L11 5800

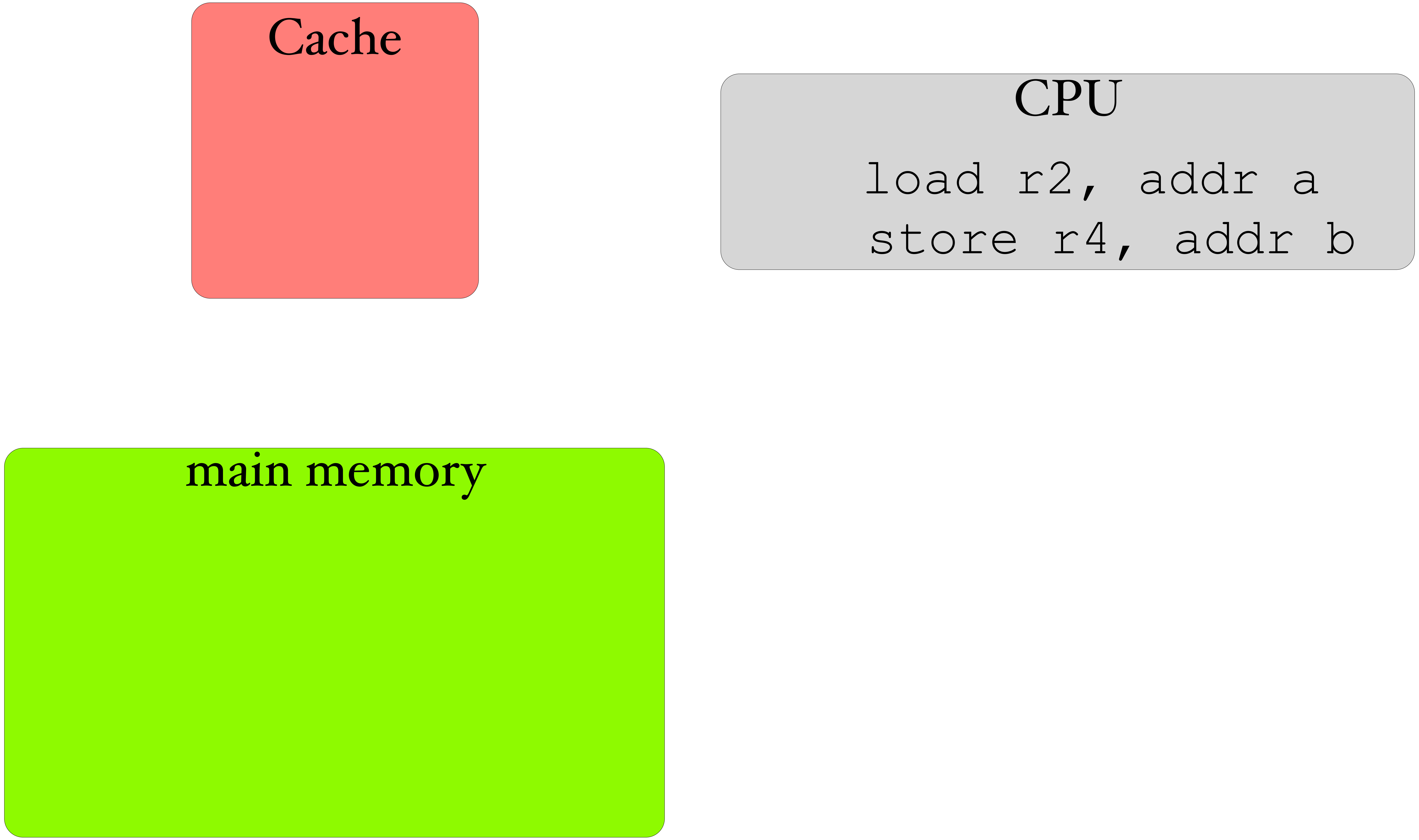
feb 27 2022

shelat

Greedy is only good for certain problems

caching

cache hit



question:

question:

How do we manage a fully-associate cache?

When it is full, which element do we replace?

problem statement

input:

output:

cache is

problem statement

input: K, the size of the cache
$d_1, d_2, \ldots, d_m$ memory accesses

output: schedule for that cache that minimizes # of cache misses while satisfying requests

cache is fully associative, line size is 1

contrast with reality

contrast with reality

In a real situation, we may not know the future memory access patterns.

Some caches have additional restrictions, like line-size, associativity, etc.

However, this algorithm can still be used to compare a real-world algorithm against the optimum cache miss rate possible.

Belady eviction rule

Belady eviction rule

Replace the element in the cache that is accessed "farthest into the future"

example

cache

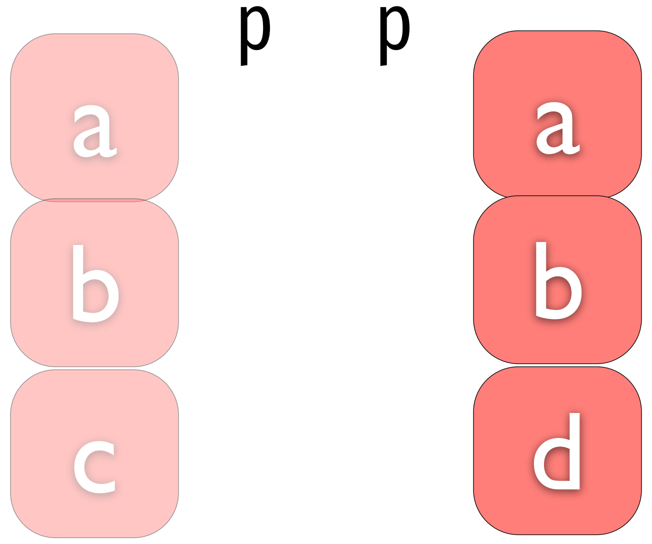
a
b
c

a b c d a d e a d b a e c e a

example

Cache operations: nop nop nop Evict c for d.

cache



Memory accesses: a b c d a d e a d b a e c e a

example

Cache operations:	nop	Evict (c,d)	Evict (b,e)
cache	a	a	a
	b	b	e
	c	d	d

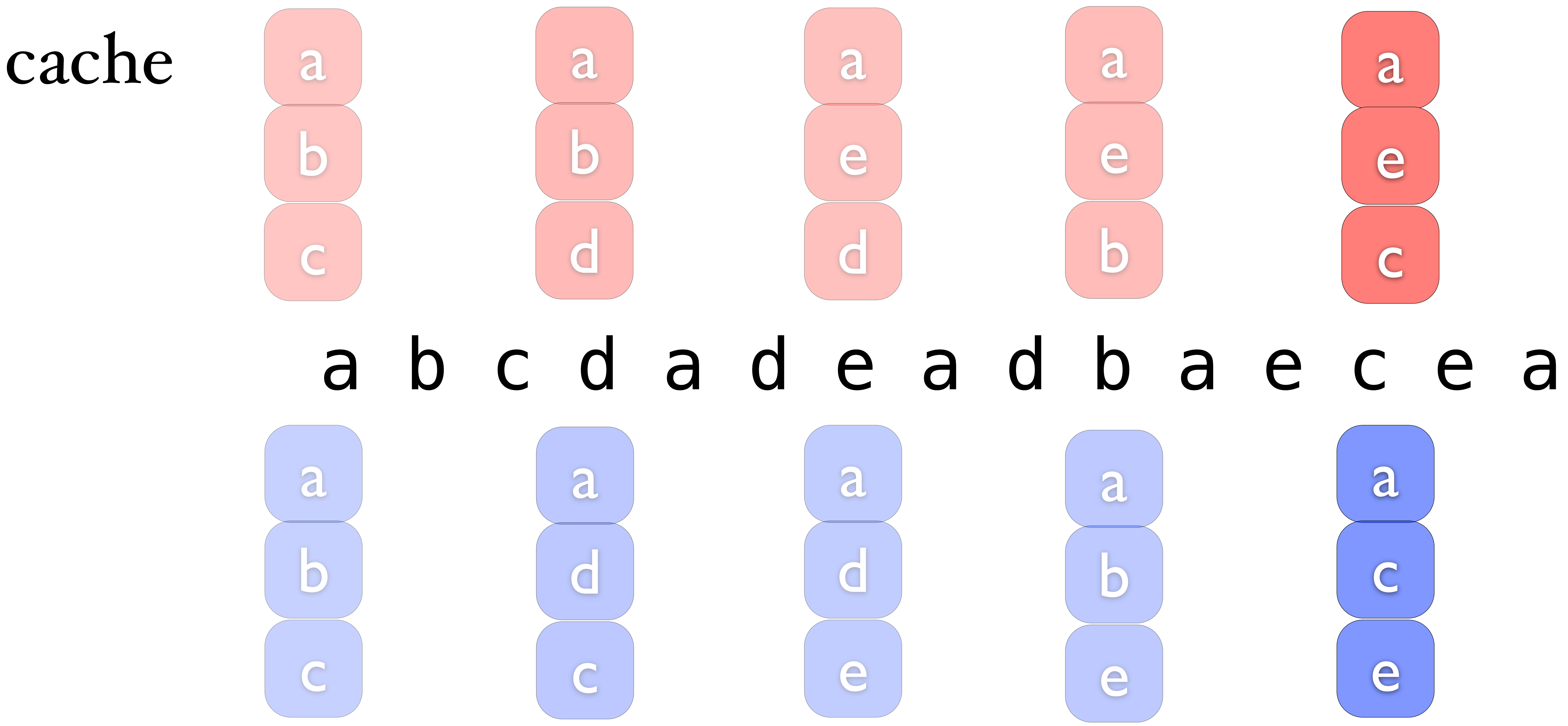
Memory accesses: a b c d a d e a d b a e c e a

example

Cache operations:	nop	Evict (c,d)	Evict (b,e)	Evict (d,b)
cache	a	a	a	a
	b	b	e	e
	c	d	d	b

Memory accesses: a b c d a d e a d b a e c e a

example



Here is an alternate optimal set of cache operations.

Surprising theorem

Surprising theorem

The schedule S_{ff} produced by the Belady “farthest in the future” eviction rule is optimal.

schedule

Schedule for access pattern $d_1, d_2, \ldots, d_n$:

Reduced schedule:

schedule

Schedule for access pattern $d_1,d_2,\ldots,d_n$:

A list of instructions for each access that is either "NOP" or "evict x for y"

Reduced schedule:

schedule

Schedule for access pattern $d_1, d_2, \ldots, d_n$:

A list of instructions for each access that is either “NOP” or “evict x for y”

Reduced schedule:

A schedule in which“evict x for y” instruction only occurs when y is accessed.

schedule

Schedule for access pattern $d_1, d_2, \ldots, d_n$:

A list of instructions for each access that is either "NOP" or "evict x for y"

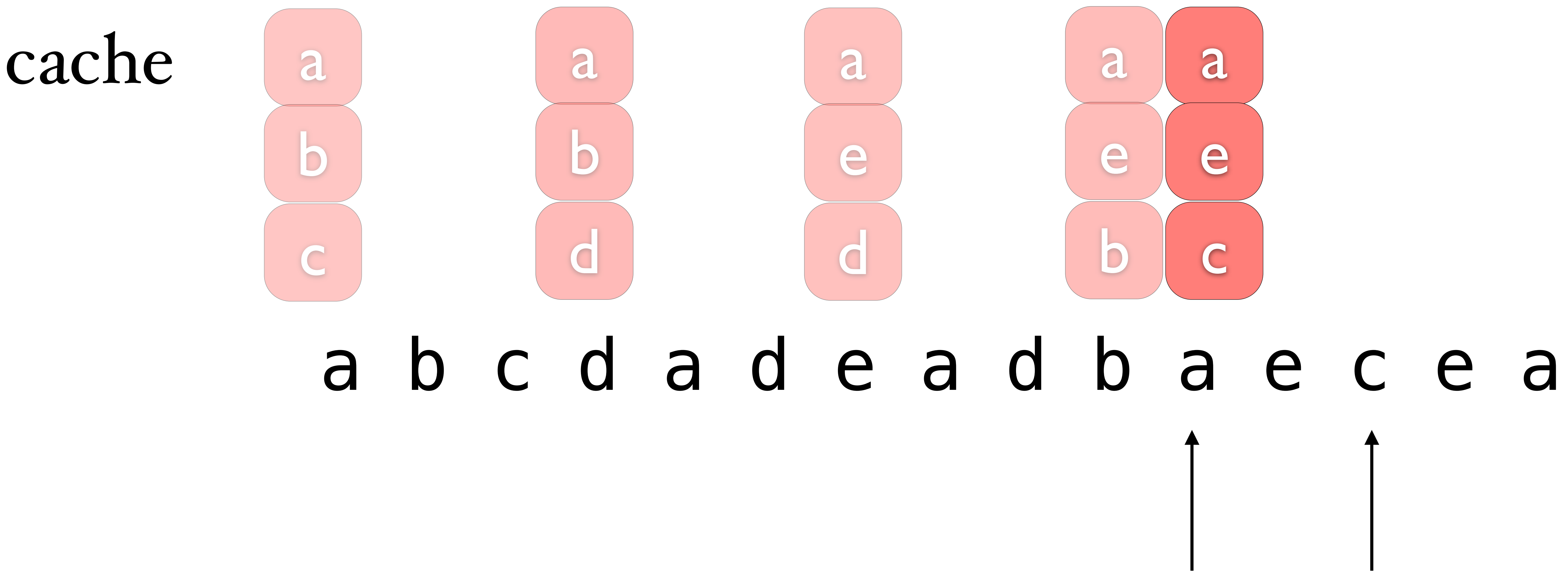
Reduced schedule:

A schedule in which"evict x for y" instruction only occurs when y is accessed.

Note: any schedule can be transformed into a reduced schedule with the same or fewer cache misses.

(Idea: starting at the end, defer "evict…t" until y is read)

Non-Reduced Schedule example



Example of a non-reduced schedule.
At this point, the cache evicts (b,c) when “a” is being accessed. It is possible to delay this eviction until “c” is accessed, thereby leading to a reduced schedule.

Exchange lemma

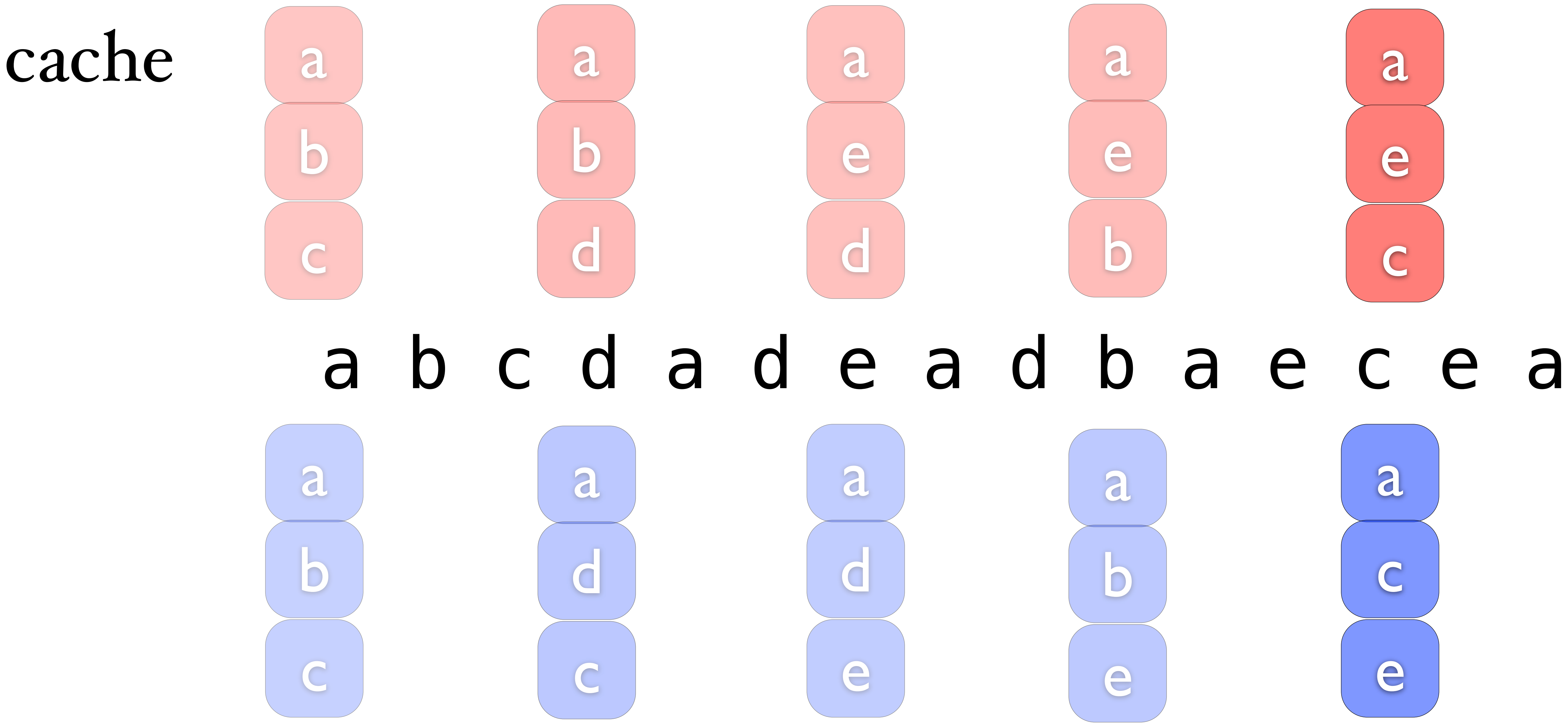
Exchange lemma

Let S be a reduced schedule that agrees with S_{ff} on the first j accesses.

Then there exists a schedule S' that agrees with S_{ff} on the first j+1 accesses and has the same or fewer misses.

What does it mean for 2 schedules to agree?

A schedule is a sequence of cache instructions:
NOP,NOP,NOP,evict(c,d),NOP,NOP,...



For example, these two schedules agree on the first three operations.

Some optimal schedule.

