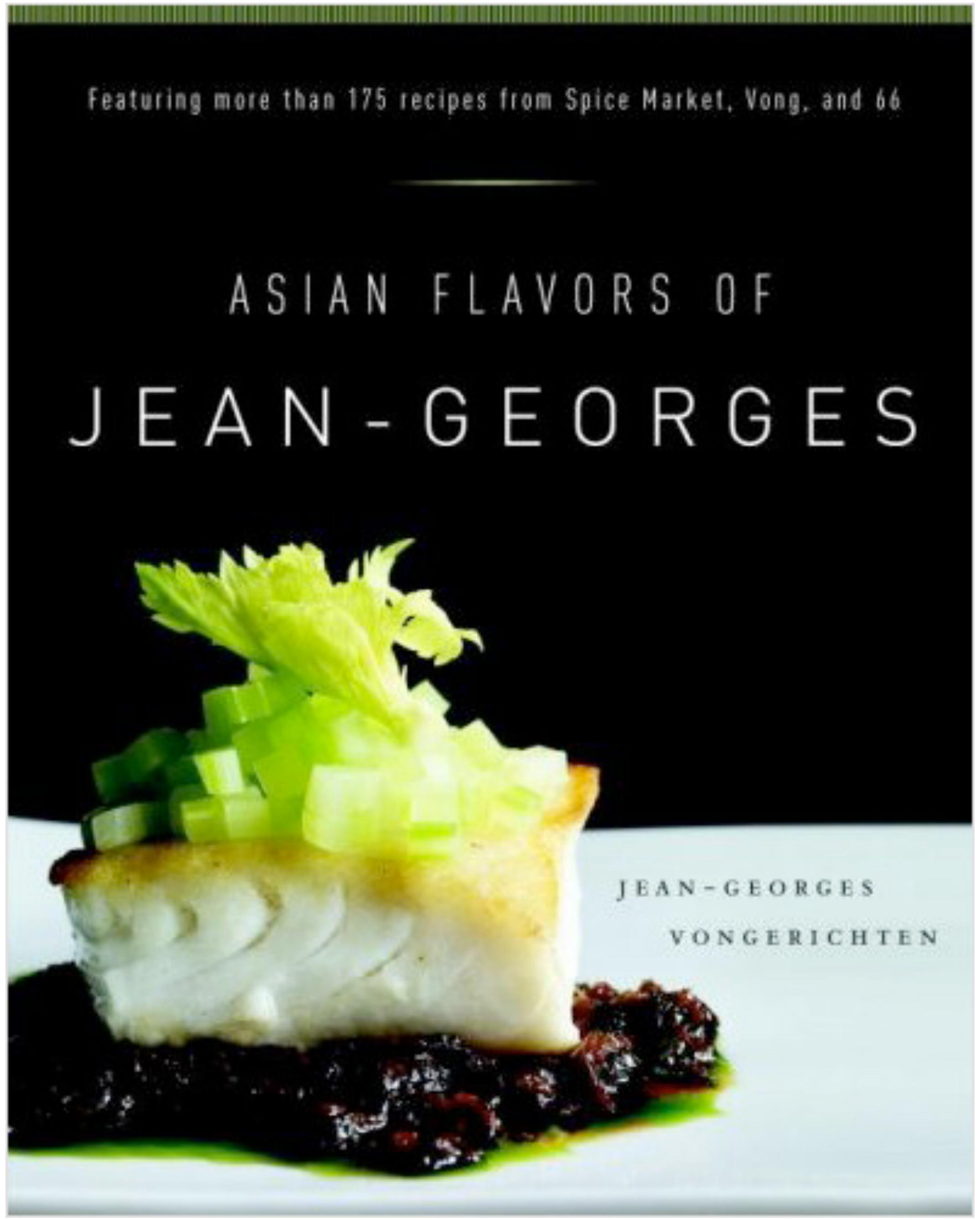
A1: Lets assume that $T(n) \leq 14n^{\log_2 3}$ $-16n$ when $n < n_0$

