2550 Intro to cybersecurity

L4: Crypto, OWF

abhi shelat

A Network is a public resource.

Goal: add privacy to a public resource.

Perfect secrecy

$(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec}, \mathcal{M}, \mathcal{K})$

is said to be PERFECTLY SECRET if

$$\forall m_1, m_2 \in \mathcal{M}, \forall c$$

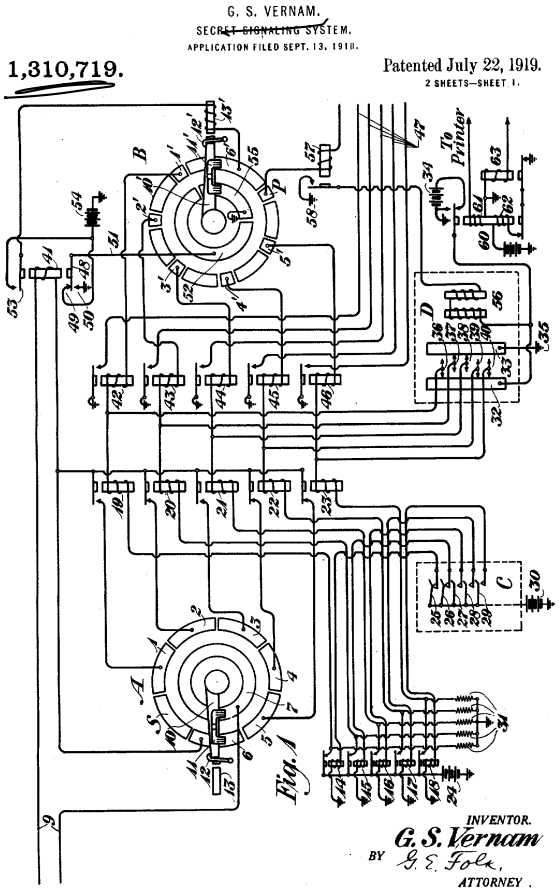
$$\Pr[k \leftarrow \mathsf{Gen} \ : \ \mathsf{Enc}_k(m_1) = c]$$

$$=$$

$$\Pr[k \leftarrow \mathsf{Gen} \ : \ \mathsf{Enc}_k(m_2) = c]$$

One-time pad (Vernam 1917)

$$
\begin{aligned}
\mathcal{M} &= \{0,1\}^n \\
\mathcal{K} &= \{0,1\}^n \\
\mathsf{Gen} &= k = k_1 k_2 \ldots k_n \leftarrow \{0,1\}^n \\
Enc_k(m_1 m_2 \ldots m_n) &= c_1 c_2 \ldots c_n \text{ where } c_i = m_i \oplus k_i \\
Dec_k(c_1 c_2 \ldots c_n) &= m_1 m_2 \ldots m_n \text{ where } m_i = c_i \oplus k_i
\end{aligned}
$$



ABCDEFGHIJKLMNOPQRSTUVWXYZ
01234567890123456789012345

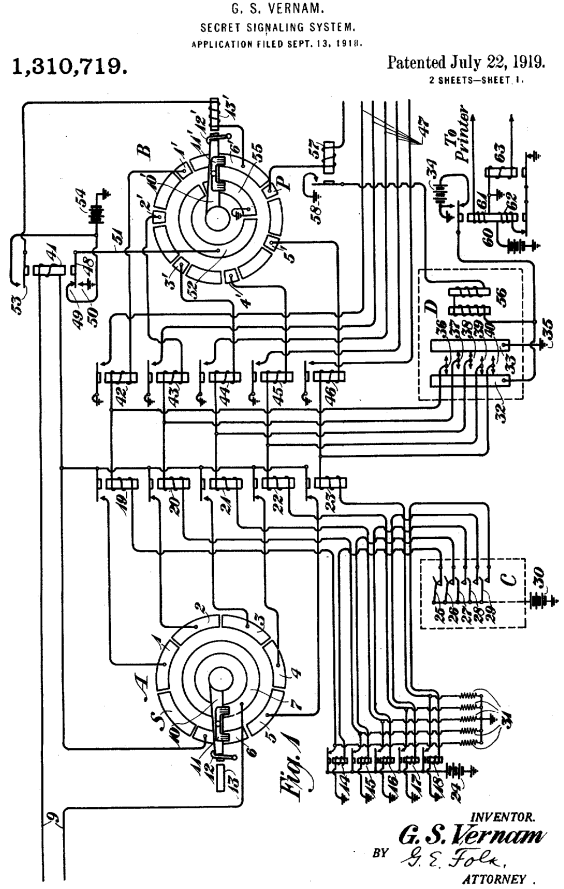
8 19 22

M = ITWASTHEBESTOFTIMES

14 18 13

K = OSNEKHRJFUTGXMGALEQ

WLJ...