S^*

S_{ff}

Some optimal schedule.

S^* S_1 S_{ff}

Agrees with S_{ff} on the first access. Can be constructed by applying the Lemma to S^* which agrees on 0 accesses.

Some optimal schedule.

S^* S_1 S_2 S_{ff}

Agrees with S_{ff} on the first access. Can be constructed by applying the Lemma to S^* which agrees on 0 accesses.

Agrees with S_{ff} on the first two accesses.

Some optimal schedule.

S^* S_1 S_2 S_3 S_{n-1} S_{ff}

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Agrees with S_{ff} on the first two accesses.

Agrees with S_{ff} on the first three accesses.

S_{ff} has the same number of cache misses as S^*.

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S_{ff} has the same number of cache misses as S^*.

$$miss(S^*) \geq miss(S_1) \geq miss(S_2) \geq \cdots \geq miss(S_n)$$

Some optimal schedule.

S^*

S_{ff}

Since S^* is optimal, this means that all of these relations need to be equality.

This also means the S_{ff} is therefore optimal.

Some optimal schedule.

S^*

S_{ff}

$$miss(S^*) \geq miss(S_1) \geq miss(S_2) \geq \cdots \geq miss(S_n) = miss(S_{ff})$$

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Proof of Lemma

Let S be a reduced sched that agrees with S_{ff} on the first j items. There exists a reduced sched **S'** that agrees with S_{ff} on the first j+1 items and has the same or fewer #misses as S.

Proof of Lemma

Let S be a reduced sched that agrees with S_{ff} on the first j items.
There exists a reduced sched **S'** that agrees with S_{ff} on the first j+1 items and has the same or fewer #misses as S.

At time j, both S and S_{ff} have the same state.
Let d be the element accessed at time j+1.

Proof of lemma

State of the cache after J operations under the two schedules.

easy case 1

Proof of lemma

State of the cache after J operations under the two schedules.

easy case 1 d is in the cache.

Proof of lemma

State of the cache after J operations under the two schedules.

easy case 1 d is in the cache.

Both S and S_{ff} agree since both do NOPs at j+1.

Proof of lemma

State of the cache after J operations under the two schedules.

easy case 2

Proof of lemma

State of the cache after J operations under the two schedules.

easy case 2 d is not in the cache, but both schedules “evict e for d.”

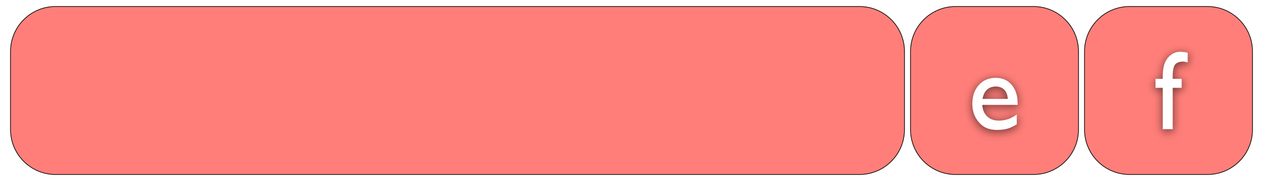
Proof of lemma

State of the cache after J operations under the two schedules.

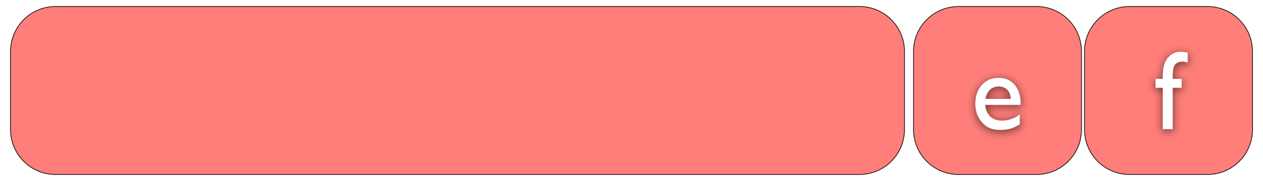
easy case 2 d is not in the cache, but both schedules “evict e for d.”

Both S and S_{ff} agree at j+1.

Proof of lemma



S



S_{ff}

case 3

Proof of lemma

case 3 S does evict(d,e), and S_{ff} does evict(f,e)

Proof of lemma

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The state of the cache after this operation:

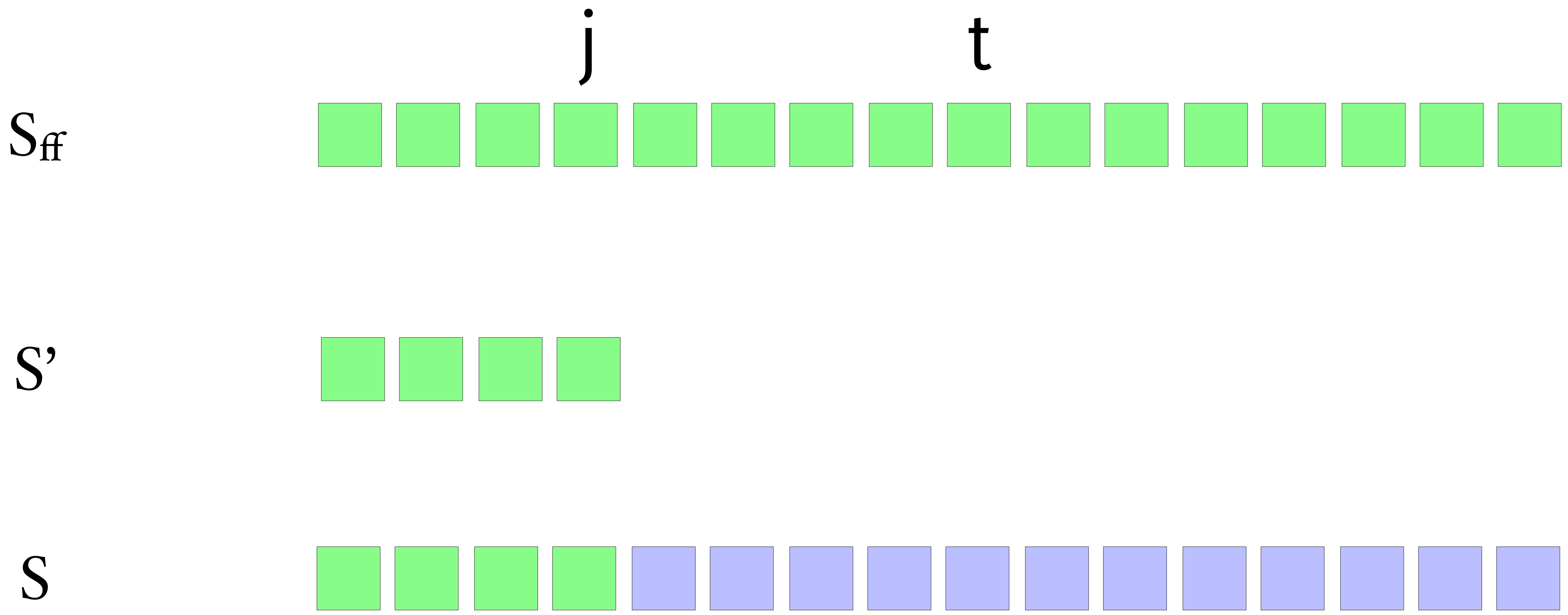
Proof of lemma

case 3 S does evict(d,e), and S_{ff} does evict(f,e)

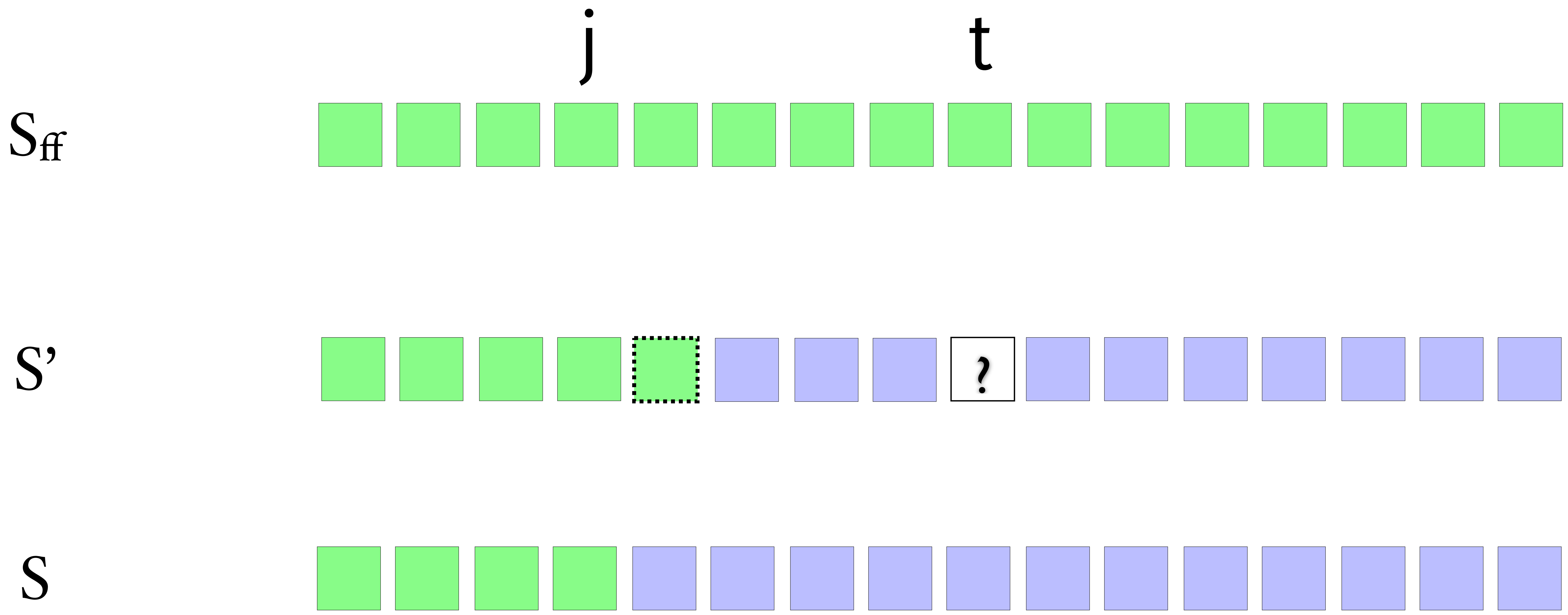
The state of the cache after this operation:

Challenge: the lemma requires us to find some schedule S' that agrees with S_{ff} and has the same or fewer misses as S.

Timeline

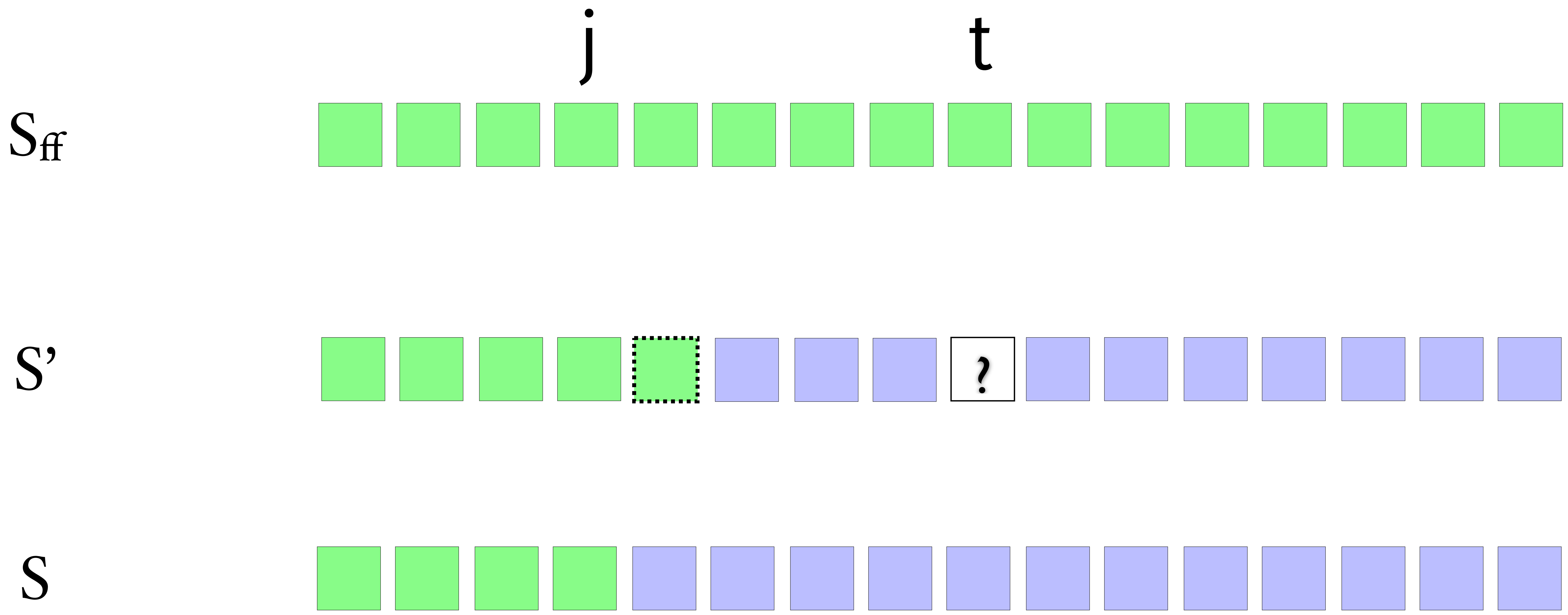


Timeline



Copy j+1 from S_{ff}. Then copy from S until t (the first time that either e or f are involved). Then copy from S until the end.

Timeline



Copy j+1 from S_{ff}. Then copy from S until t (the first time that either e or f are involved). Then copy from S until the end.

Challenge: Argue that S’ has the same misses as S.

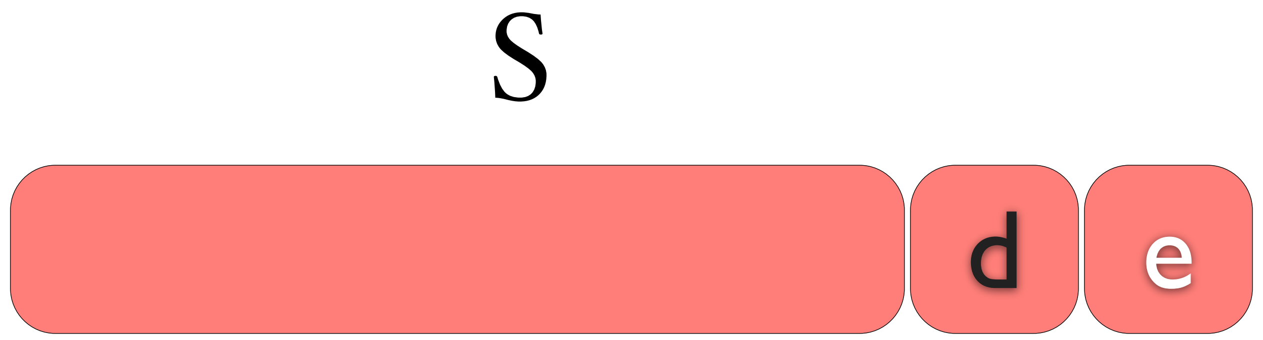
Proof of lemma

Let t be the first access that either e or f are involved.

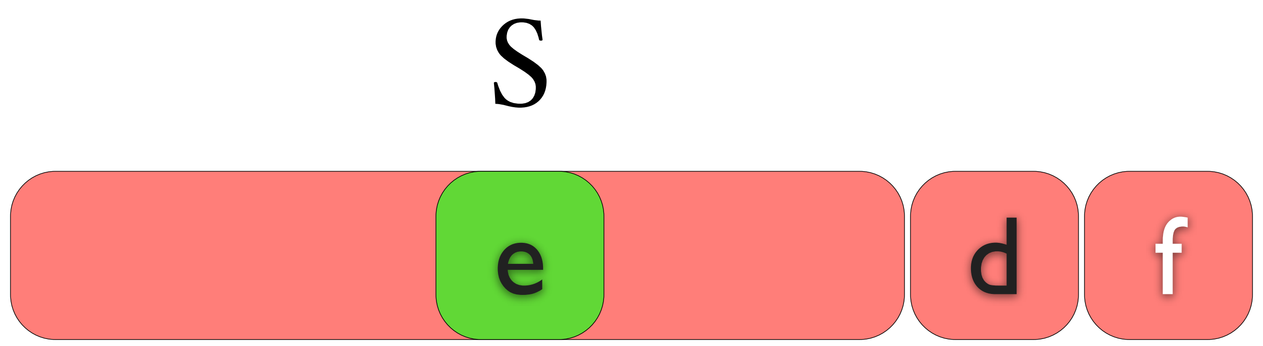
What if t is “access e”:

Proof of lemma

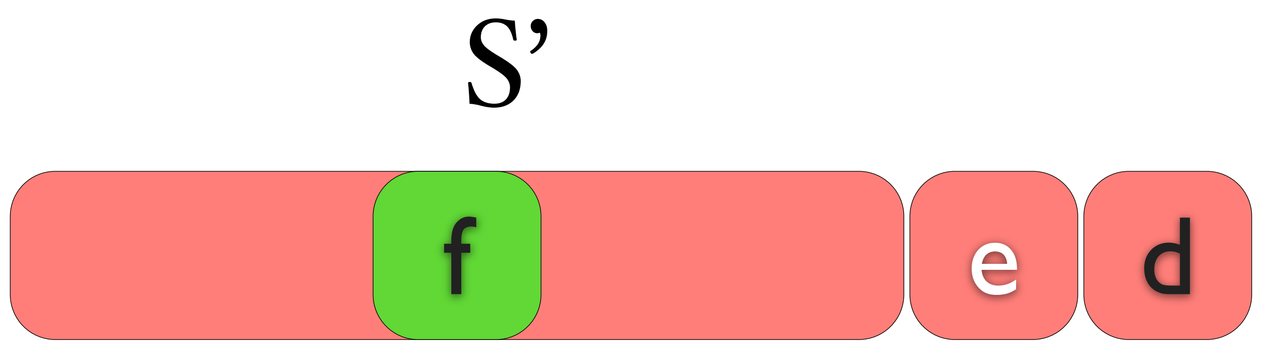
What if t = access e:



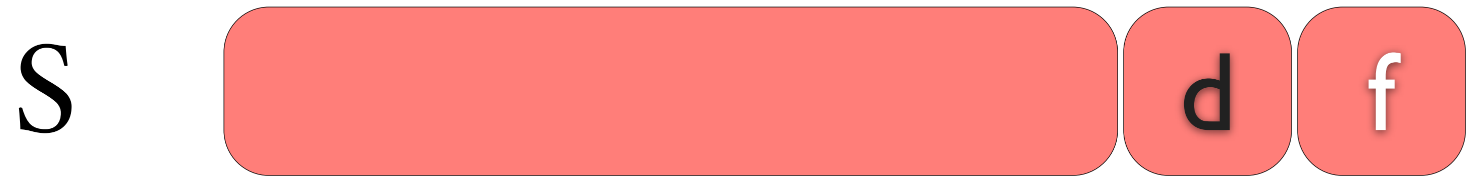
S needs to evict some element to load e.
If it evicts(f,e), then S' can do a NOP.