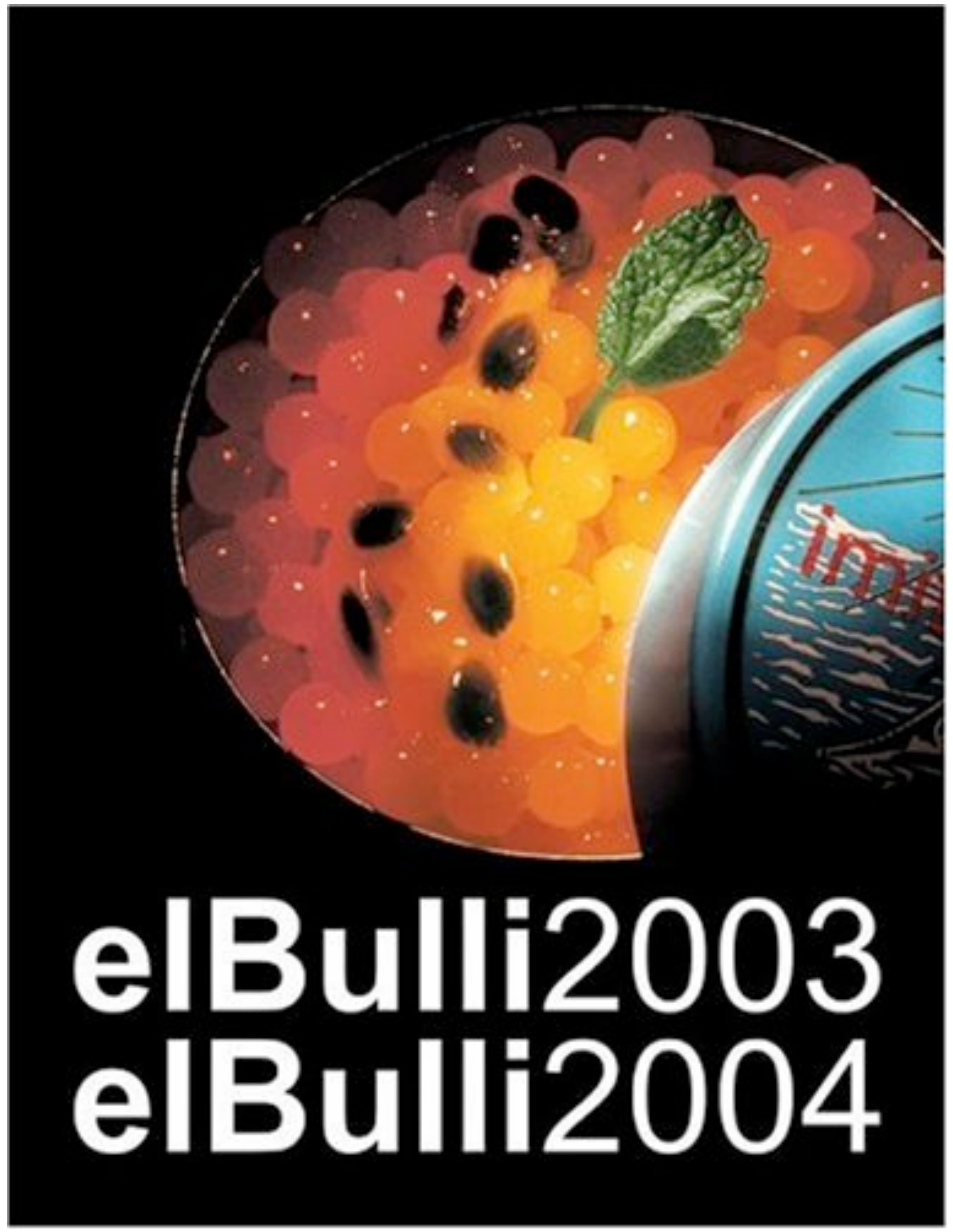
$$T(n) = 8T(n/2) + \Theta(n^2)$$ (guess +chk)