ABCDEFGHIJKLMNOPQRSTUVWXYZ
0123456789012345678901234 5

ITWASTHEBESTOFTIMES

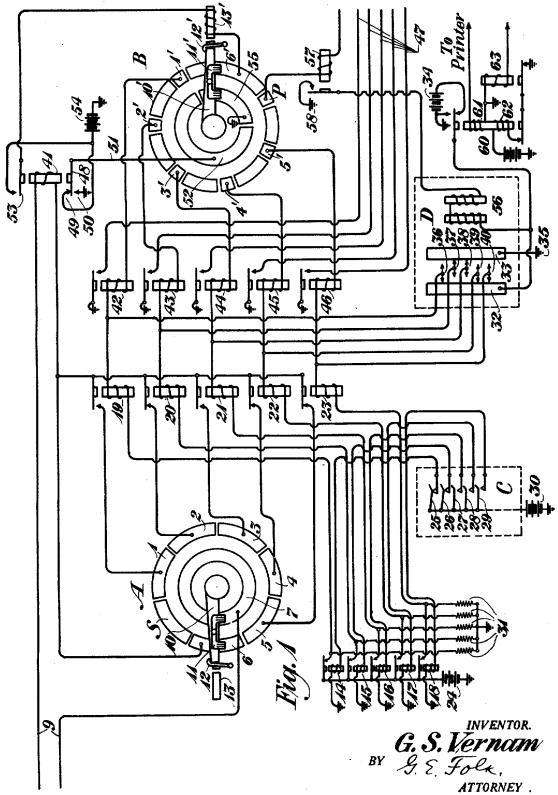
OSNEKHRJFUTGXMGALEQ

WLJECAYNGYLZLRZIXII

G. S. VERNAM.
SECRET SIGNALING SYSTEM.
APPLICATION FILED SEPT. 13, 1918.

1,310,719. Patented July 22, 1919.

2 SHEETS—SHEET 1.



INVENTOR.
G. S. Vernam
BY G. E. Folk,
ATTORNEY.

ABCDEFGHIJKLMNOPQRSTUVWXYZ
0123456789012345678901234 5

12 1 6
C = MBGZBWSNDHCVCGFN

K MVCLWOFLMDZBRYMP~~DVEE~~

M = AGE...

One-time pad (Vernam 1917)

- key size has to be equal to msg length
- hard to implement key mgmt.

$$
\begin{aligned}
\mathcal{M} &= \{0,1\}^n \\
\mathcal{K} &= \{0,1\}^n \\
\mathsf{Gen} &= k = k_1 k_2 \ldots k_n \leftarrow \{0,1\}^n \\
Enc_k(m_1 m_2 \ldots m_n) &= c_1 c_2 \ldots c_n \text{ where } c_i = m_i \oplus k_i \\
Dec_k(c_1 c_2 \ldots c_n) &= m_1 m_2 \ldots m_n \text{ where } m_i = c_i \oplus k_i
\end{aligned}
$$

- Shannon shows this is unavoidable for perfect security.

FAILURE CASES: ① key runs out. what to do??
⇒ key reuse, which breaks security

Uniform distribution on strings of len n

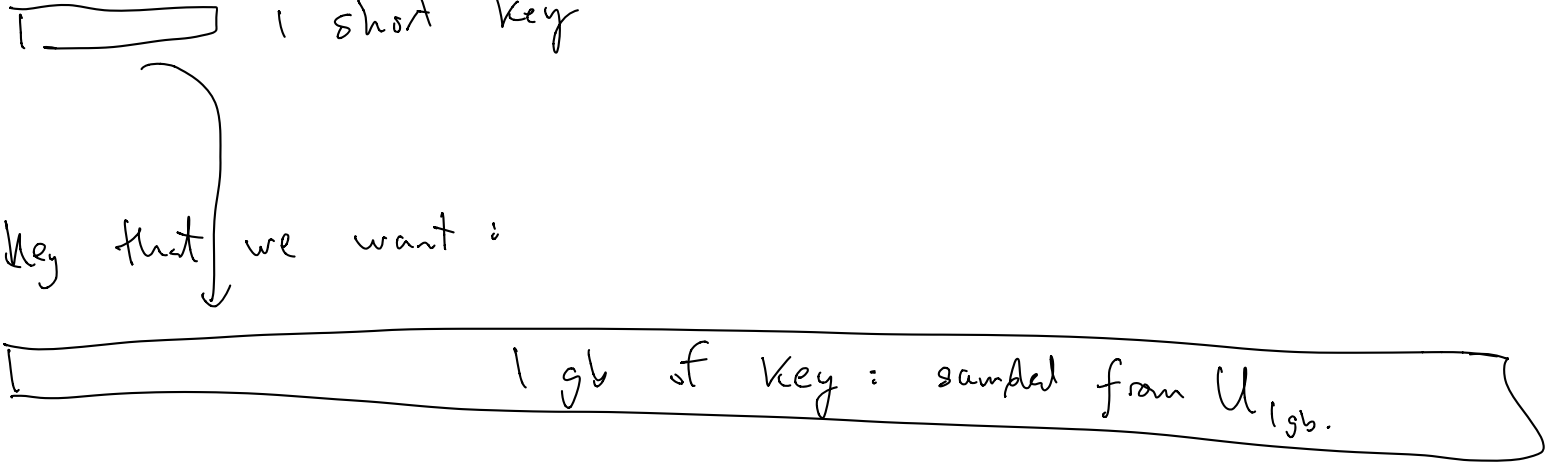
$$K = U_n = \{t \leftarrow \{0,1\}^n\}$$

random strings of length n.

- the keys in the onetime pad are drawn from U_n.