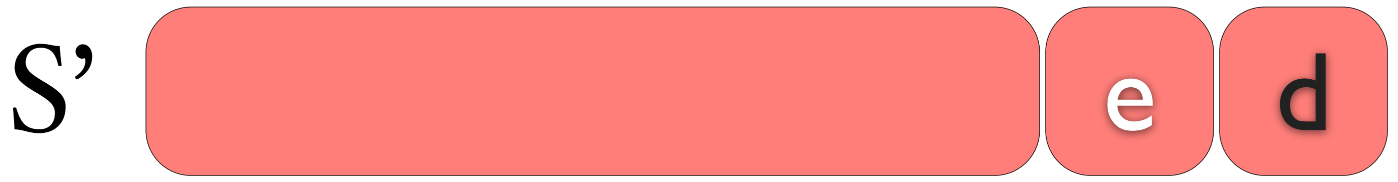


If it evicts(h,e) $h \neq f$, S' can evict(h,f) and maintain equality of the cache.



Proof of lemma





what if t=access f ?

Proof of lemma

what if t=access f ?

This case is impossible because f is accessed “farthest in the future.”

Proof of lemma

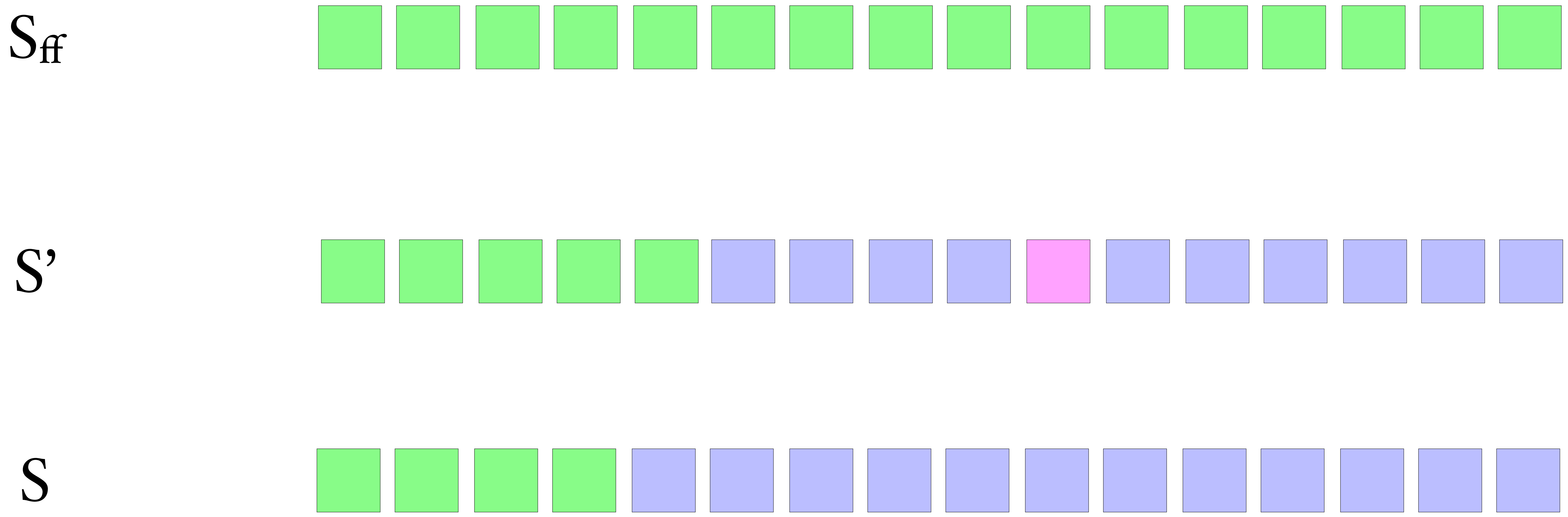
what if t is evict(f,x) ?

Proof of lemma

what if t is evict(f,x) ?

Then S’ can evict(e,x) and have the same cache state.

What have we shown



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$$S^*$$

$$S_{ff}$$

Recap

The greedy algorithm is quite simple.

But the analysis for why the solution works is more subtle and complicated.

In this case, we had to apply the exchange lemma multiple times to prove optimality.

Huffman Coding

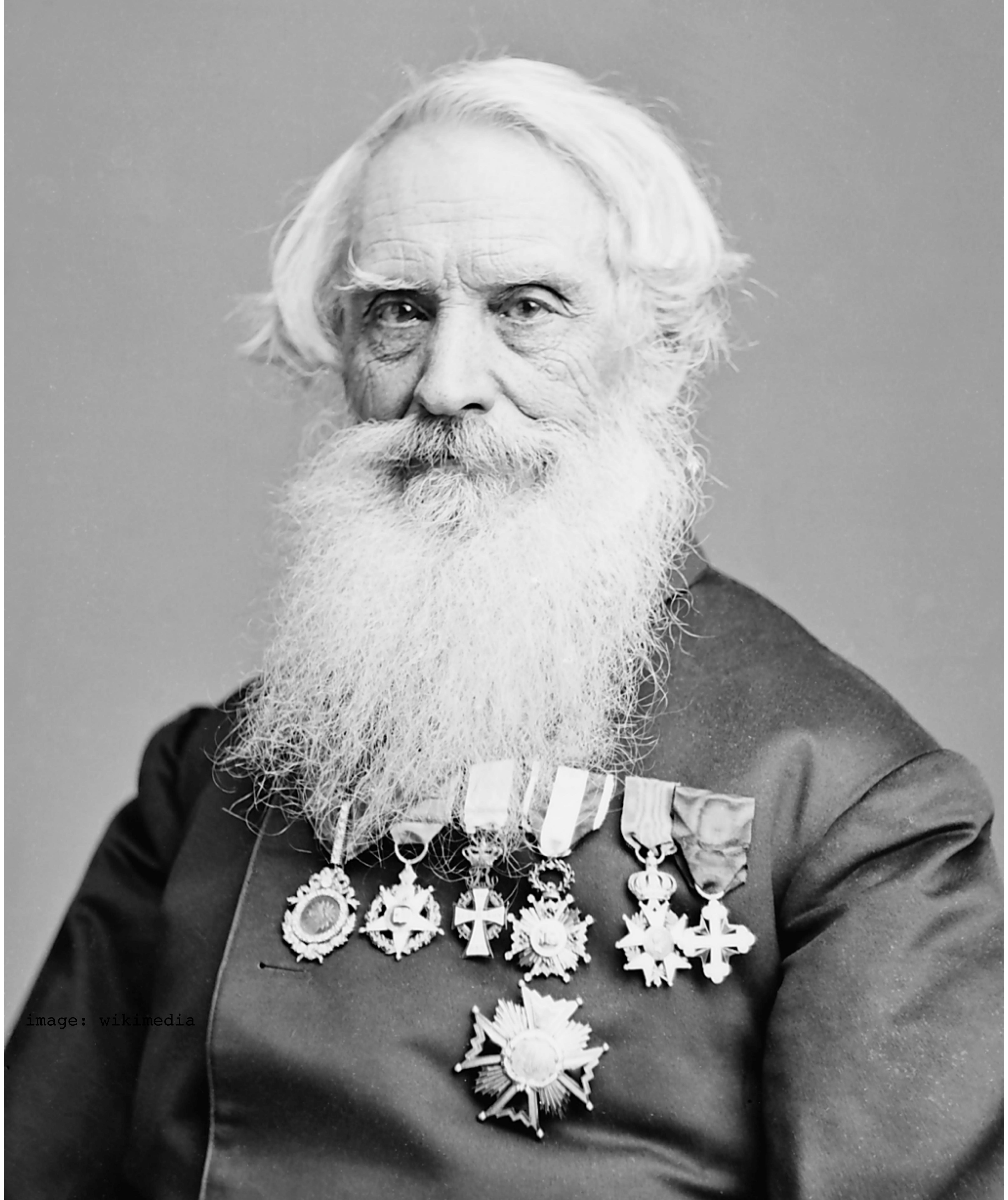
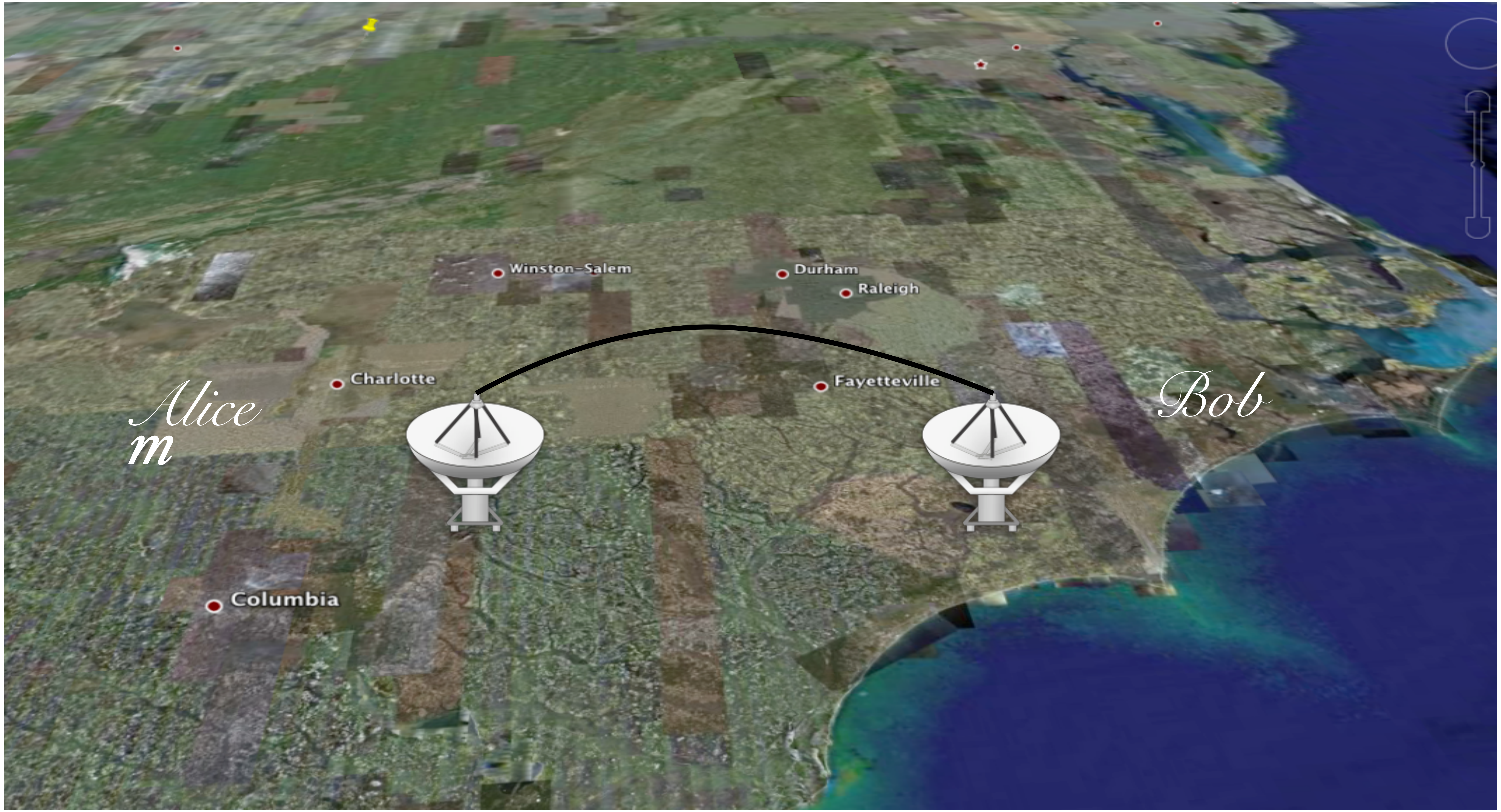
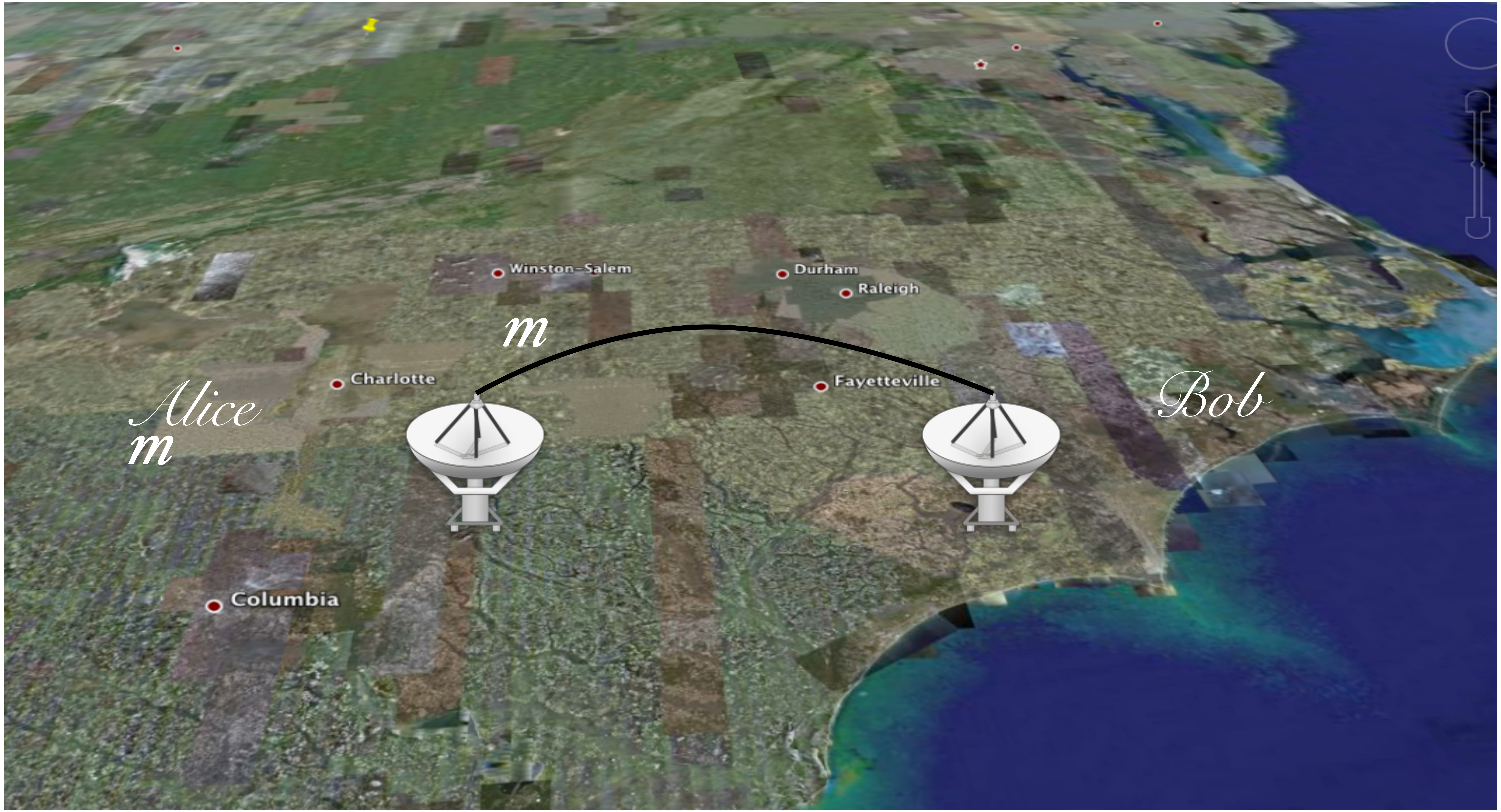