home

cookbook

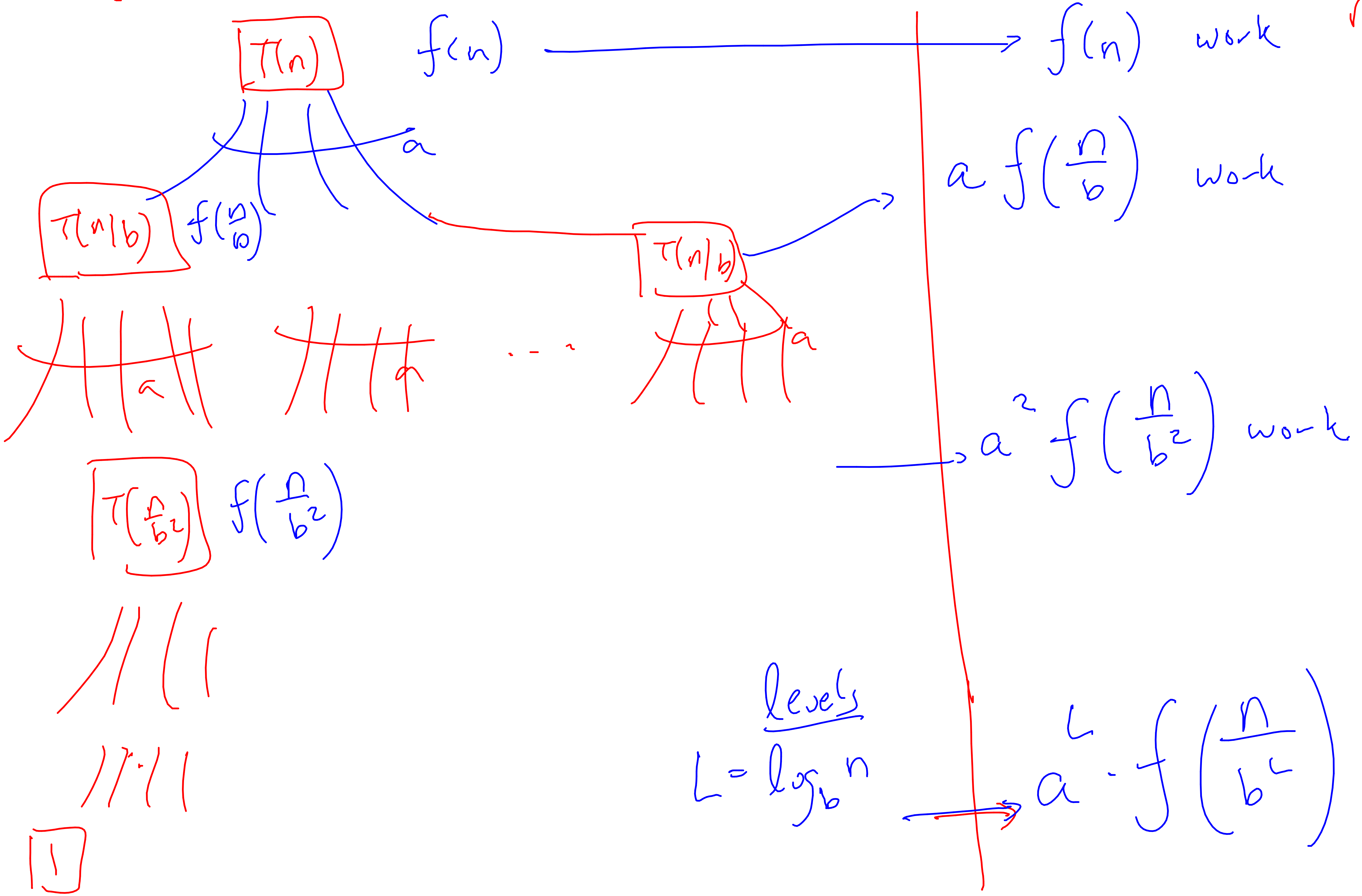
Masters' thm



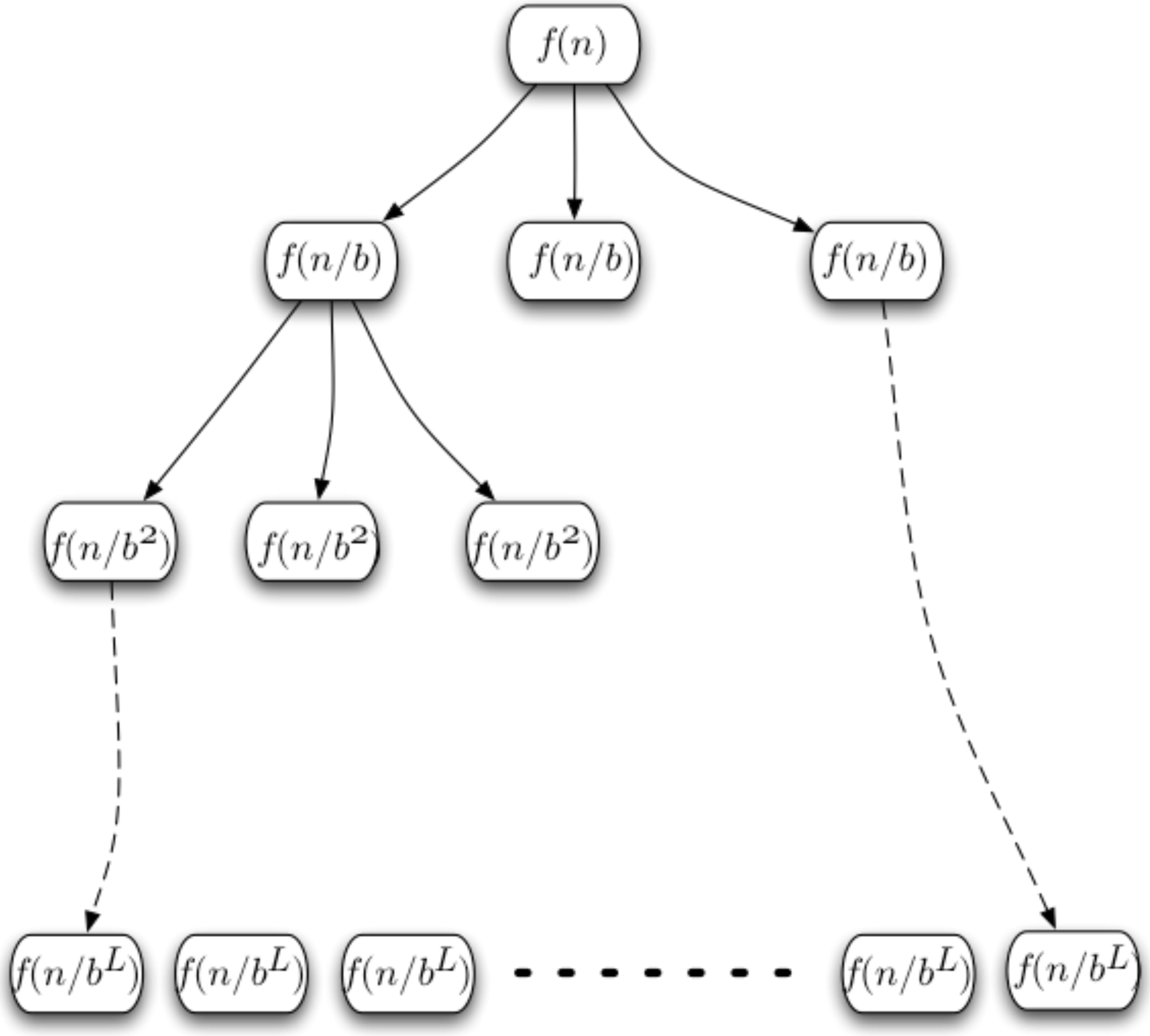


