— This is a pledged take-home midterm. Do not discuss the midterm with anyone except the course staff. You can consult your personal course notes and the course website, but all other forms of assistance are prohibited.

— The exam is due **Fri Mar 18 at 5pm**. Submit early/often. *No late submissions will be accepted!* Submit PDF solutions to https://church.cs.virginia.edu/16s-4102/.

— You will be graded on *clarity*, *correctness*, and *precision*.

PROBLEM 1 *Fast Exponentiation*

Given a pair of positive integers (a, n), devise a divide and conquer algorithm that computes a^n using only $\Theta(\log n)$ calls to a multiplication routine. (This method is the core operation of encryption schemes used in HTTPS/SSH/VPN on the internet.)

PROBLEM 2 *Fewer busses*

Design a new North-South bus system that allows any commuter to get from stop i to $j > i$ in at most 3 "hops", but only requires $\Theta(n \log \log n)$ bus segments. Prove that your system satisfies the 3-hop property and has the stated number of segments.

Hint: Consider the recurrence $T(n) = \sqrt{n}T(\sqrt{n}) + \Theta(n)$. How does this suggest you break this problem into smaller ones?

PROBLEM 3 *Best trading period*

You manage a hedge fund. Given an array of numbers $a_1, \ldots, a_n$ that summarizes your fund's daily earnings (or losses), devise a $\Theta(n)$-time algorithm that computes your most lucrative consecutive trading period; i.e., your algorithm must find the pair of days (i, j) where $i \leq j$ that maximizes the sum $s = \sum_{k=i}^{j} a_k$. Output (i, j, s). Describe your algorithm first as a DP equation, then pseudo-code and finally analyze the running time.

PROBLEM 4 *Mashup*

For simplicity, let x and y be *snippets of music* represented as strings over the alphabet of notes $\{A, B, C, \ldots, G\}$. For example, $x = ABC$ and $y = ADEF$. We say that z is a *loop* of x if z is a prefix of x^k for some integer $k > 0$. For example, $z = ABCABCA$ is a loop of x since it is a prefix of $x^3 = ABCABCABC$.

A song s is a *mashup* of loops z and w if it is an interleaving of z and w. For example, one mashup of the loops $z = ABCABCA$ and $w = ADEFADE$ could be $ABCADEABFADECA$. Given song s and snippets (or hooks) of music x, y, devise an algorithm that determines if s is a mashup of loops of x and y.

Describe a dynamic programming solution to this problem as we have done in class by specifying a variable and then providing an equation that relates that variable to smaller instances of itself. Describe an algorithm based on your equation from above. Analyze the running time.