image: wikimedia



Winston-Salem
Durham
Raleigh
Charlotte
Fayetteville
Columbia
Alice
m
Bob



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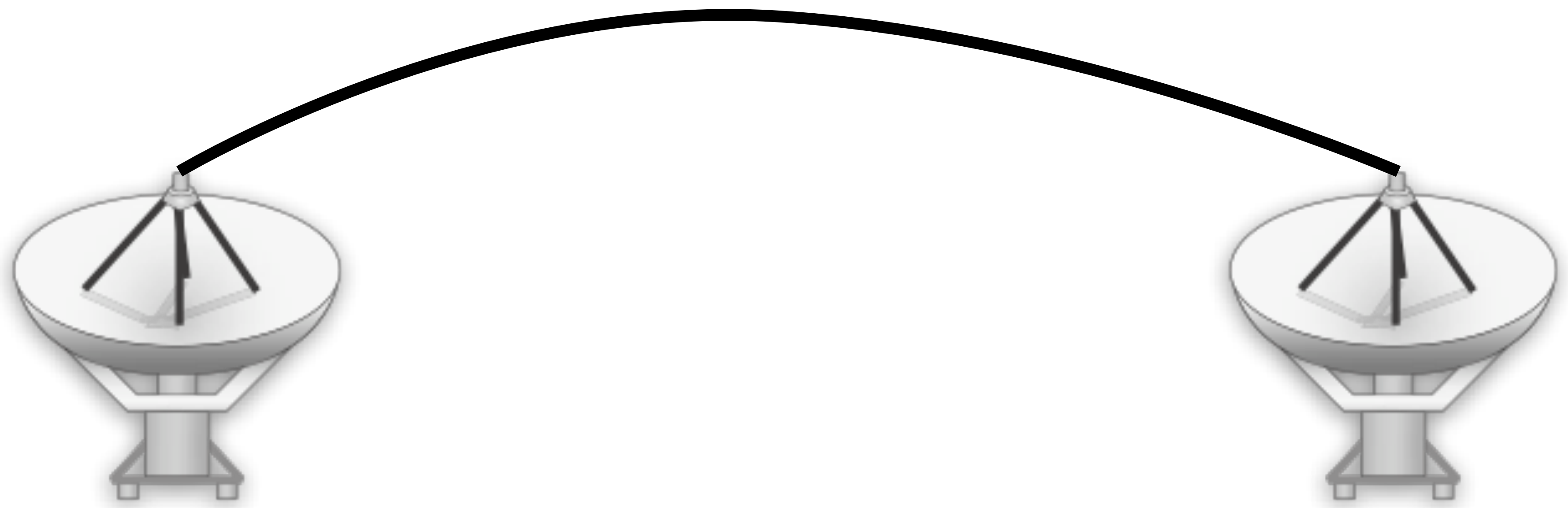
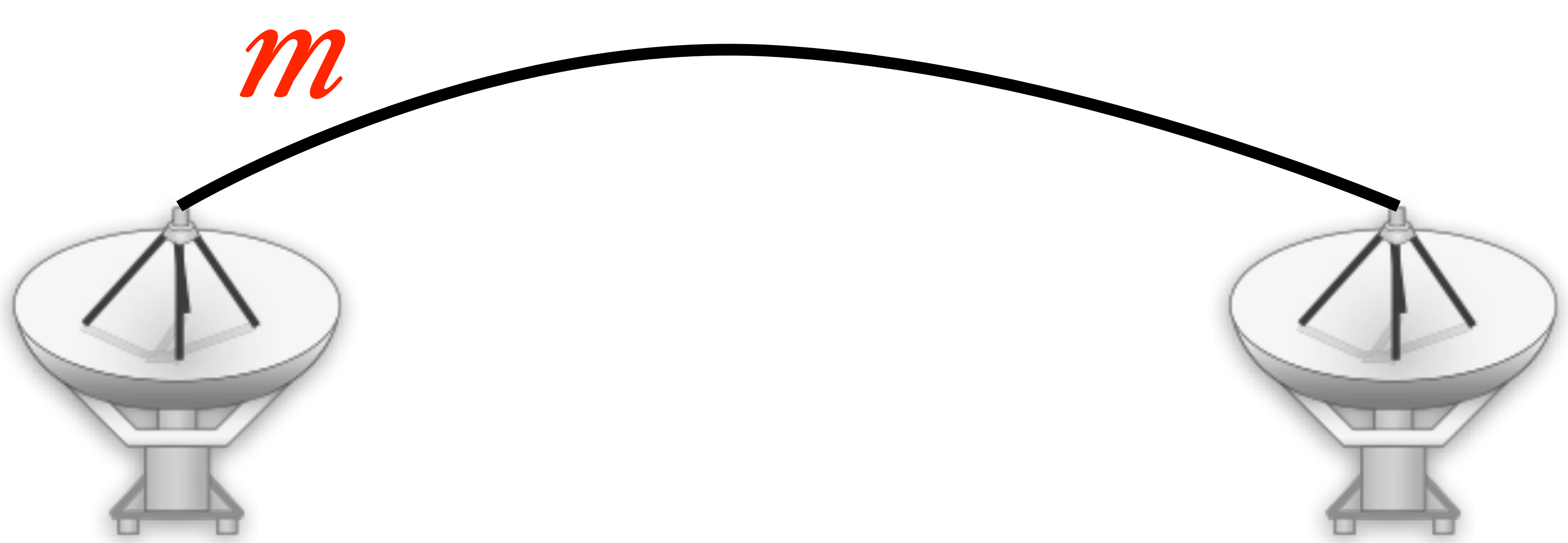


MOSCOW — President Vladimir V. Putin's typically theatrical order to withdraw the bulk of Russian forces from Syria, a process that the Defense Ministry said it began on Tuesday, seemingly caught Washington, Damascus and everybody in between off guard — just the way the Russian leader likes it.

By all accounts, Mr. Putin delights at creating surprises, reinforcing Russia's newfound image as a sovereign, global heavyweight and keeping him at the center of world events.



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Characters in the msg

$c \in C$	f_c	T
e:	235	
i:	200	
o:	170	
u:	87	
p:	78	
g:	47	
b:	40	
f:	24	

881

$c \in C$	f_c	T	ℓ_c
e:	235	000	3
i:	200	001	3
o:	170	010	3
u:	87	011	3
p:	78	100	3
g:	47	101	3
b:	40	110	3
f:	24	111	3

881

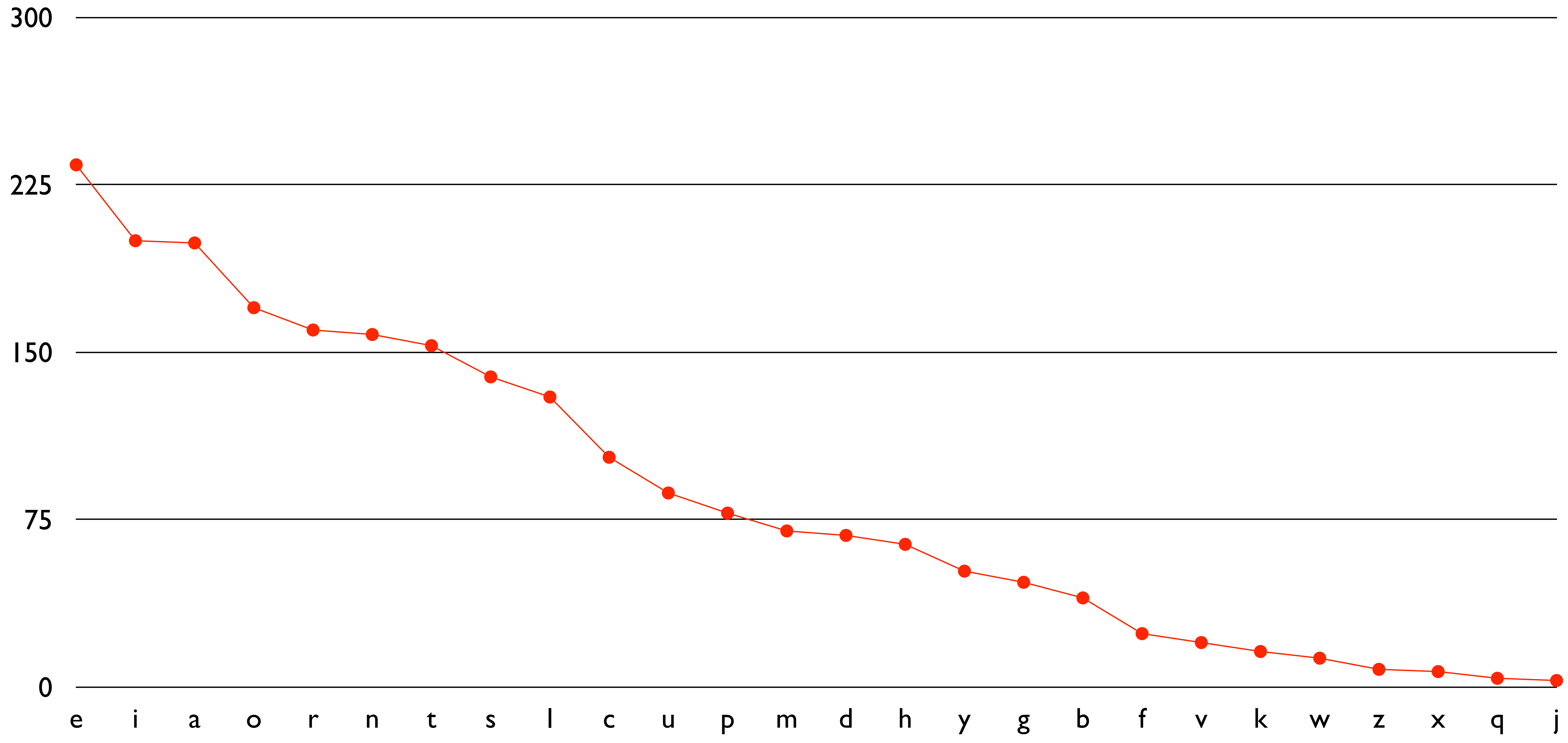
def: cost of an encoding

$$B(T, \{f_c\}) = \sum_{c \in C} f_c \cdot \ell_c$$

$c \in C$	f_c	T	ℓ_c
e:	235	000	3
i:	200	001	3
o:	170	010	3
u:	87	011	3
p:	78	100	3
g:	47	101	3
b:	40	110	3
f:	24	111	3
	881		

character frequency

e: 234803
i: 200613
a: 198938
o: 170392
r: 160491
n: 158281
t: 152570
s: 139238
l: 130172
c: 103307
u: 87211
p: 78077
m: 70504
d: 68007
h: 64165
y: 51527
g: 47011
b: 40351
f: 24110
v: 20103
k: 16012
w: 13825
z: 8439
x: 6926
q: 3729
j: 3075

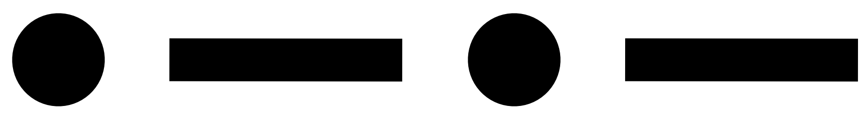


Morse code

Image http://en.wikipedia.org/wiki/Morse_code

Morse code


International Morse Code
- 1 dash = 3 dots.
- The space between parts of the same letter = 1 dot.
- The space between letters = 3 dots.
- The space between words = 7 dots.
A • —
B — • • •
C — • — •
D — • •
E •
F • • — •
G — — •
H • • • •
I • •
J • — — —
K — • —
L • — • •
M — —
N — •
O — — —
P • — — •
Q — — • —
R • — •
S • • •
T —
U • • —
V • • • —
W • — —
X — • • —
Y — • — —
Z — — • •
. • — • — • —
, — — • • — —
? • • — — • •
/ — • • — •
@ • — — • — •
1 • — — — —
2 • • — — —
3 • • • — —
4 • • • • —
5 • • • • •
6 — • • • •
7 — — • • •
8 — — — • •
9 — — — — •
0 — — — — —



def: prefix-free code

def: prefix-free code

$$\forall x, y \in C, x \neq y \implies \textsc{code}(x) \text{ not a prefix of } \textsc{code}(y)$$

def: prefix code

$$\forall x, y \in C, x \neq y \implies \textsc{code}(x) \text{ not a prefix of } \textsc{code}(y)$$

```
e: 235    0
i: 200    10
o: 170    110
u: 87     1110
p: 78     11110
g: 47     111110
b: 40     1111110
f: 24     11111110
```

Example of a prefix free code

decoding a prefix code

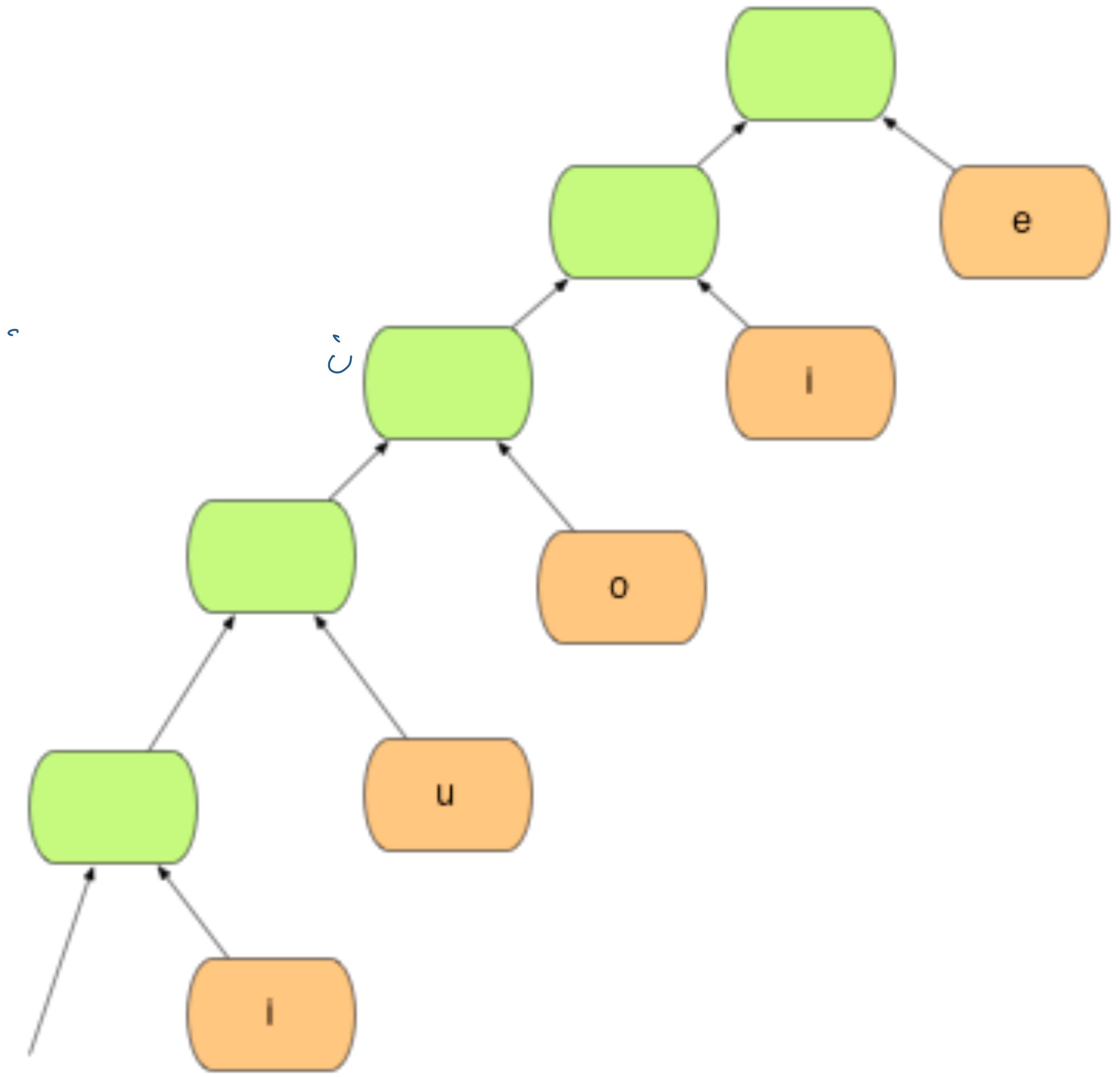
e:	235	0
i:	200	10
o:	170	110
u:	87	1110
p:	78	11110
g:	47	111110
b:	40	1111110
f:	24	11111110

111111010111110

Prefix code to binary tree

e:	235	0
i:	200	10
o:	170	110
u:	87	1110
p:	78	11110
g:	47	111110
b:	40	1111110
f:	24	11111110

1111111010111110

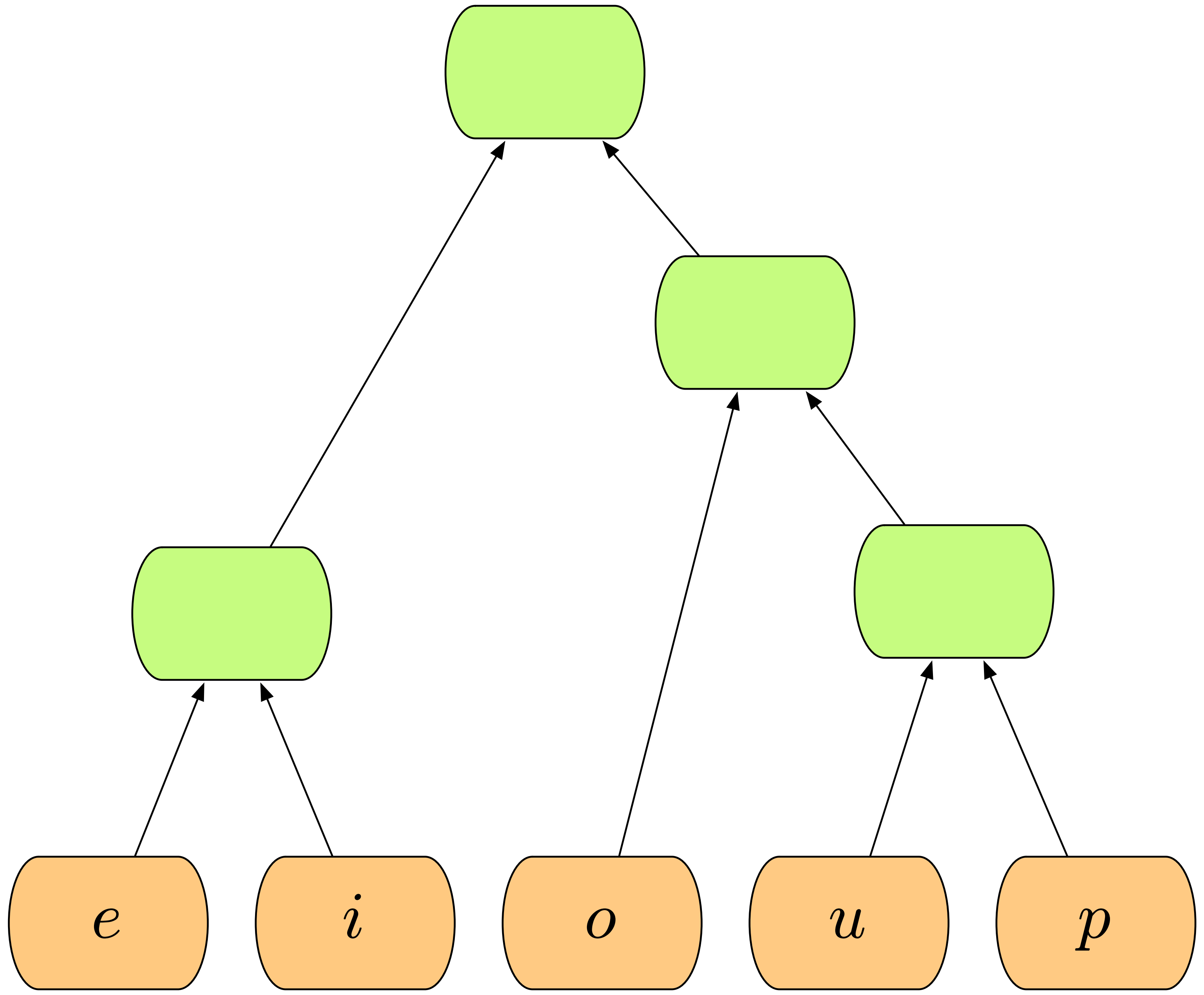


prefix code

binary tree

The prefix-free code and the binary tree are different representations of the same object.