$$T(n) = aT(n/b) + f(n)$$

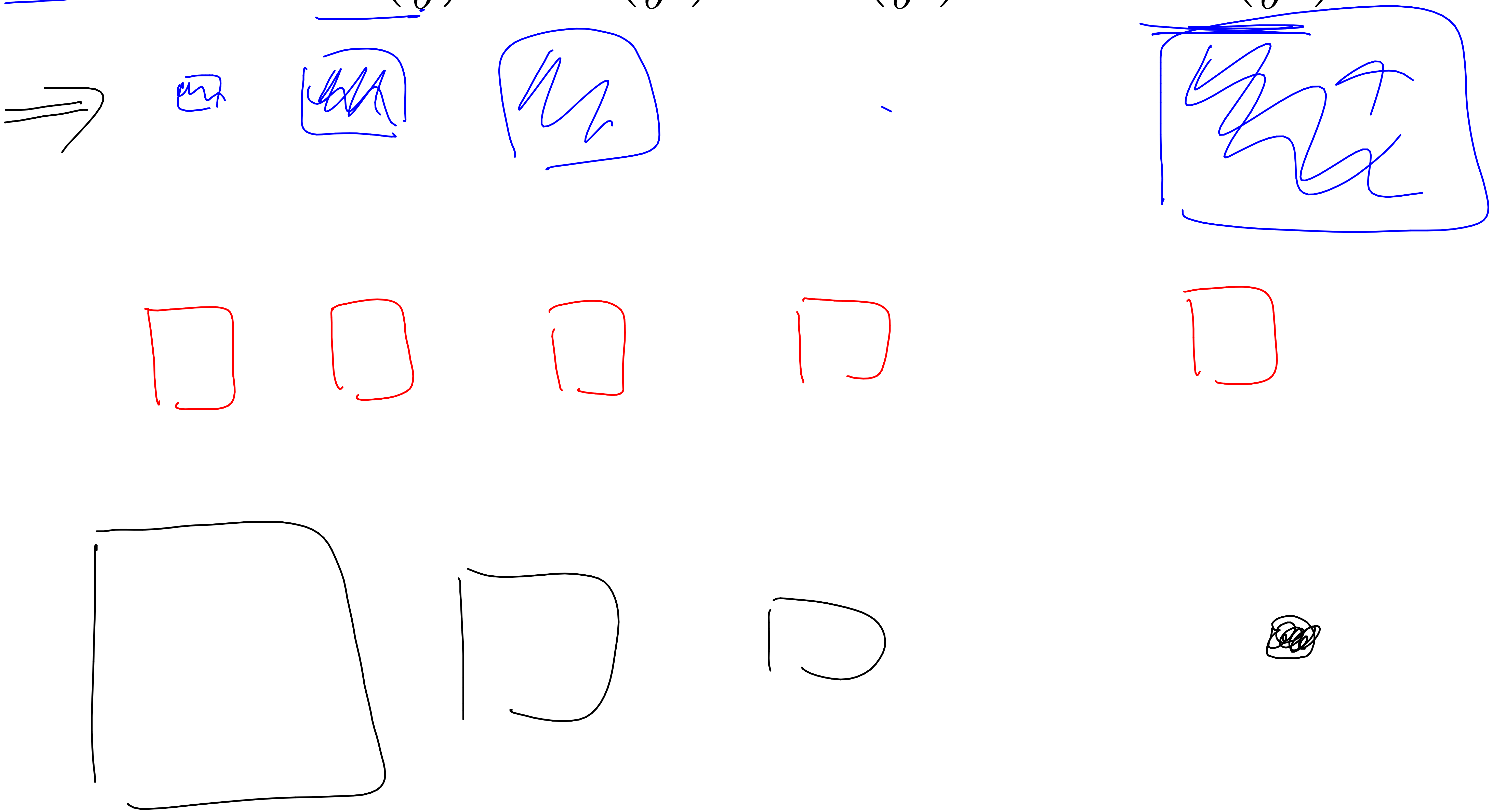
find a solution for recurrences of this form