Goal: 1 key to many keys

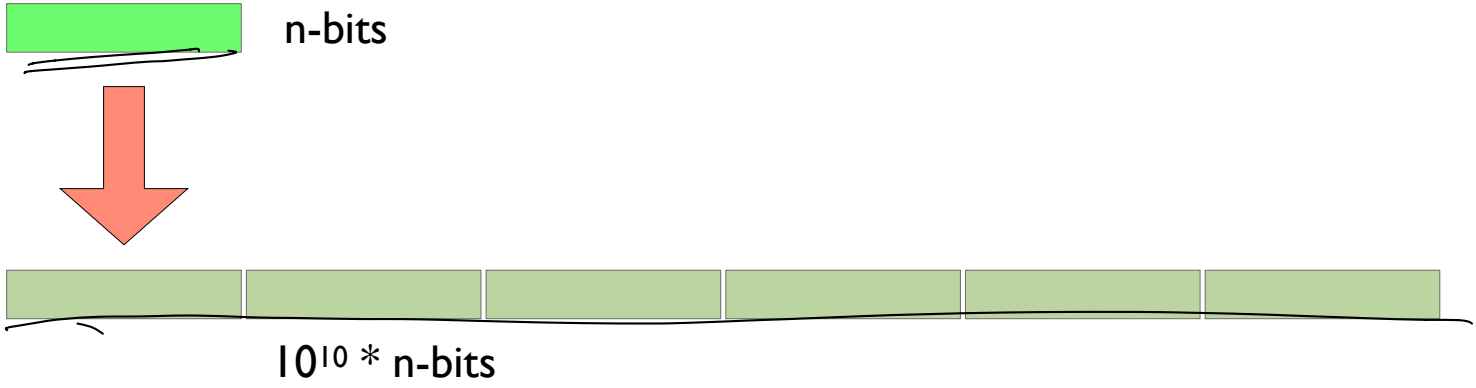
1 short key ⇒ produce many long keys.



Goal: 1 key to many keys

10^{10} * n-bits

τ should look like U_n.



This is the idea behind a stream cipher.

Stream cipher

Gen: pick an n-bit binary string k

Enc(k,m): Output G(k) + m

Dec(k,m): Output G(k) + m

what security properties are needed for this to work?

① the output of our generator should appear uniformly random...

Vigenere cipher

```
ABCDEFGHIJKLMNOPQRSTUVWXYZ
01234567890123456789012345
```

MSG: THEMODERNSTUDYOF ...

$\oplus$

KEY: ABHIABHIABHIABHIABHIABHI

ciphertext: TILUOELZNTACDZVN...

Insecure b/c key is not random !!

10^{10} * n-bits

Key

G

Pseudo-random generator.

stream

10^{10} * n-bits

should appear to be the same as a random string $\{0,1\}^{10^{10}n}$

$$U_{10^{10}n}$$

what does it mean for a process $\{X\}$ to be pseudo-random?

$$\{X_n\}_{n \in \mathbb{N}}$$

$$x \leftarrow X_n$$

$$\{X_n\}_{n\in\mathbb{N}}$$

$$x \leftarrow X_n$$

$\{U_n\}_{n\in\mathbb{N}}$

random process

roughly same # of 0s and 1s $\rightarrow$ 0101010101 ..

same # of 00 and 11, 01, 10 $\rightarrow$ 00100111|0010

# of 000, 001, 010 ... , 111

$$\{X_n\}_{n \in \mathbb{N}}$$

$$x \leftarrow X_n$$

roughly same # of 0s and 1s

roughly same # of 00s and 11s

$$\{X_n\}_{n \in \mathbb{N}}$$

$$x \leftarrow X_n$$

roughly same # of 0s and 1s

roughly same # of 00s and 11s

roughly same # of any pattern

"any efficient statistical test should pass on both $\{U_n\}$ and $\{X_n\}$"

$$\{X_n\}_{n\in\mathbb{N}} \quad \text{and} \quad \{U_n\}$$

$$x \leftarrow X_n$$

ARE

INDISTINGUISHABLE

TO THE

ADVERSARY.

roughly same # of 0s and 1s

roughly same # of 00s and 11s

roughly same # of any pattern

given any prefix, hard to guess next bit

{ Computational

"Indistinguishability" }

provides a precise way of formulating pseudo-randomness

next slide has 2 pics

are they the same
or different?

same or different?

twice the time.

same or different?

same or different?

twice the time.

same or different?

same or different?

twice the time.

same or different?

lesson:

Ability to answer correctly...

Depends on the amount of computational power that you have

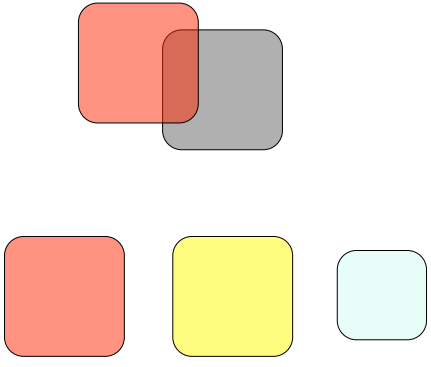
http://i.telegraph.co.uk/telegraph/multimedia/archive/01553/toss_1553792c.jpg

NEW PROBLEM:

consider all drawings consisting of boxes.