use tree to encode



$c \in C$	f_c	T	ℓ_c
e:	235	00	2
i:	200	01	2
o:	170	10	2
u:	87	110	3
p:	78	111	3

goal

GIVEN THE

goal

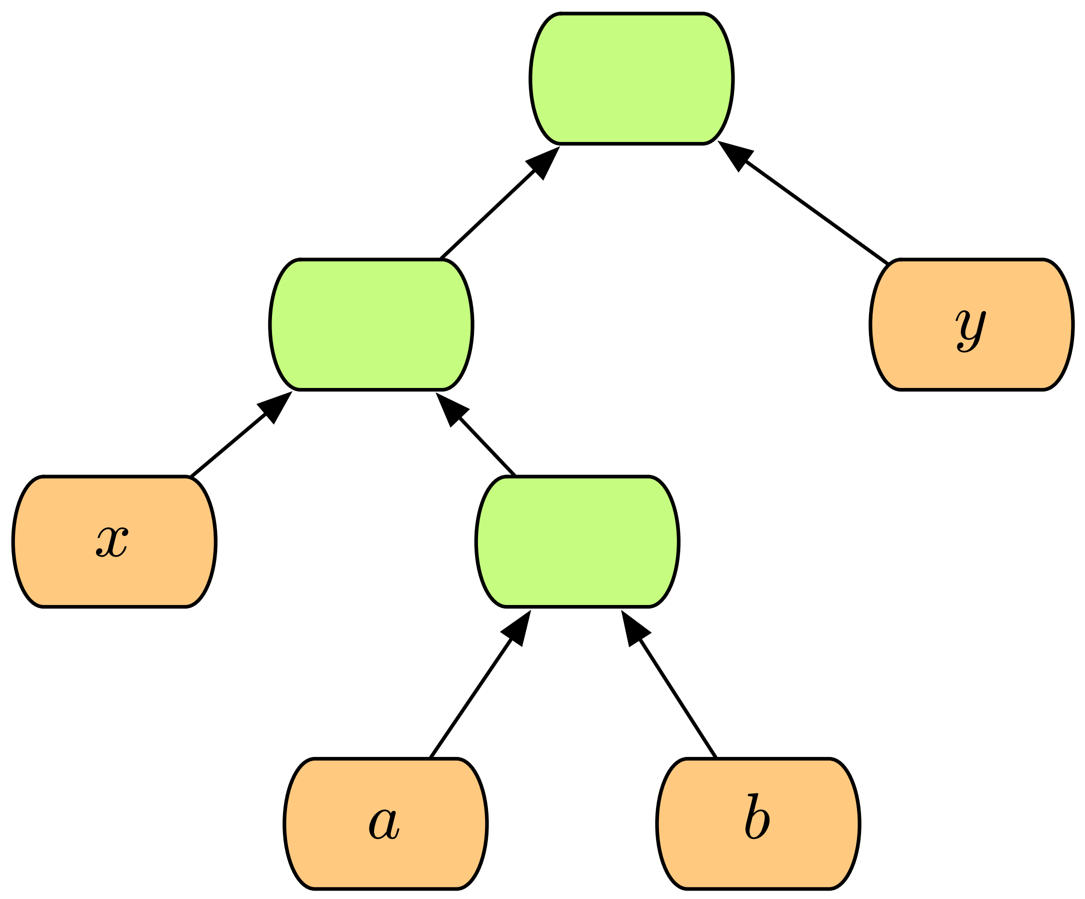
(all frequencies are > 0)

GIVEN THE CHARACTER FREQUENCIES $\{f_c\}_{c\in C}$

PRODUCE A PREFIX CODE T WITH SMALLEST COST

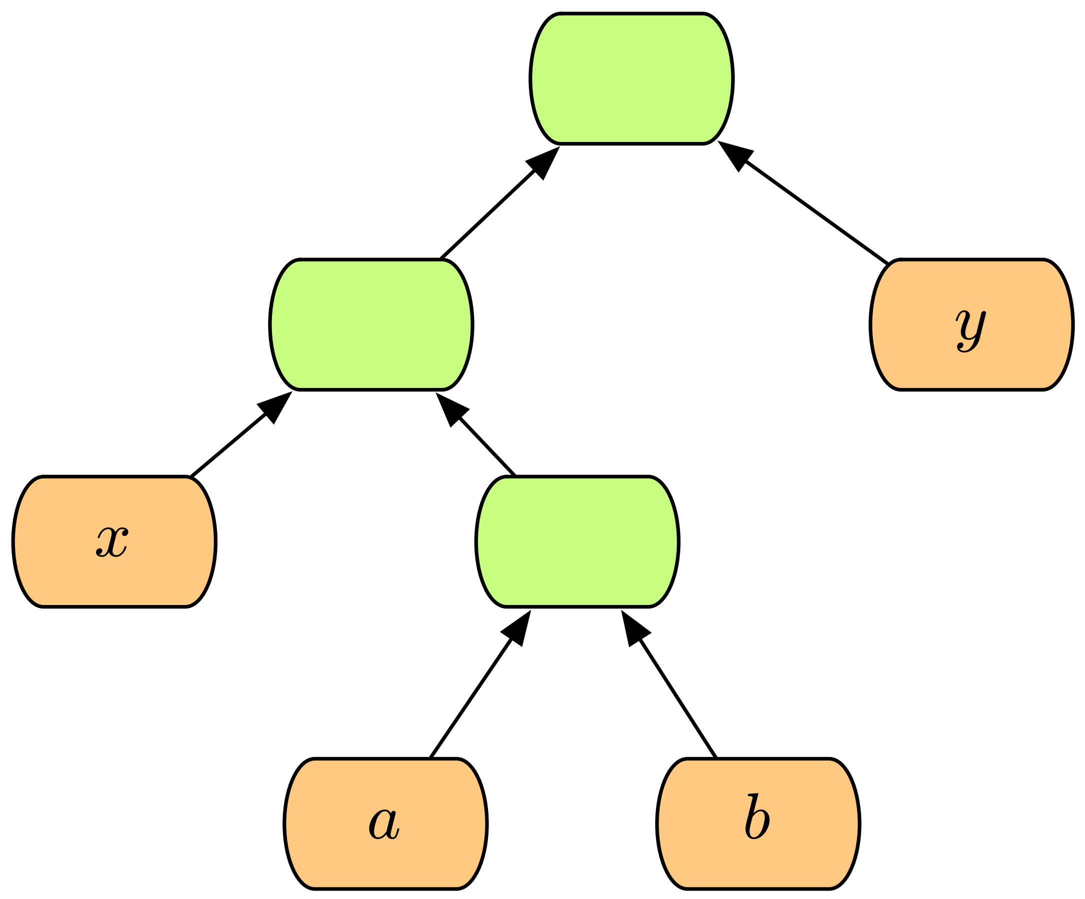
$$\min_T B(T, \{f_c\})$$

property



LEMMA: OPTIMAL TREE MUST BE FULL.

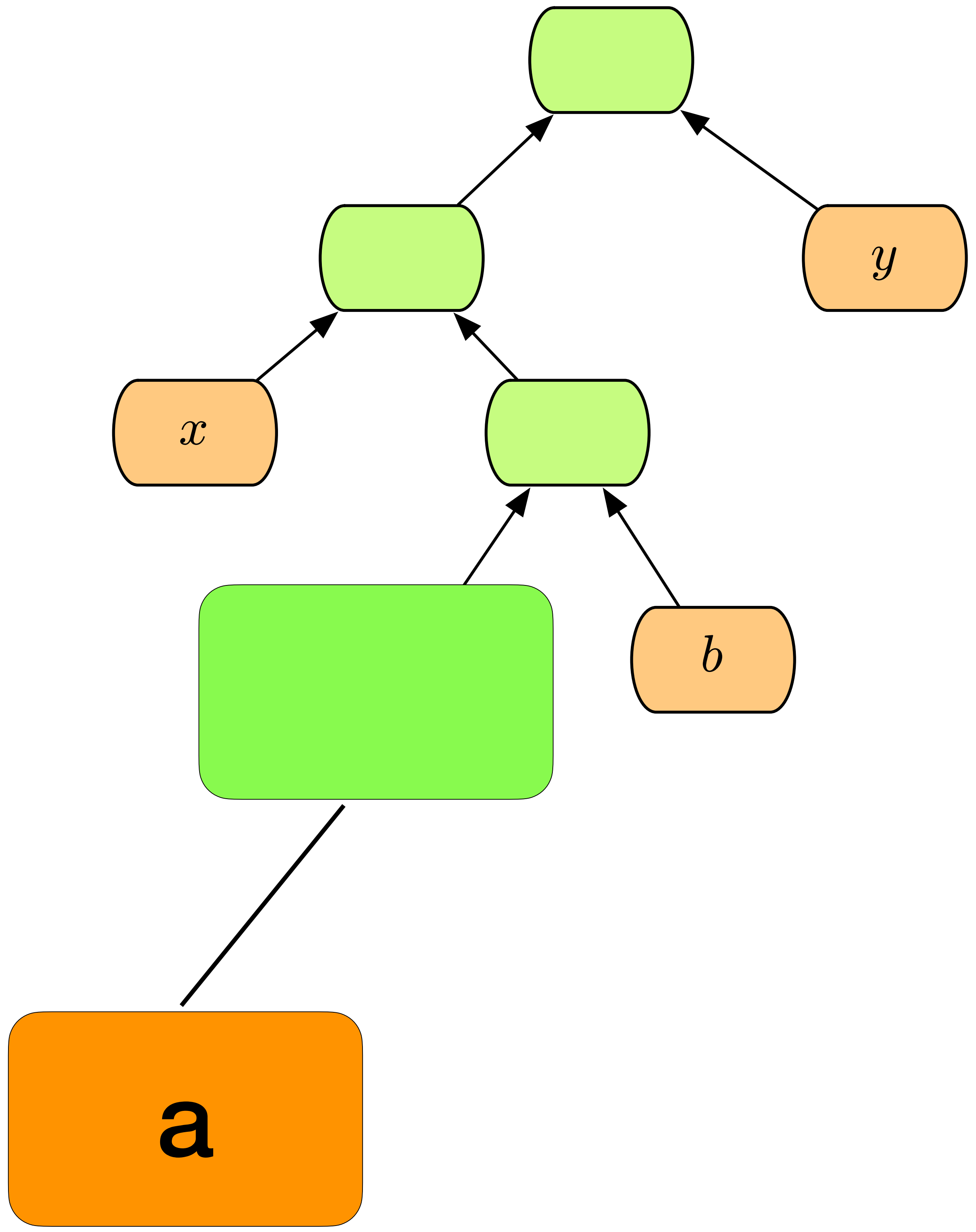
property



LEMMA: OPTIMAL TREE MUST BE FULL.

A full tree has nodes with either 0 or 2 children.

property



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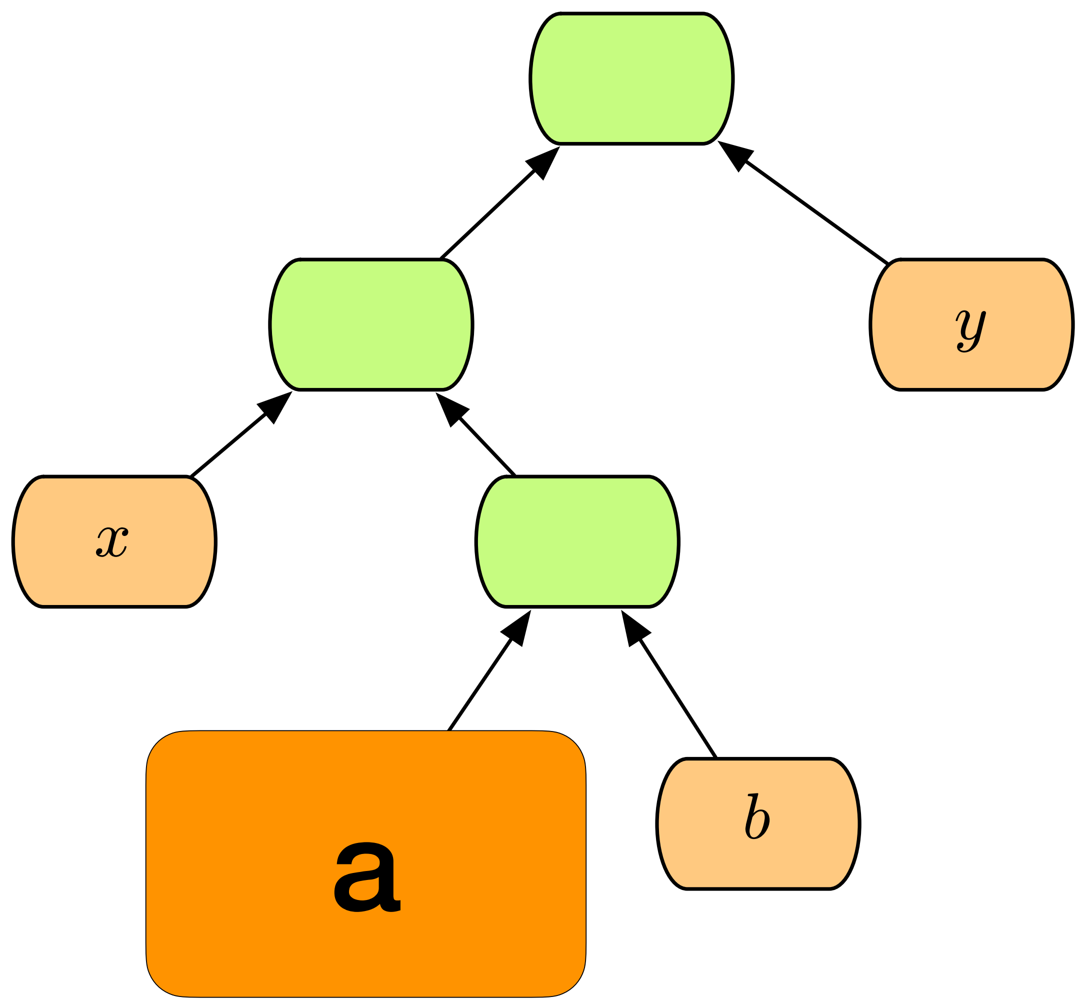
A full tree has nodes with either 0 or 2 children.

Consider a node with only 1 child.

property



LEMMA:OPTIMAL TREE MUST BE FULL.

A full tree has nodes with either 0 or 2 children.

Consider a node with only 1 child.

The length of the code for this child can be reduced by replacing the parent with the child.

Thus, the cost of the code can be reduced or remain equal if the parent is replaced by the child

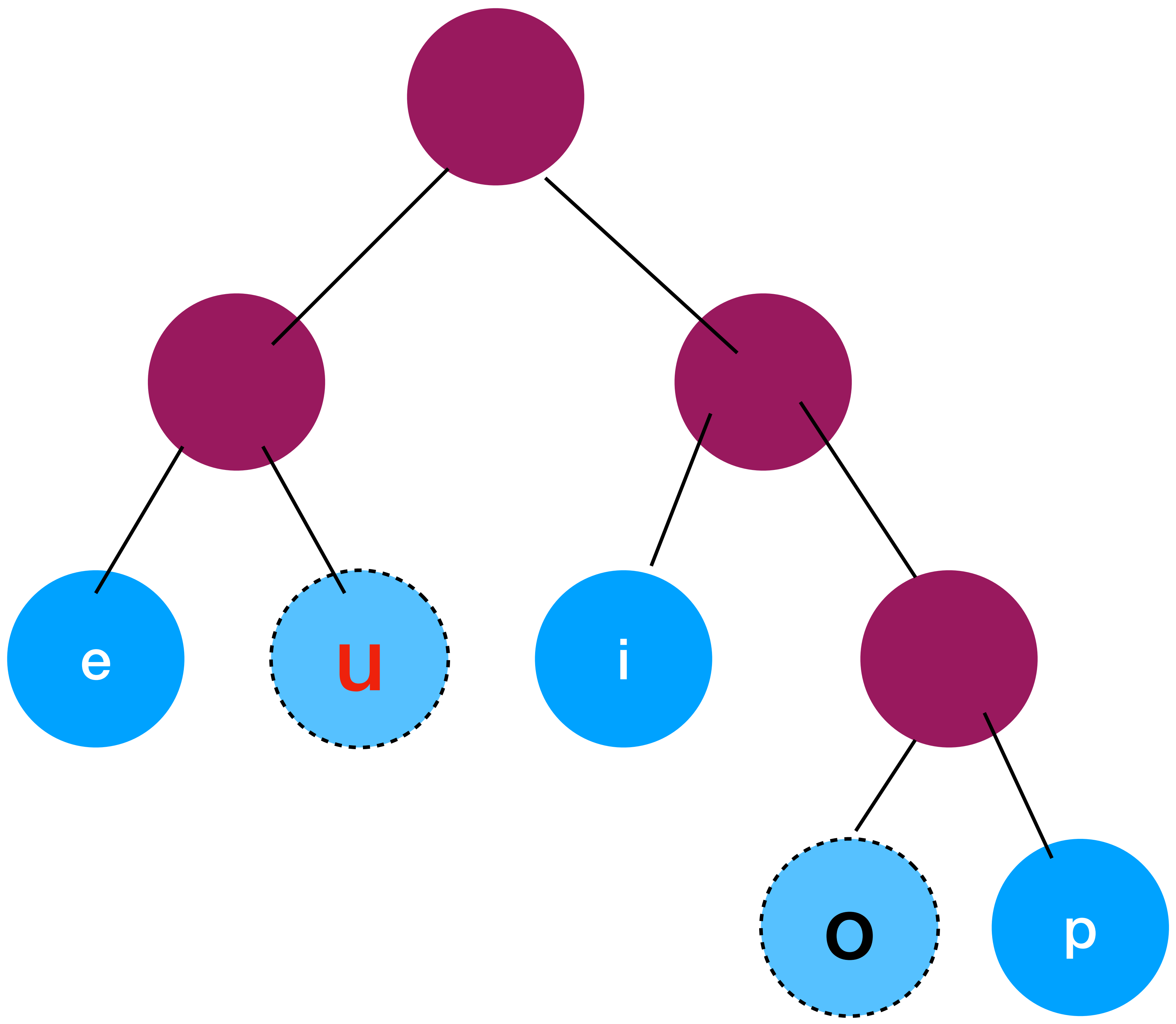
divide & conquer Tug of War?