$$T(n) = aT(n/b) + f(n)$$



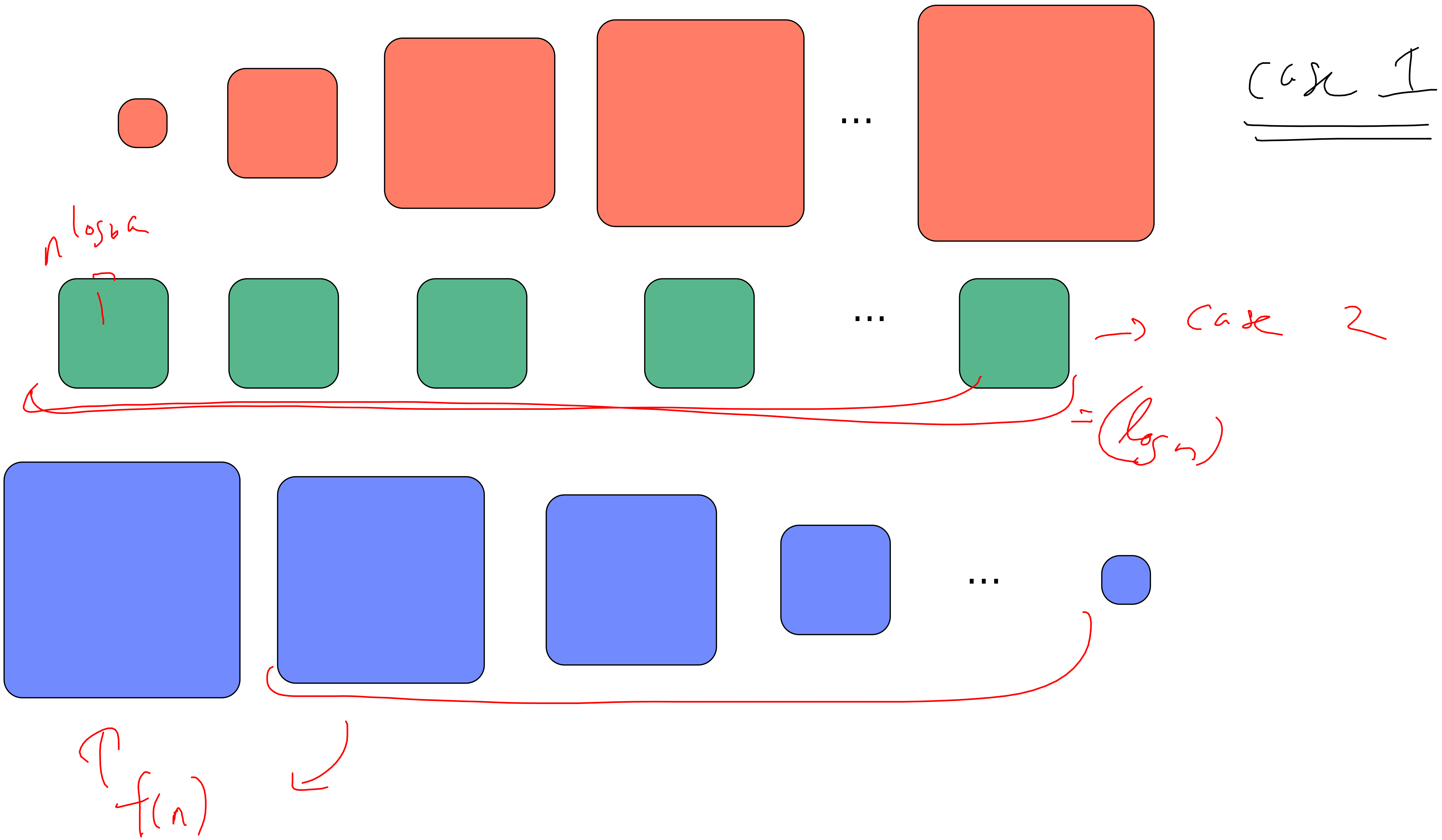
$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$



all the work happens @ the leaf.