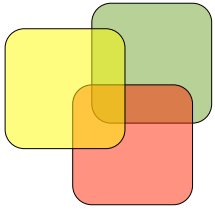
evens

# of boxes that overlap another box is even



odds

of ... is odd

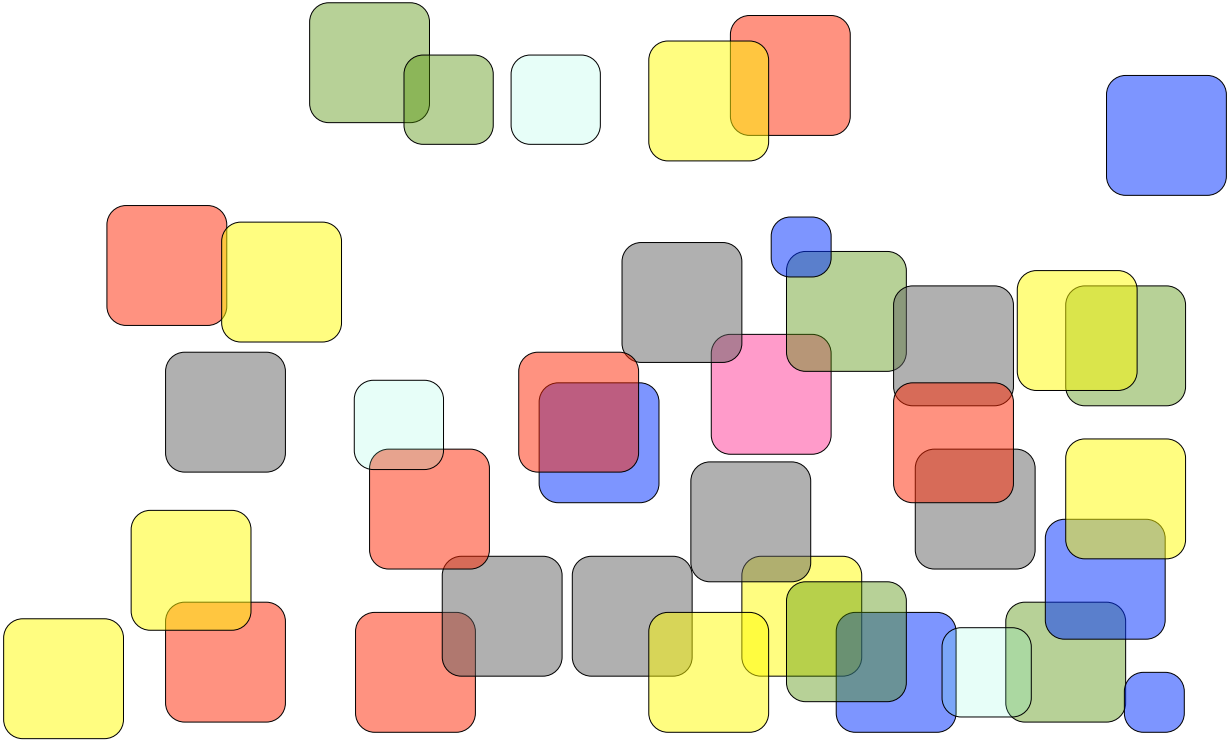


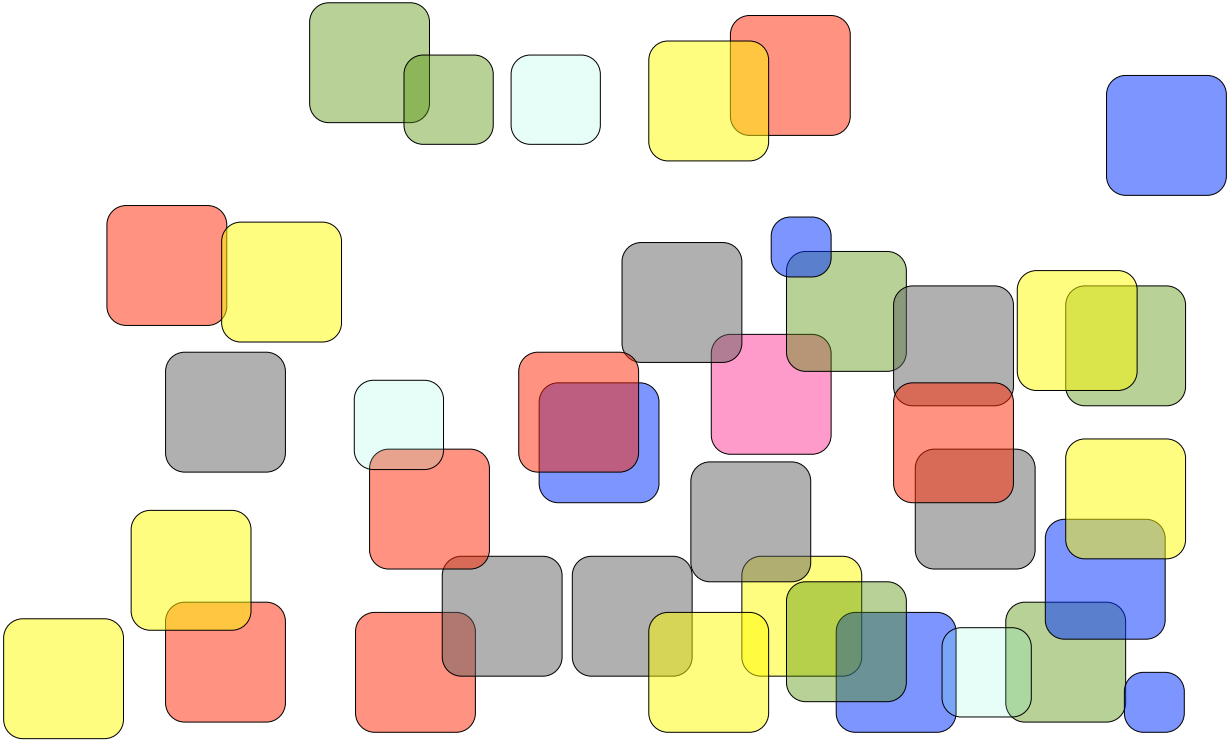
GAME:

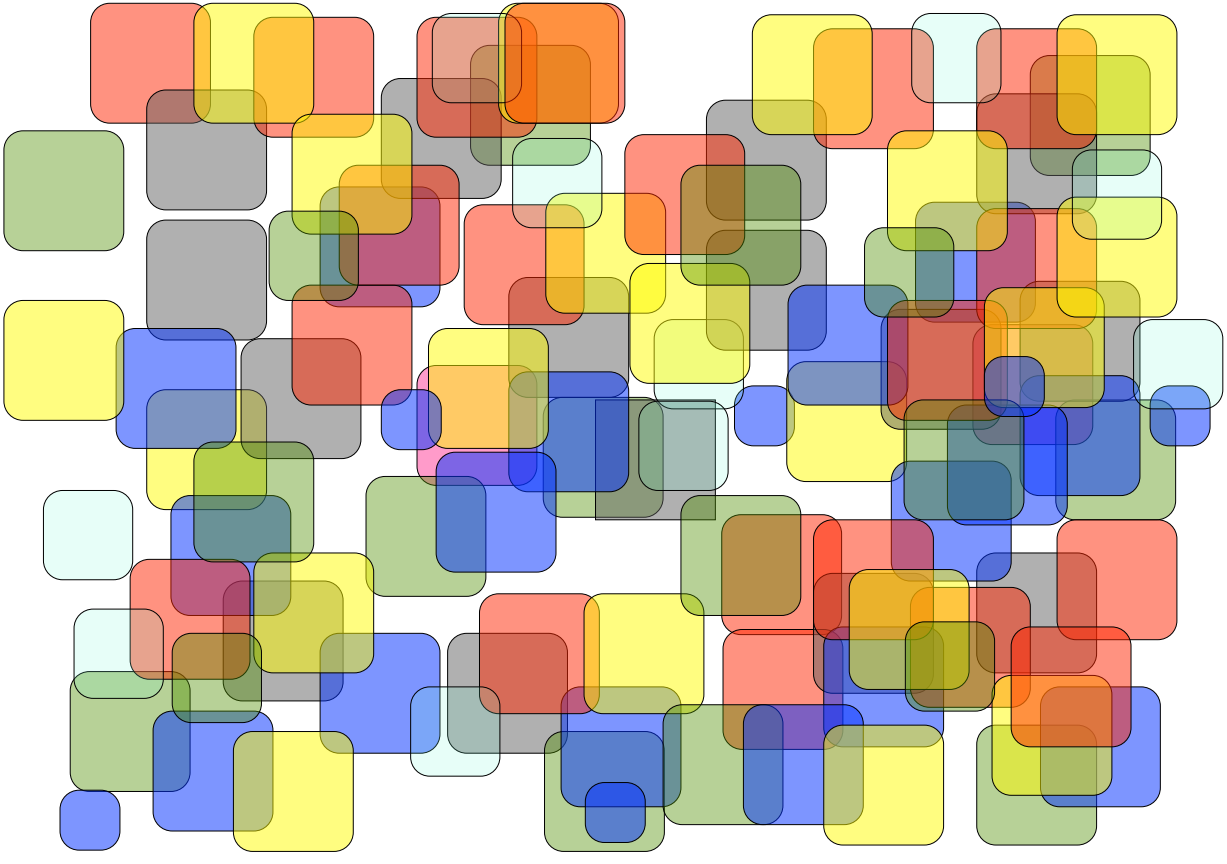
I will pick a sample from either evens or odds, and you will have to guess which one.

READY?