Consider a "Tug of War" strategy in which we balance the weights of the teams and recurse.

```
e: 32
i: 25
o: 20
u: 18
p: 5
```

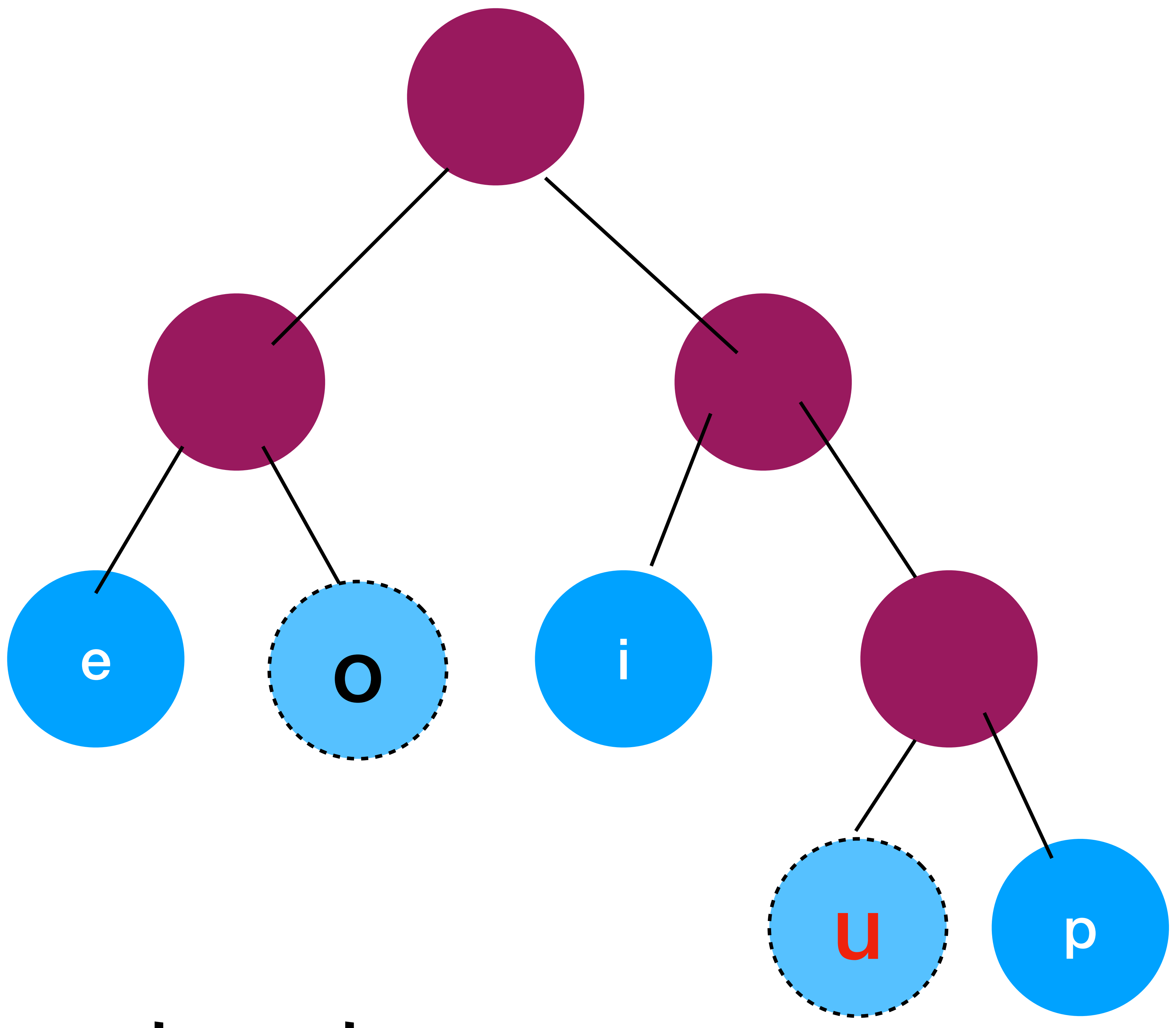
counter-example

```
e: 32  2: 64
i: 25  2: 50
o: 20  3: 60
u: 18  2: 36
p: 5   3: 15
          225
```



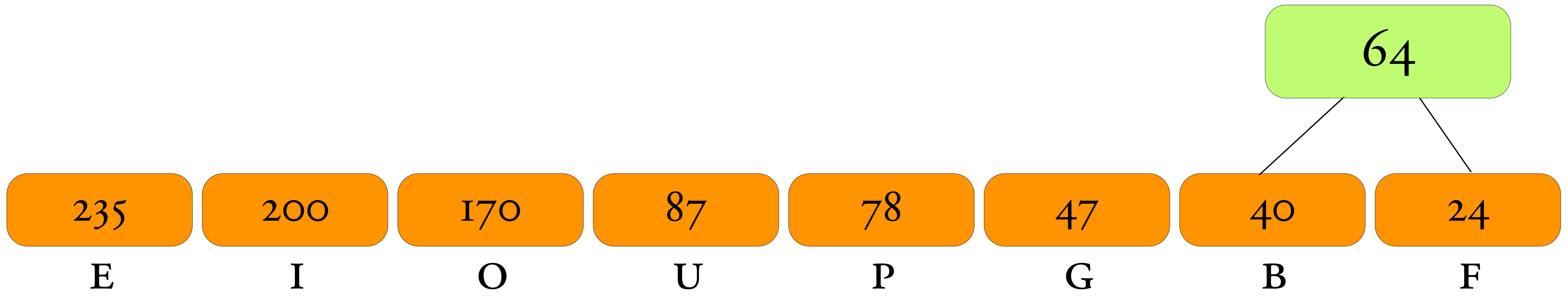
counter-example

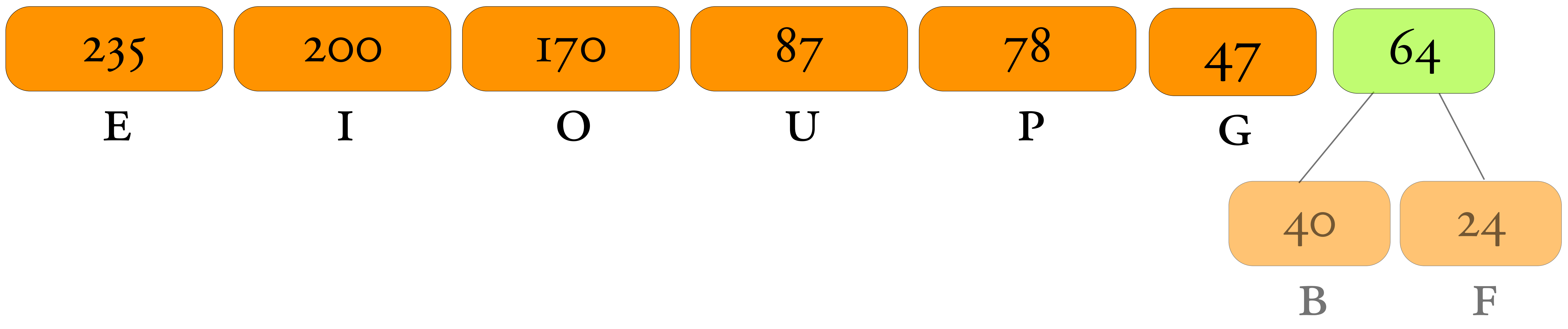
```
e: 32  2: 64
i: 25  2: 50
o: 20  2: 40
u: 18  3: 54
p: 5   3: 15
          223
```



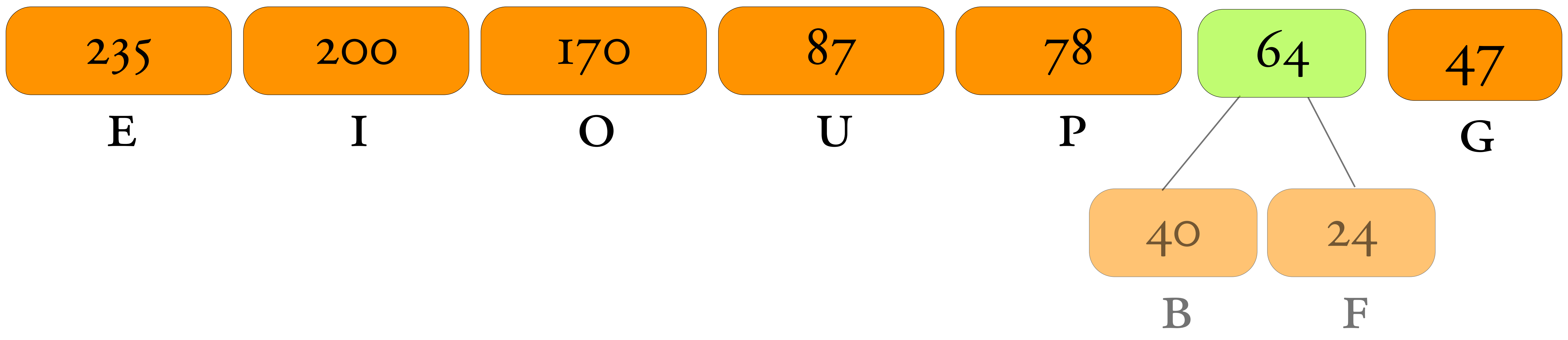
By switching {u,o}, the cost of the code can be reduced.
It can be reduced further with an optimal code.

Huffman construction





235
E
200
I
170
O
87
U
78
P
47
G
64
40
B
24
F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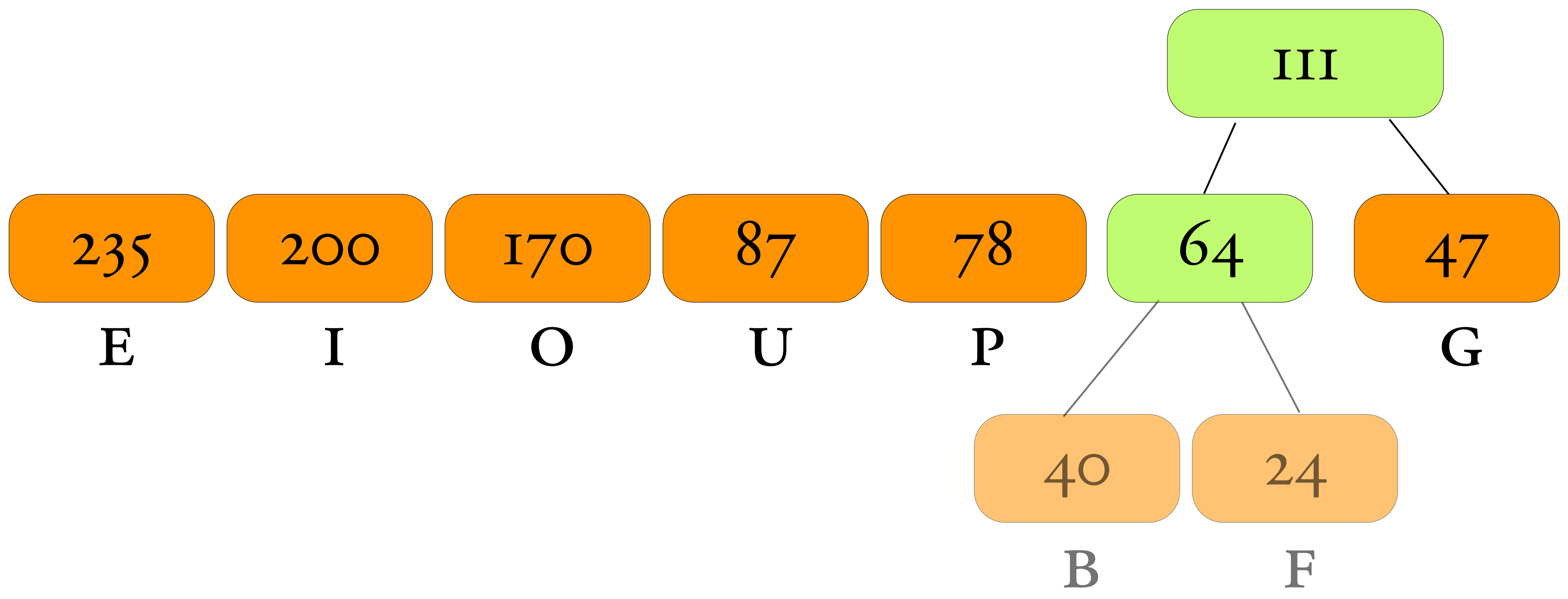
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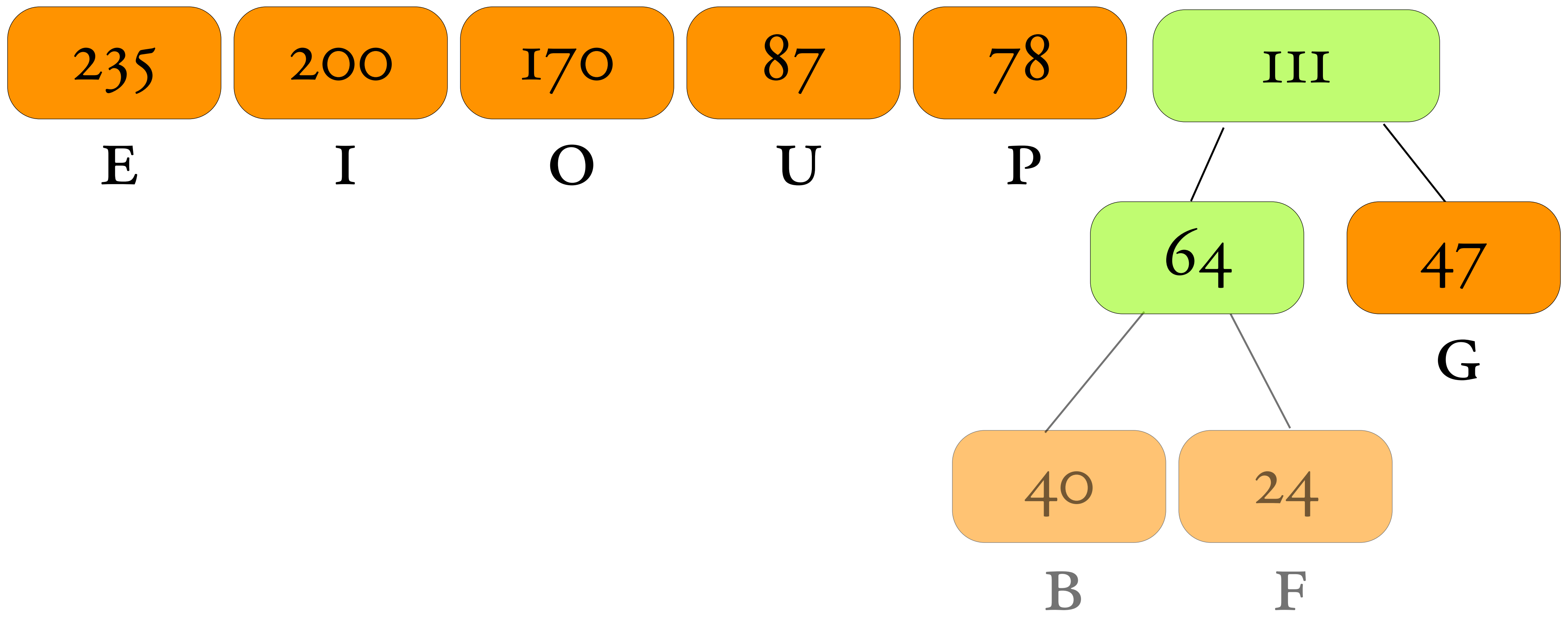