the same

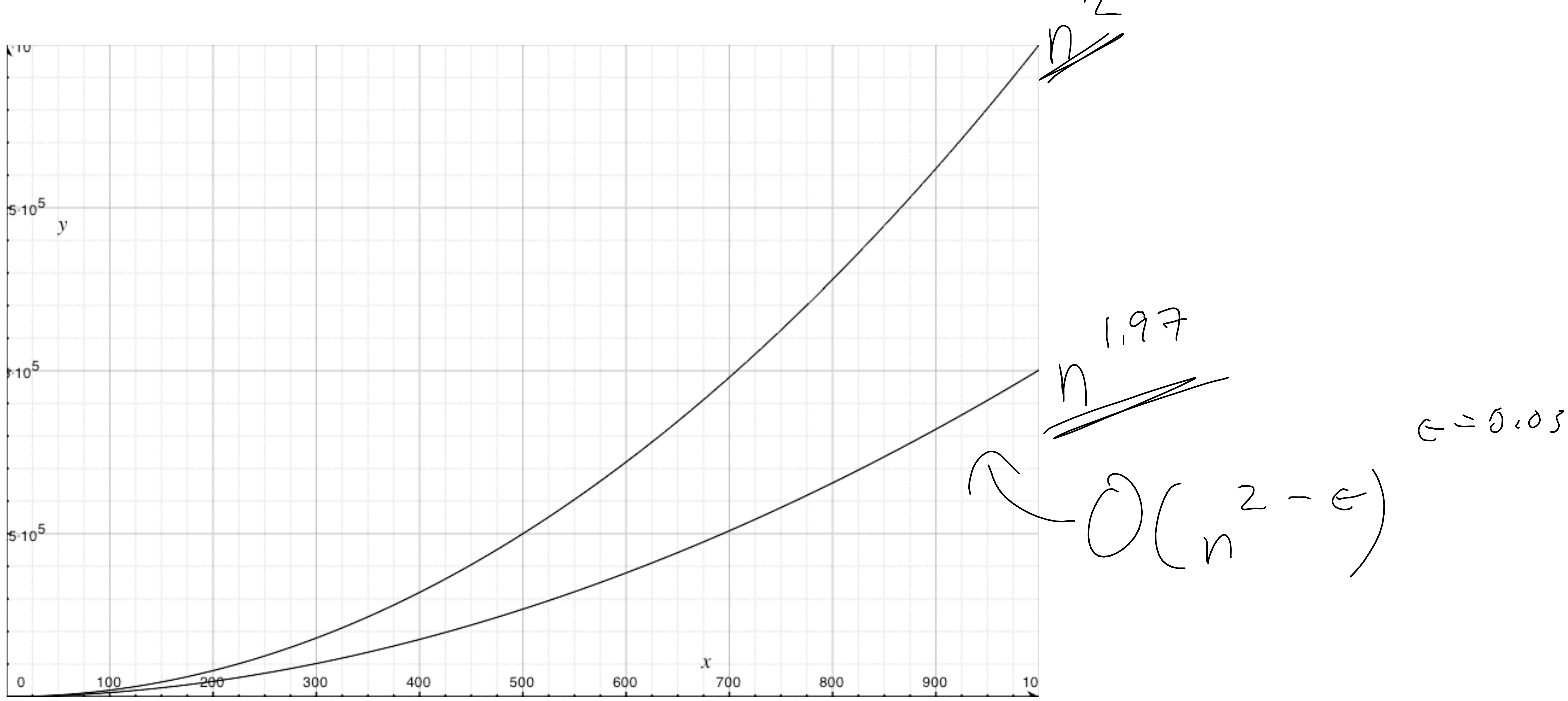
$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$



$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1: $f(n) = O(n^{\log_b a - \epsilon})$

When $f(n) = O(n^{\log_b a - \epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.