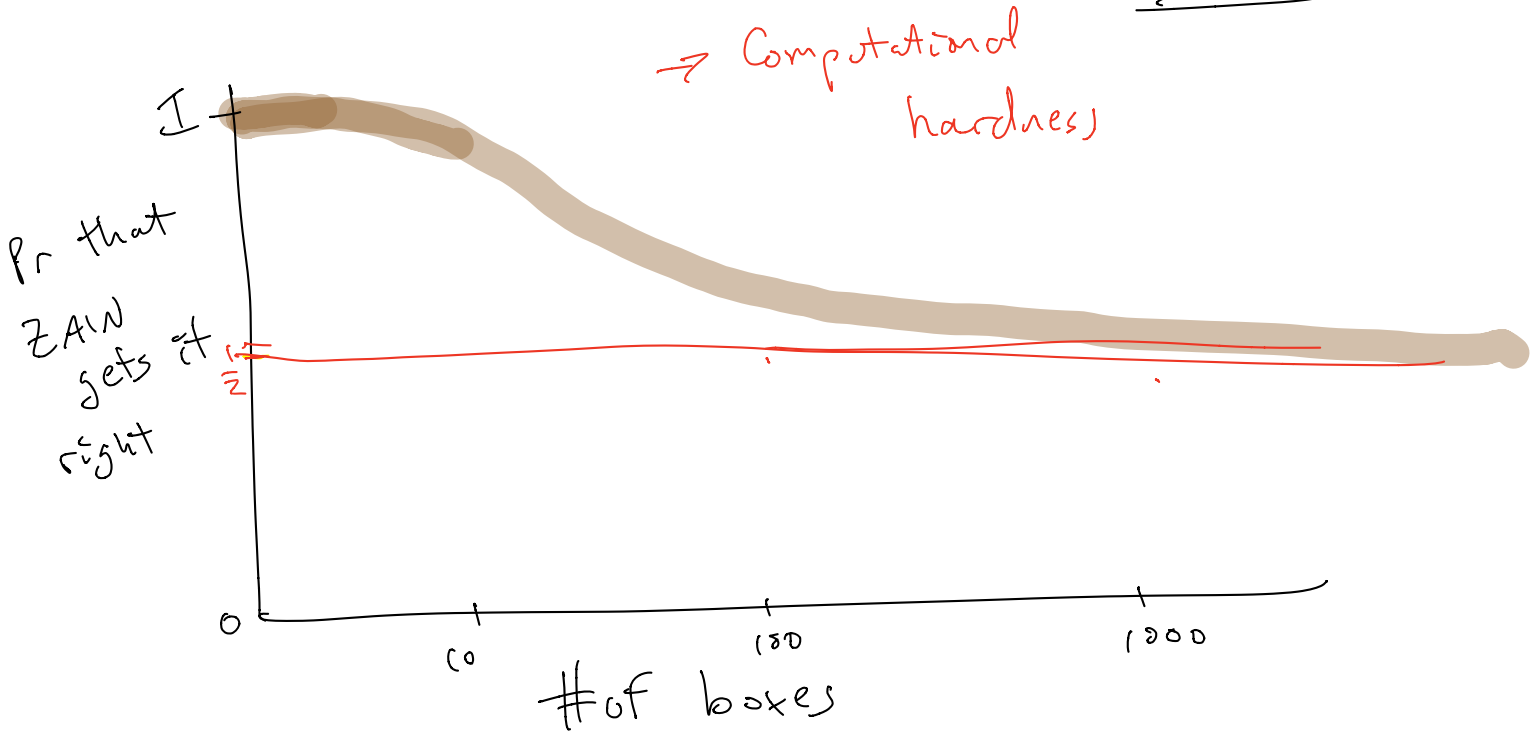
READY?







This game is parameterized by its size: i.e, # of boxes.



This game is parameterized by its size: i.e, # of boxes.

evens_n odds_n

As the game size increases, it becomes intractable to distinguish b/w evens and odd

parameterized experiment

$$\underbrace{\{X_n\}_{n\in\mathbb{N}}}$$

a sequence of probability distributions
where X_n is a distribution over strings of length
$\ell(n)$

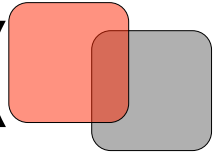
ensembles

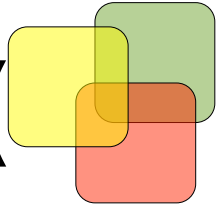
Computational Indistinguishability

"let there be two parameterized experiments, *X* and *Y*. as the experiment size increases, no p.p.t. algorithm *D* succeeds in distinguishing *X* from *Y*."

probabilistic polynomial time } "efficient"

what does it mean
for an algorithm D to
distinguish a sample?

D()= “evens”

D()= “odds”

Two ensembles are comp. indistinguishable

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$$\{X_n\}_{n \in N} \approx \{Y_n\}_{n \in N}$$

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$$\{X_n\}_{n \in N} \approx \{Y_n\}_{n \in N}$$

if for all non-uniform p.p.t. alg D,
there exists a negligible function $\epsilon(n)$
such that for all n

Two ensembles are comp. indistinguishable

formally

$$\{X_n\}_{n\in N} \approx \{Y_n\}_{n\in N}$$

if for all non-uniform p.p.t. alg D,
there exists a negligible function $\epsilon(n)$
such that for all n

$$|\Pr[t \leftarrow X_n, D(t) = 1] - \Pr[t \leftarrow Y_n, D(t) = 1]| \leq \epsilon(n).$$

what does it mean
for a process $\{x\}$ to be
pseudo-random?