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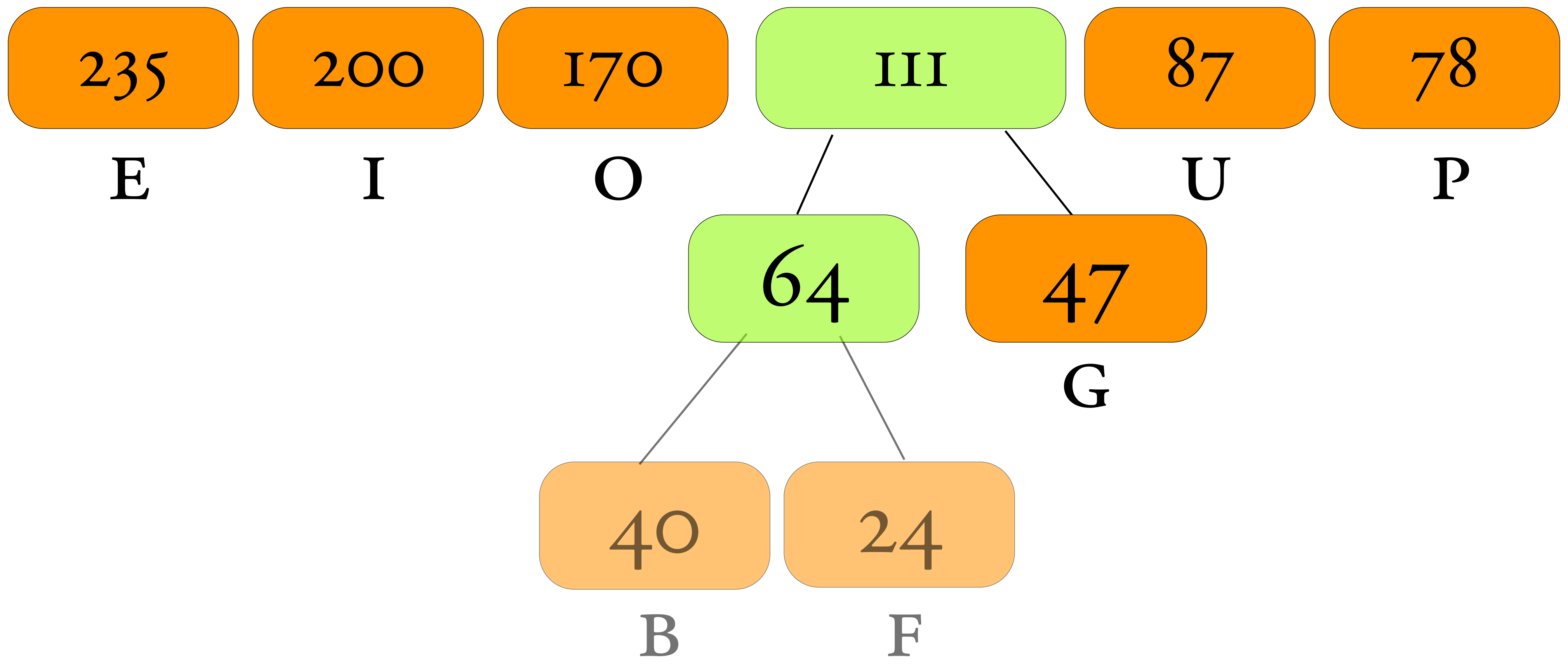
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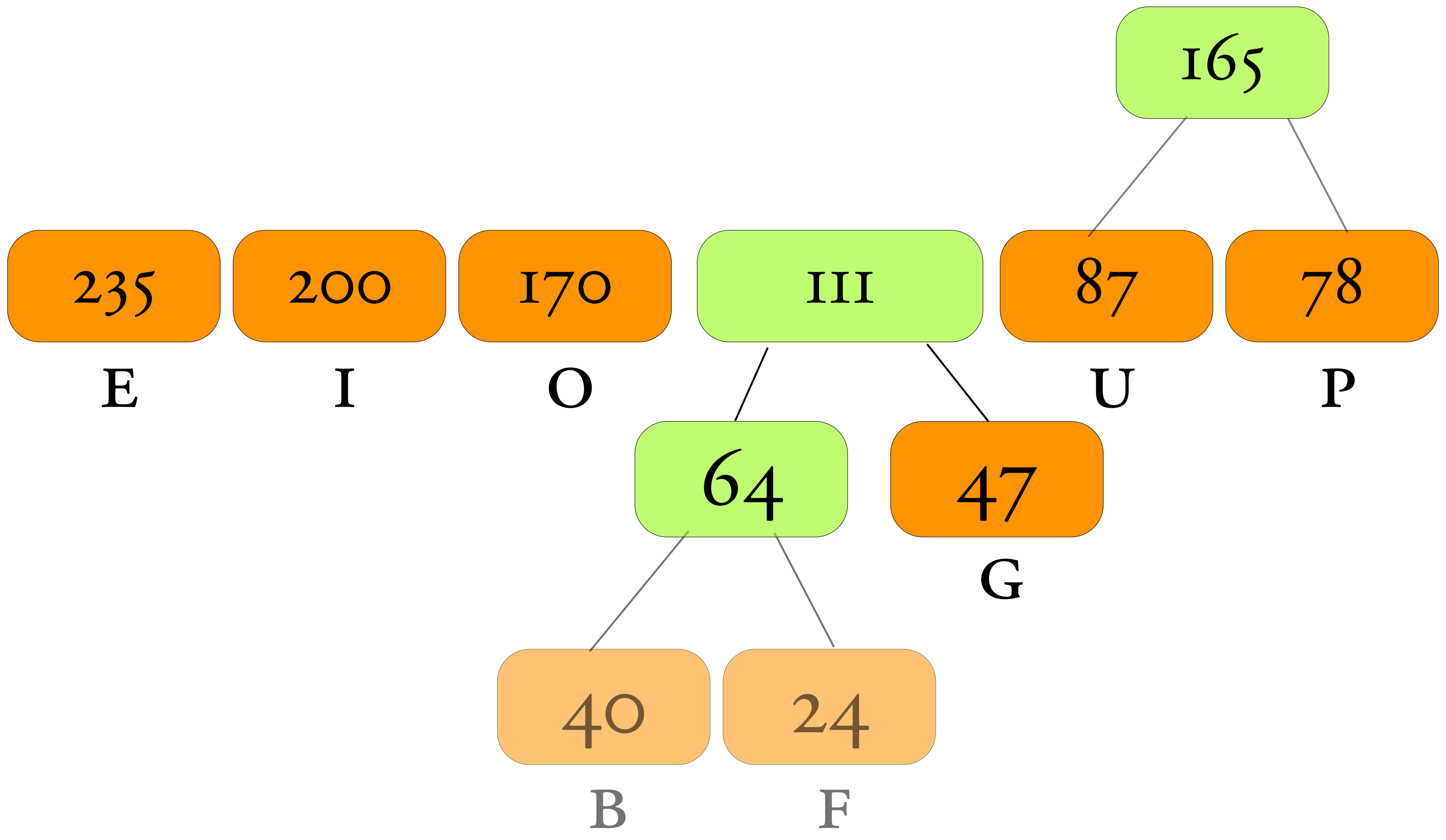
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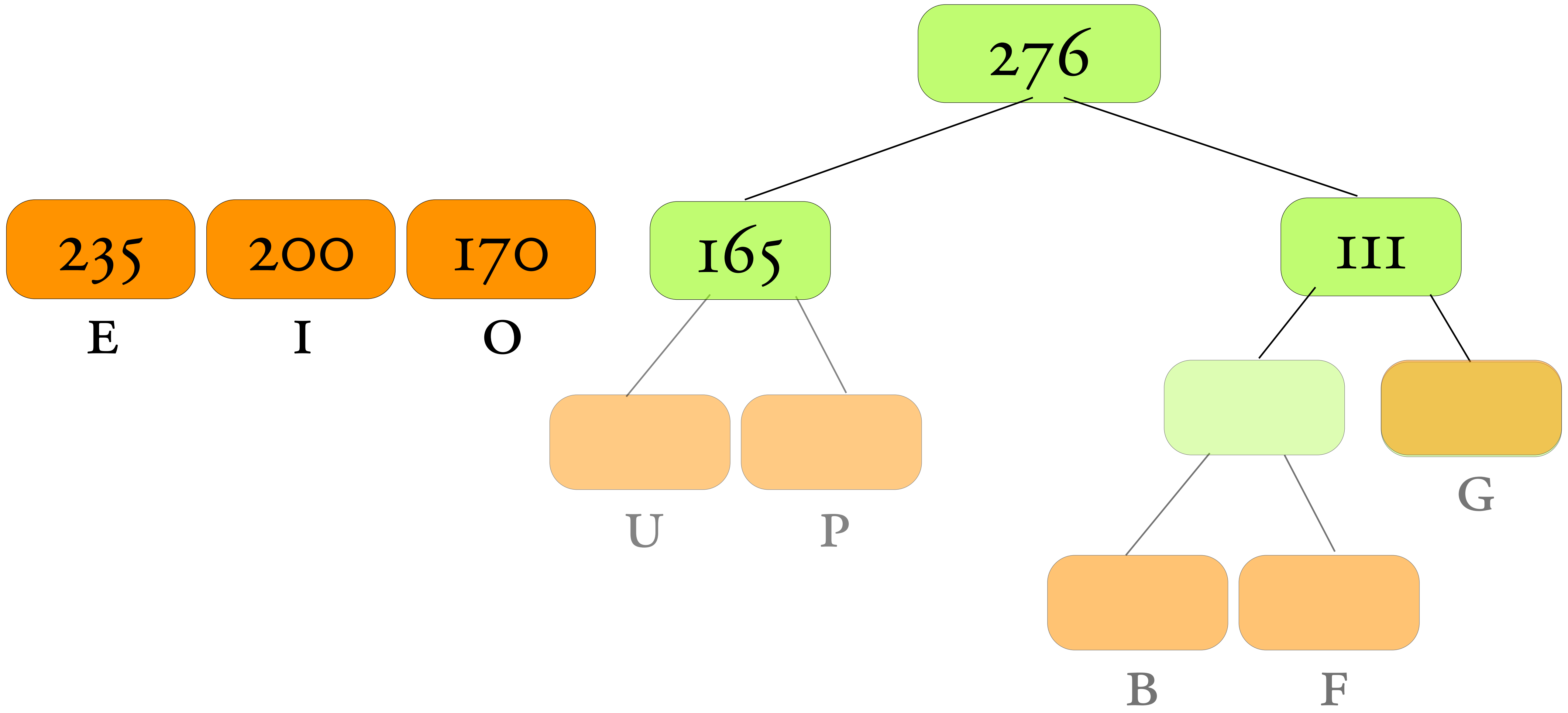
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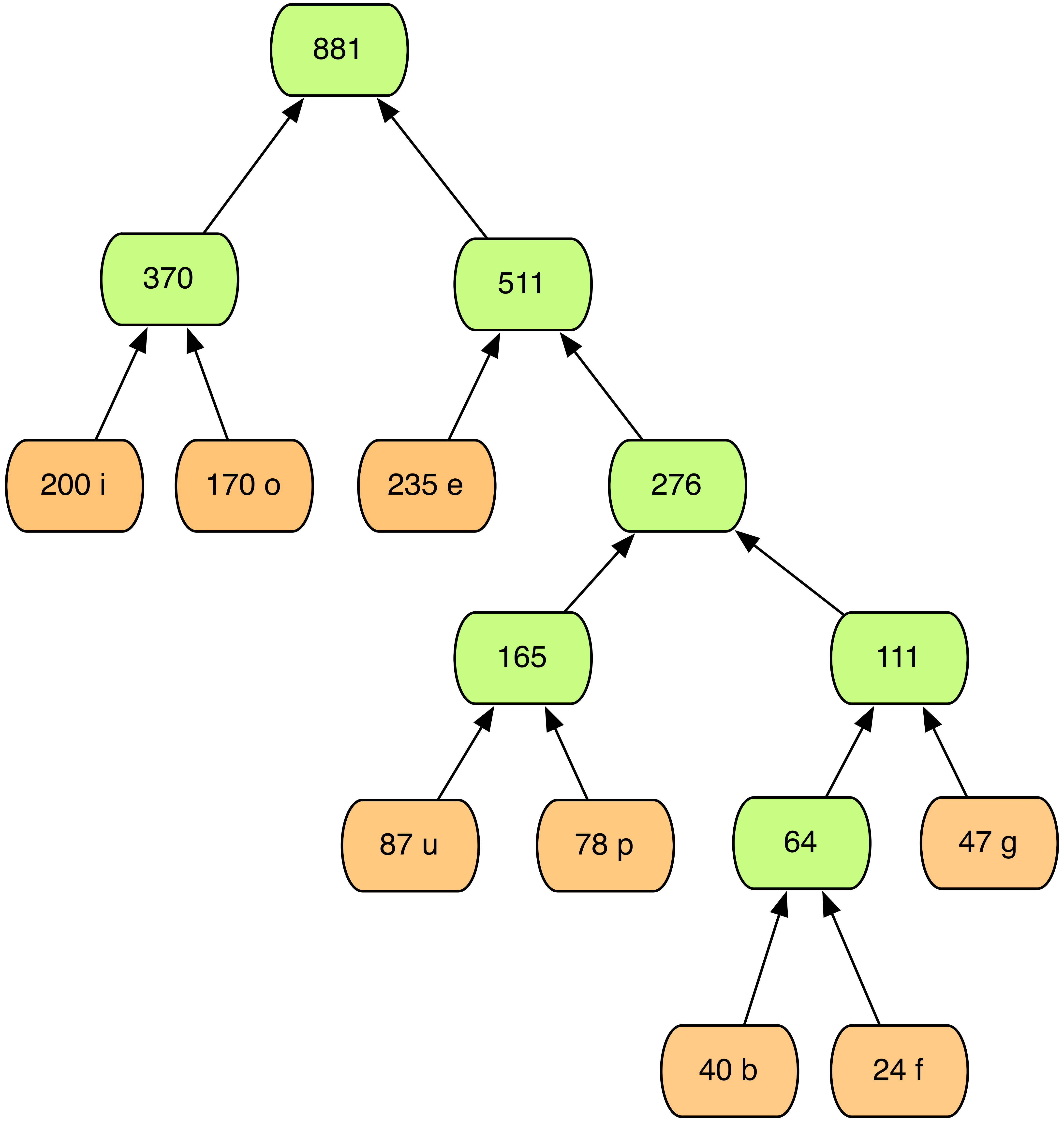
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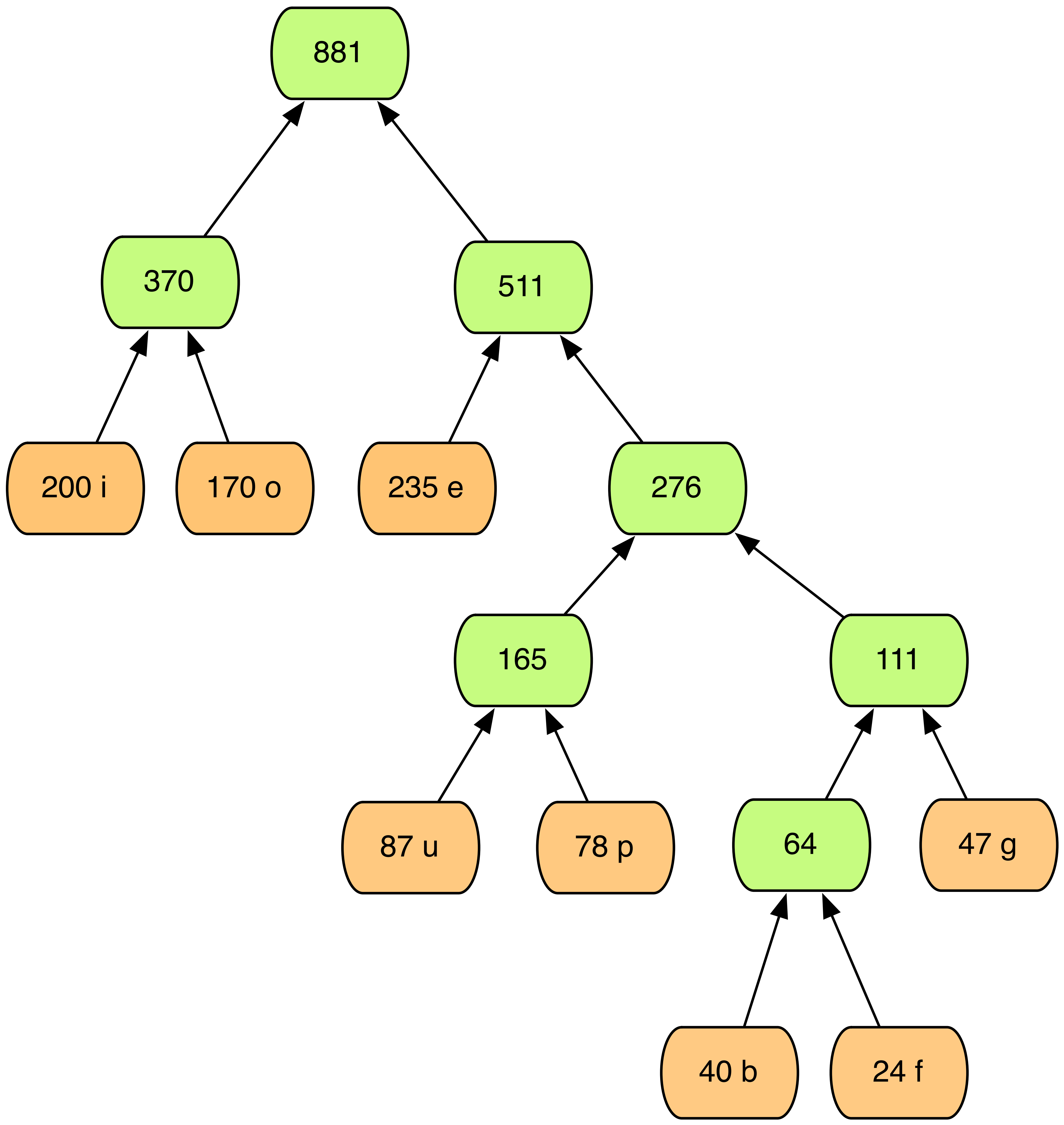
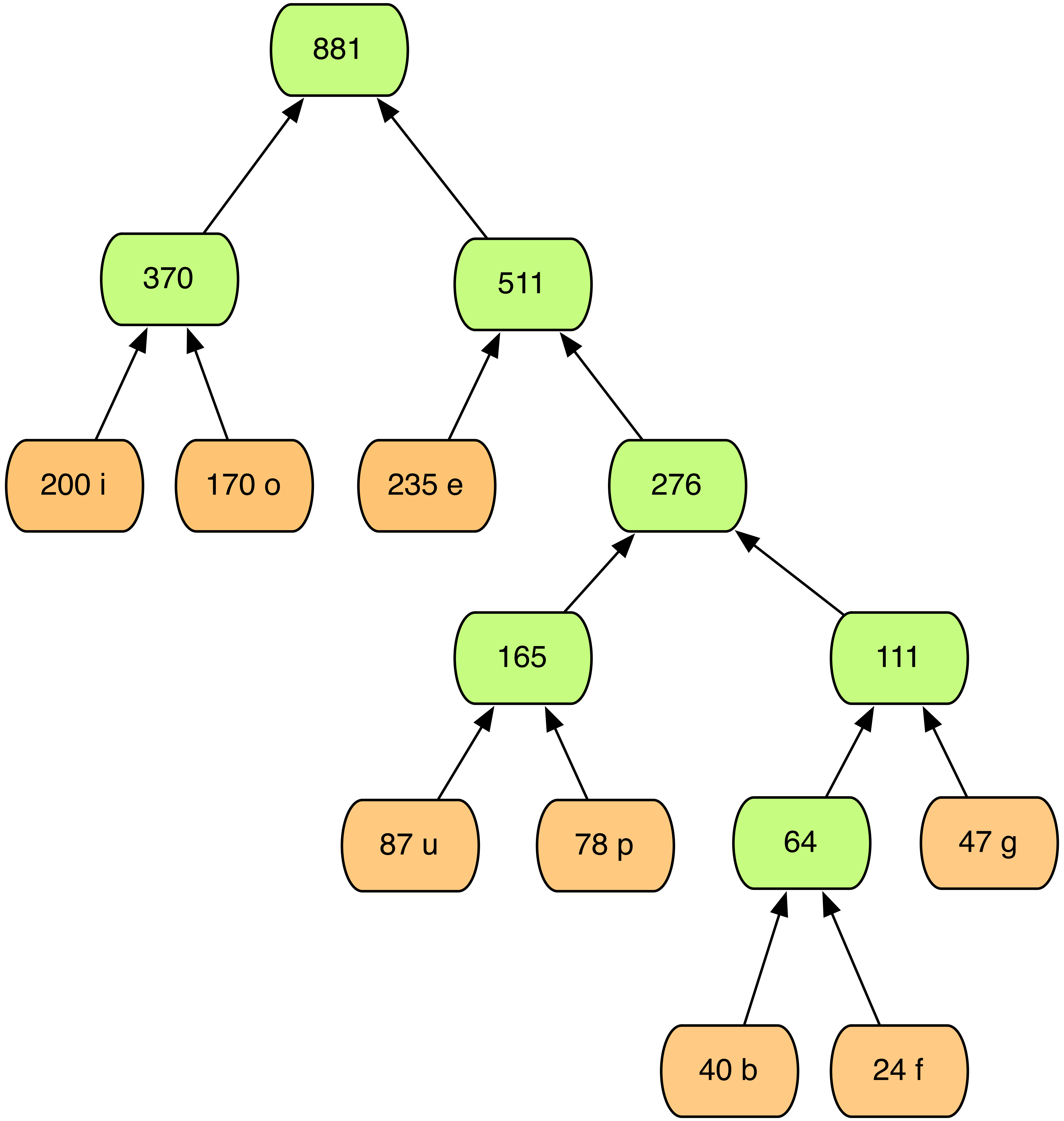