5·10^5
y
5·10^5
x
0
100
200
300
400
500
600
700
800
900
n^2
n^1,97
O(n^{2-ε})
ε = 0,03

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1: $$f(n) = O(n^{\log_b a - \epsilon})$$

example:

$$T(n) = 4T(n/2) + 3n \Rightarrow T(n) = \Theta(n^{\log_2 4}) = \Theta(n^2)$$

GOAL: $f(n) = 3n \overset{?}{=} O(n^{\log_2 4 - \epsilon}) = O(n^{2-\epsilon})$

$a = 4 \quad b = 2$

Yes. set eps = 0.91

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

$$\frac{n}{b^L} = 1$$

case 1: $f(n) = O(n^{\log_b a - \epsilon}) \Rightarrow f(n) \leq c \cdot n^{\log_b(a) - \epsilon}$

$$T(n) \leq c \cdot n^{\log_b a - \epsilon} + a \cdot c\left(\frac{n}{b}\right)^{\log_b a - \epsilon} + a^2 \cdot c\left(\frac{n}{b^2}\right)^{\log_b a - \epsilon} + \cdots + a^{L-1} \cdot c\left(\frac{n}{b^{L-1}}\right)^{\log_b(a) - \epsilon} + a^L f(1)$$

$$\leq cn^{\log_b(a) - \epsilon}\left[1 + \frac{a}{b^{\log_b a - \epsilon}} + \frac{a^2}{(b^2)^{\log_b a - \epsilon}} + \cdots + \frac{a^{L-1}}{(b^{L-1})^{\log_b(a) - \epsilon}}\right] + a^L f(1)$$

$$\leq cn^{\log_b(a) - \epsilon}\left[1 + b^{\epsilon} + b^{2\epsilon} + \cdots + b^{\epsilon(L-1)}\right] + n^{\log_b a} f(1)$$

$$\leq cn^{\log_b(a) - \epsilon}\left[(n^{\epsilon} - 1)(c'')\right] + n^{\log_b(a)} \cdot f(1)$$

$(b^{\epsilon} - 1)$

$c' = c + f(1)$

$$\leq c' n^{\log_b(a)} - c'' n^{\log_b(a) - \epsilon} \Rightarrow T(n) = O(n^{\log_b(a)})$$

$\Omega(-)$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

$$\frac{n}{b^L} = 1$$

case 1: $f(n) = O(n^{\log_b a - \epsilon}) \Rightarrow f(n) \le c \cdot n^{\log_b(a) - \epsilon}$

$$T(n) \le c \cdot n^{\log_b a - \epsilon} + a \cdot c\left(\frac{n}{b}\right)^{\log_b a - \epsilon} + a^2 \cdot c\left(\frac{n}{b^2}\right)^{\log_b a - \epsilon} + \cdots + a^{L-1} \cdot c\left(\frac{n}{b^{L-1}}\right)^{\log_b(a) - \epsilon} + a^L f(1)$$

$$\le cn^{\log_b(a) - \epsilon}\left[1 + \frac{a}{b^{\log_b a - \epsilon}} + \frac{a^2}{(b^2)^{\log_b a - \epsilon}} + \cdots + \frac{a^{L-1}}{(b^{L-1})^{\log_b(a) - \epsilon}}\right] + a^L f(1)$$

$$\le cn^{\log_b(a) - \epsilon}\left[1 + b^{\epsilon} + b^{2\epsilon} + \cdots + b^{\epsilon(L-1)}\right] + n^{\log_b a} f(1)$$

$$\le cn^{\log_b(a) - \epsilon}\left[(n^{\epsilon} - 1)(c''')\right] + n^{\log_b(a)} \cdot f(1)$$

$$c''' = (b^{\epsilon} - 1)$$

$$\left(c \cdot n^{\log_b(a) - \epsilon}\right)\left(-c'''\right)$$

$$1 + b^{\epsilon} + b^{2\epsilon} + \cdots + b^{(L-1)\epsilon} = \left(\frac{b^{\epsilon L} - 1}{b^{\epsilon} - 1}\right) = \left(\frac{n^{\epsilon} - 1}{b^{\epsilon} - 1}\right)$$

---

① $$\frac{a^i}{(b^i)^{\log_b(a)-\epsilon}} = \frac{a^i}{\left(b^{\log_b(a)-\epsilon}\right)^i} = \frac{a^i}{\frac{a^i}{b^{\epsilon i}}} = b^{\epsilon i}$$

② $$a^L = a^{\log_b n} = \left(b^{\log_b a}\right)^{\log_b n} = \left(b^{\log_b n}\right)^{\log_b a} = n^{\log_b a}$$

③ $$b^{\epsilon L} = \left(b^{\log_b n}\right)^{\epsilon} = n^{\epsilon}$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1 (cont):

$$T(n) \leq cn^{\log_b a - \epsilon}\left[1 + b^{\epsilon} + b^{2\epsilon} + \cdots + b^{\epsilon(L-1)}\right] + n^{\log_b a}$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 2: $f \in \Theta(n^{\lg_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log n)$

case 3: $f \in \Omega(n^{\lg_b a + \epsilon})$ and... (special conditions on f)