$$\{X_n\}_{n \in \mathbb{N}}$$

$$x \leftarrow X_n$$

$$\{X_n\}_{n \in \mathbb{N}}$$

$$x \leftarrow X_n$$

roughly same # of 0s and 1s

$$\{X_n\}_{n \in \mathbb{N}}$$

$$x \leftarrow X_n$$

roughly same # of 0s and 1s

roughly same # of 00s and 11s

$$\{X_n\}_{n \in \mathbb{N}}$$

$$x \leftarrow X_n$$

roughly same # of 0s and 1s

roughly same # of 00s and 11s

roughly same # of any pattern

$$\{X_n\}_{n \in \mathbb{N}}$$

$$x \leftarrow X_n$$

roughly same # of 0s and 1s

roughly same # of 00s and 11s

roughly same # of any pattern

given any prefix, hard to guess next bit

indistinguishability provides a precise way of formulating pseudo-randomness

pseudo-random

An ensemble {X} is said to be

pseudo-random

pseudo-random

if

$$\{X\}_{n\in\mathbb{N}} \approx \{U_n\}_{n\in\mathbb{N}}$$

An ensemble {X} is said to be

pseudo-random

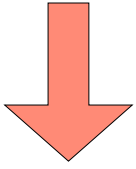
if

$$\{X\}_{n\in\mathbb{N}} \approx \{U_n\}_{n\in\mathbb{N}}$$

Original goal

small key

n-bits



large one-time pad.

10^{10} * n-bits

indistinguishable from a random key.

Pseudo-random generator

A function $G : \{0,1\}^* \to \{0,1\}^*$

is a pseudo-random generator if

G can be computed in p.p.t.

$|G(x)| > \ell(|x|)$ for some $\ell(y) > y$

$\{x \leftarrow U_n \ : \ G(x)\}_{n \in \mathbb{N}}$ is pseudo-random

How can we build pseudo-random generators?

7

$$7 * 7 =$$

49

$$\underbrace{7 \cdot 7 \cdots 7}_{x \text{ times}}$$

$$7^x$$

7^{1000}

1253256639965718318107554832382734206164985075080986171463495007520970596317381164324488390543515207631986159195515940766858289
8946726302276179083827085457983001511124666120398462435892983257161571801470409630566809750761327366302322689525054138592715842
6088684494082416768617708189592286936039922311125683719215046689156738352590137241554510185855964549927575493247391132548534378
4979788060849510858742020118363623157274201095547829887915300882897118445505002304856384131899471321422439473341992593007356224
9293741945365006149030210512792031443040163685567754913633748132181134967842707609143734504539933734861126116805592935540299282
3192491190360027036112283180935872775214517464013178274657100736321564606838252739601156414628445543663144696050650160812621814
3270626661951727017802002866450238230831859280613713103008292840711412077312806000001

7^{10000}

95587288567444097027721621933777274477156508800837670232809747736882802688984737969351872846915678176303391547032418309440558367061778809896056537787
80399681208530465536899923872083728356691350372430337696619926639213646136205780780799564658253882535717364116449482061428736198739029989324013720616
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91762387934893989942083292678611416543611768850763424813437908028375226699829575967165231253786105772454128891004817031720210914641548363847541833079
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WolframAlpha computational knowledge engine

7^100000000000000000

Examples Random

Input:

$$7^{100\,000\,000\,000\,000\,000}$$

Power of 10 representation:

$$10^{10^{10^{1.228577610529823}}}$$

Number length:

84 509 804 001 425 684 decimal digits

$\approx 8.45098 \times 10^{16}$ digits

Last few digits:

...0000000001

Computed by Wolfram *Mathematica*

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Inverse problem is EASY:

Given: 1231928

Find X such that $7^X = 1231928$

logarithm, base 7 of

7.206957107766616386323701465694329662925565456980879196705 1

```
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```

$$f(x) = 7^x$$

Easy

$$X \qquad\qquad Y$$

Also easy

$$f^{-1}(Y)$$

log base 7

$$7^x$$

$$\underbrace{7 \cdot 7 \cdots 7}_{x \text{ times}}$$

$$f(x) = 7^x \bmod p$$

$$7^2 = \underbrace{7 \cdot 7}_{49} \mod \underline{131}$$

$49 \mod 131 = 49$

$$7^2 = 7 \cdot 7 \mod 131$$

$$= 49$$

$$7^3 = \underbrace{49 \cdot 7}_{} \bmod 131$$

$$131 \overline{)343} \bmod 131$$

2, r. 81

262

81

$$7^3 = 49 \cdot 7 \mod 131$$

$$\begin{array}{r|l} & 2 \\ \hline 131 & 343 \\ & 262 \\ \hline & 81 \end{array} \mod 131$$

$$7^3 = 49 \cdot 7 \mod 131$$

$$\begin{array}{r|l} & 2 \\ \hline 131 & 343 \\ & 262 \\ \hline & 81 \end{array} \mod 131$$

$7^4 = 81 \cdot 7$

$$= 81 \mod 131$$

Pick a large prime number, ~200 digits long

p=

```
52061717268811022582985085340962890043057948799610254543088975844032721944045806199348262783479007502882378061899682808553821447440104558871825720697952817636569237467061923678480735550795549941664404569955484 99
```

$$(7, x, p) \rightarrow 7^x \bmod p$$

Remarkably easy

Assailing U.S. and Kiev, Putin Keeps Open Option of Force, By STEVEN LEE MYERS, ELLEN BARRY and ALAN COWELL 56 minutes ago, President Vladimir V. Putin of Russia on Tuesday described events in Ukraine as an unconstitutional coup, expressed contempt toward the United States and said that any potential use of military force would be a last resort.

$7^x \bmod p$

```
sage:
power_mod(7,661615980609061103328546834632981100327606021844328118170610327602021315328013021032801317061210331202327112150001443267213283846986697832766969327789698283443269767669783266658282893298110032657665783267798769767632535433100611181702153298041144328115021606010211163287089801061006143286463281181706103312023283181616059733121032851802160098213301021600150599020033021902111715330610328608149806110132981532981033181100121116170617181706121098083300121812443302211315021616020033001211170210131633171219981500331705013286110617020032841698170215329811003315980600331704981632981121331312170211170598083318160133120233100609061698152133031215000133201218090032990132973308981616331502161215164 6,p)

497929711688584217527863332300629329098654983269423357729595289026203941126793387158661751180509701235256633429962635553676002083869590769727522267027725835398089223356845936769316025070772103626851899346393242 9
```

$Y = 12345$

$$(7, p, Y) \rightarrow x$$

such that $7^x \bmod p = Y$

discrete logarithm

Incredibly hard!

World record in discrete logarithms in GF(p)

232 digits (768 bits)

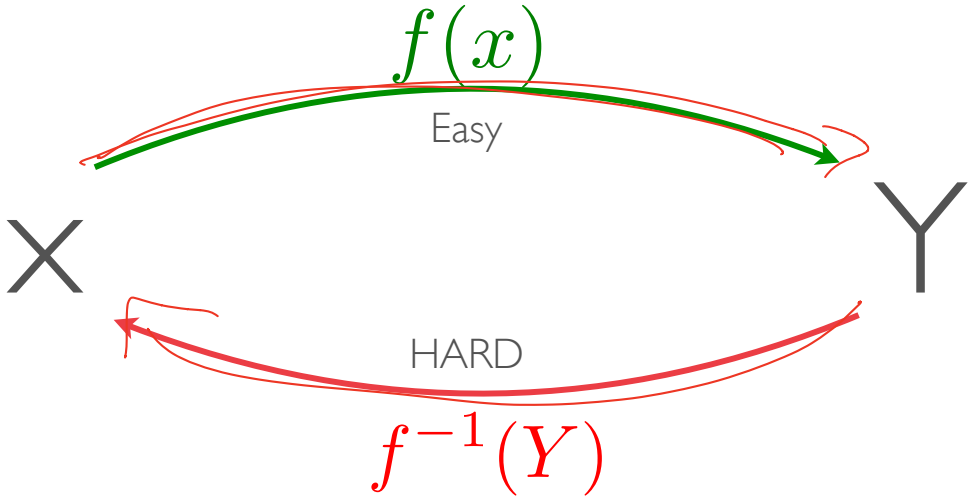
6600yrs of CPU time

Intel Xeon E5-2660 at 2.2 GHz

2016

Thorsten Kleinjung, Claus Diem, Arjen K. Lenstra, Christine Priplata, and Colin Stahlke

$$X \xrightarrow[7^x \bmod p = Y]{f(x)} Y$$



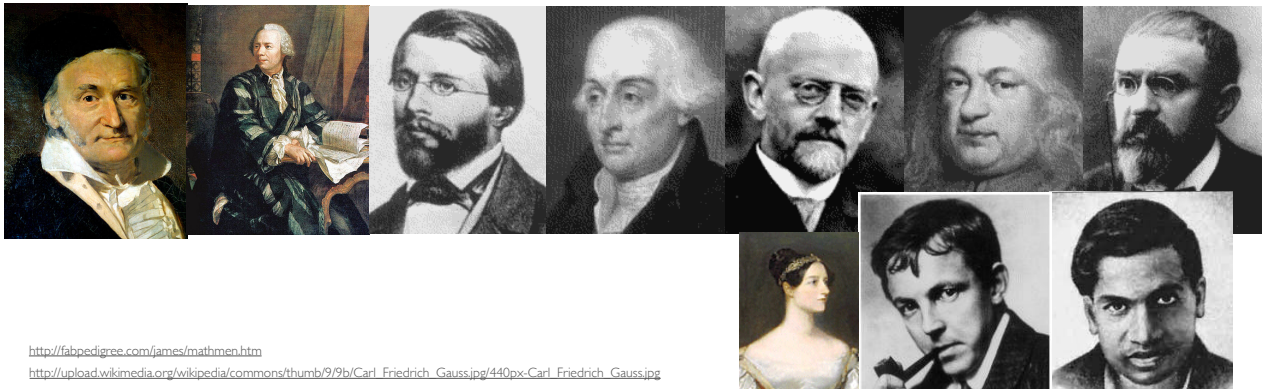
compute an x such that $7^x = Y \bmod p$

http://www.fitzmuseum.cam.ac.uk/gallery/chinesevases/sortingfragments.html

$7^x \bmod p.$

$$X \xrightarrow[\text{Easy}]{f(x)} Y$$

$$X \xleftarrow[\text{HARD}]{f^{-1}(Y)} Y$$



http://fabpedigree.com/james/mathmen.htm

http://upload.wikimedia.org/wikipedia/commons/thumb/9/9b/Carl_Friedrich_Gauss.jpg/440px-Carl_Friedrich_Gauss.jpg