$T(n) = \Theta(f(n))$

example 2: $T(n) = 8T(n/2) + \Theta(n^2)$

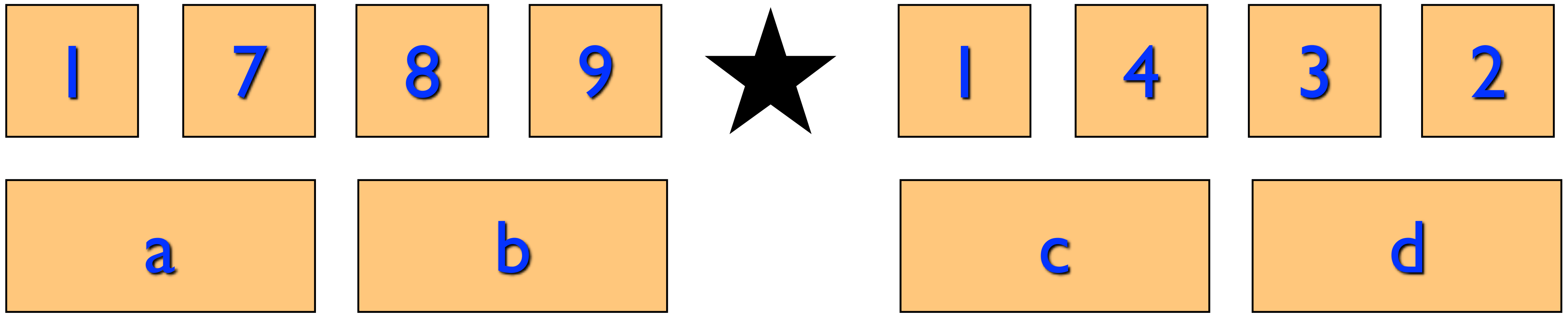
$f(n) = \Theta(n^2)$

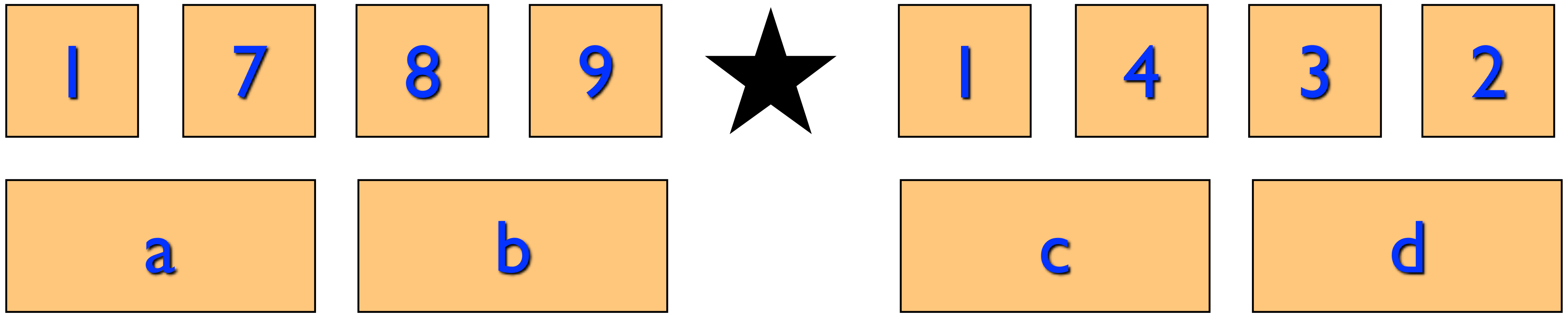
$b = 2 \quad a = 8$

$n^{\log_2 8} = n^3$

Yes $f(n) = \Theta(n^2) = O(n^{3-\epsilon})$ case I ✓

$T(n) = \Theta(n^3)$


1
7
8
9
1
4
3
2
a
b
c
d



$$T(n) = 4T(n/2) + 3O(n)$$

$$T(n) = 2T(n/2) + n^3$$

$f(n) = n^3$

$a = 2 \quad b = 2$

$n^{\log_2 2} = n^1$

$f(n) \in \Omega(n^{1+\epsilon}) \Rightarrow$ case 3.

$T(n) = \Theta(n^3)$

$$T(n) = 7T(n/2) + O(n^2)$$

example:

$$T(n) = T\left(\frac{14}{17}n\right) + 24$$


8
13
FIFA

$$T(n) = 2T\left(\sqrt{n}\right) + \lg n$$