e:	235	01
i:	200	11
o:	170	10
u:	87	0011
p:	78	0010
g:	47	0000
b:	40	00011
f:	24	00010



e:	235	01	470
i:	200	11	400
o:	170	10	340
u:	87	0011	348
p:	78	0010	312
g:	47	0000	188
b:	40	00011	200
f:	24	00010	120

2378

objective

objective

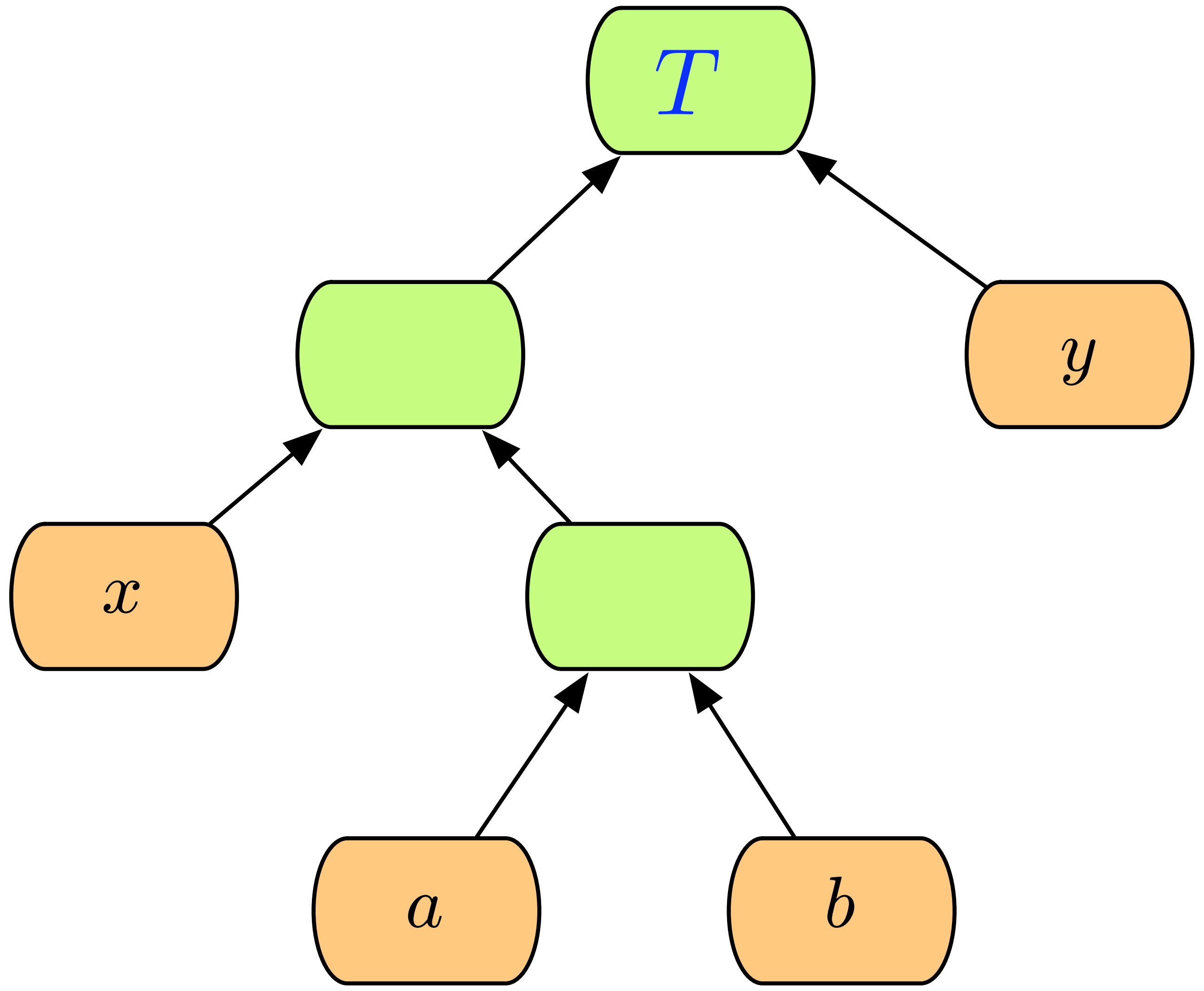
The goal is to prove that the procedure outlined produced an optimal code. Taking a greedy step to make the problem one size smaller is optimal.

exchange argument

LEMMA:

exchange argument

LEMMA: Let $x, y \in C$ be characters with smallest frequencies f_x, f_y. There exists an optimal prefix code T'' for C in which x, y are siblings. That is, the codes for x, y have the same length and only differ in the last bit.



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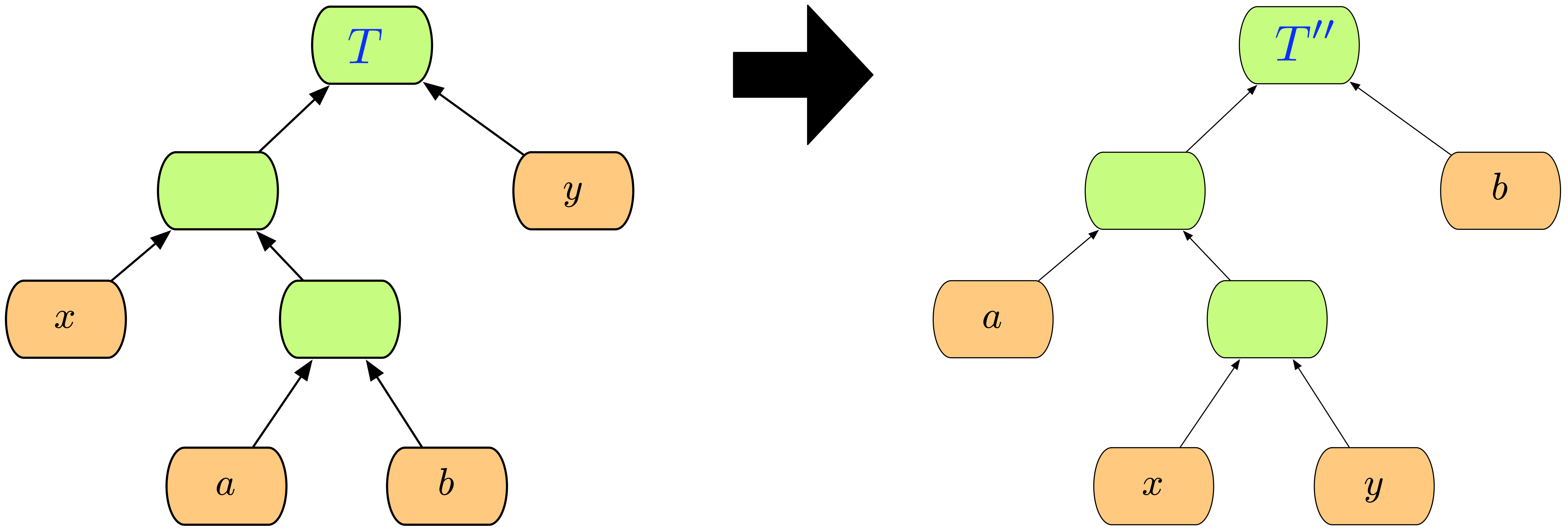


Idea: take an arbitrary optimal tree T for a prefix code and modify it into another optimal tree in which x,y are sibling children at the lowest level of the tree.

exchange argument

LEMMA: Let $x, y \in C$ be characters with smallest frequencies f_x, f_y. There exists an optimal prefix code T'' for C in which x, y are siblings. That is, the codes for x, y have the same length and only differ in the last bit.

PROOF:

exchange argument

LEMMA: Let $x, y \in C$ be characters with smallest frequencies f_x, f_y. There exists an optimal prefix code T'' for C in which x, y are siblings. That is, the codes for x, y have the same length and only differ in the last bit.

PROOF:

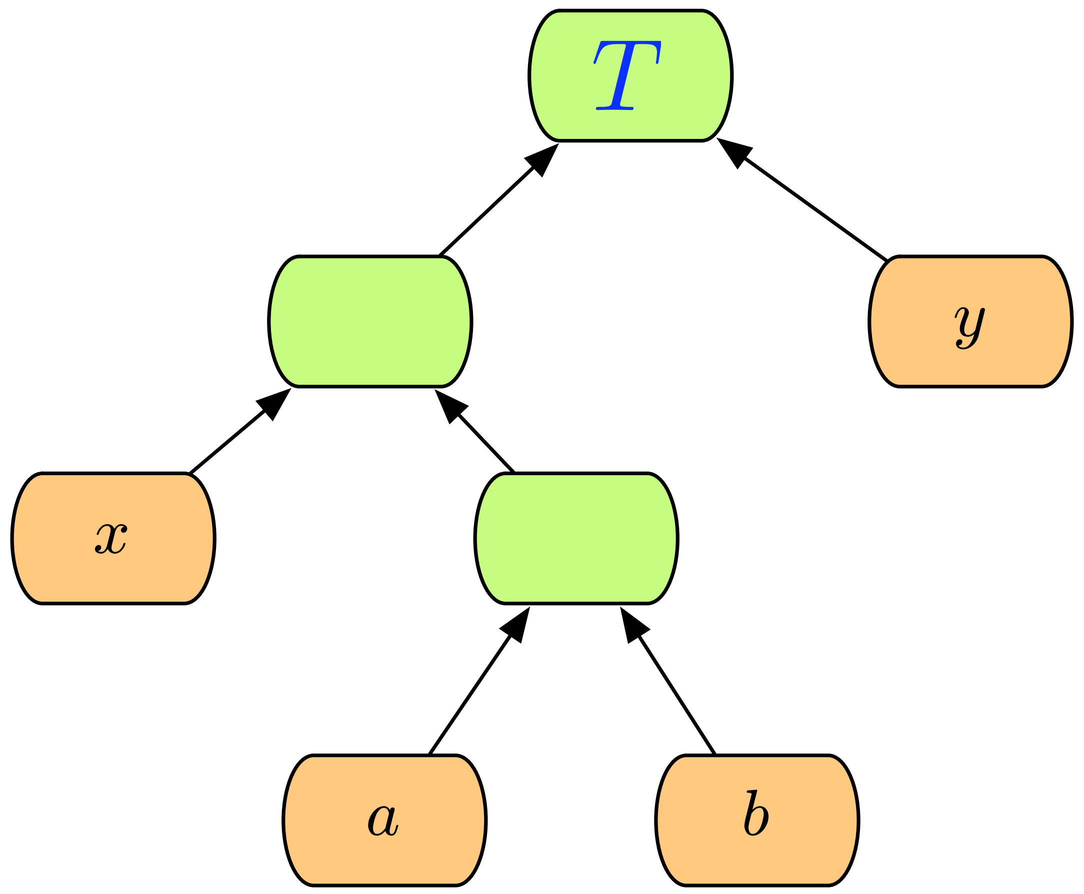
Let T be an optimal code. If x, y are siblings in T, then the lemma holds.

Otherwise, since T is full, let a, b be the sibling nodes with the largest depth. (Q: Why do a, b exist?)

exchange argument

LEMMA: Let $x, y \in C$ be characters with smallest frequencies f_x, f_y. There exists an optimal prefix code T'' for C in which x, y are siblings. That is, the codes for x, y have the same length and only differ in the last bit.

EXAMPLE OF SUCH A TREE



exchange argument

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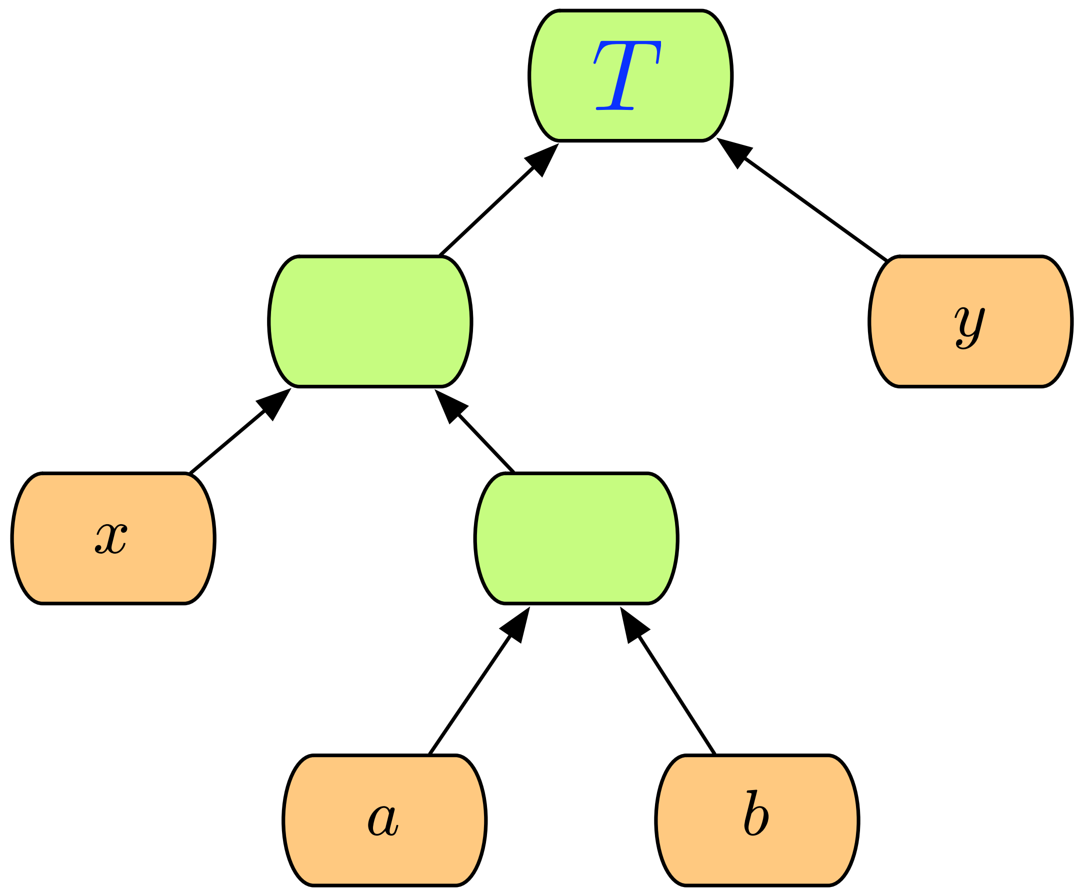
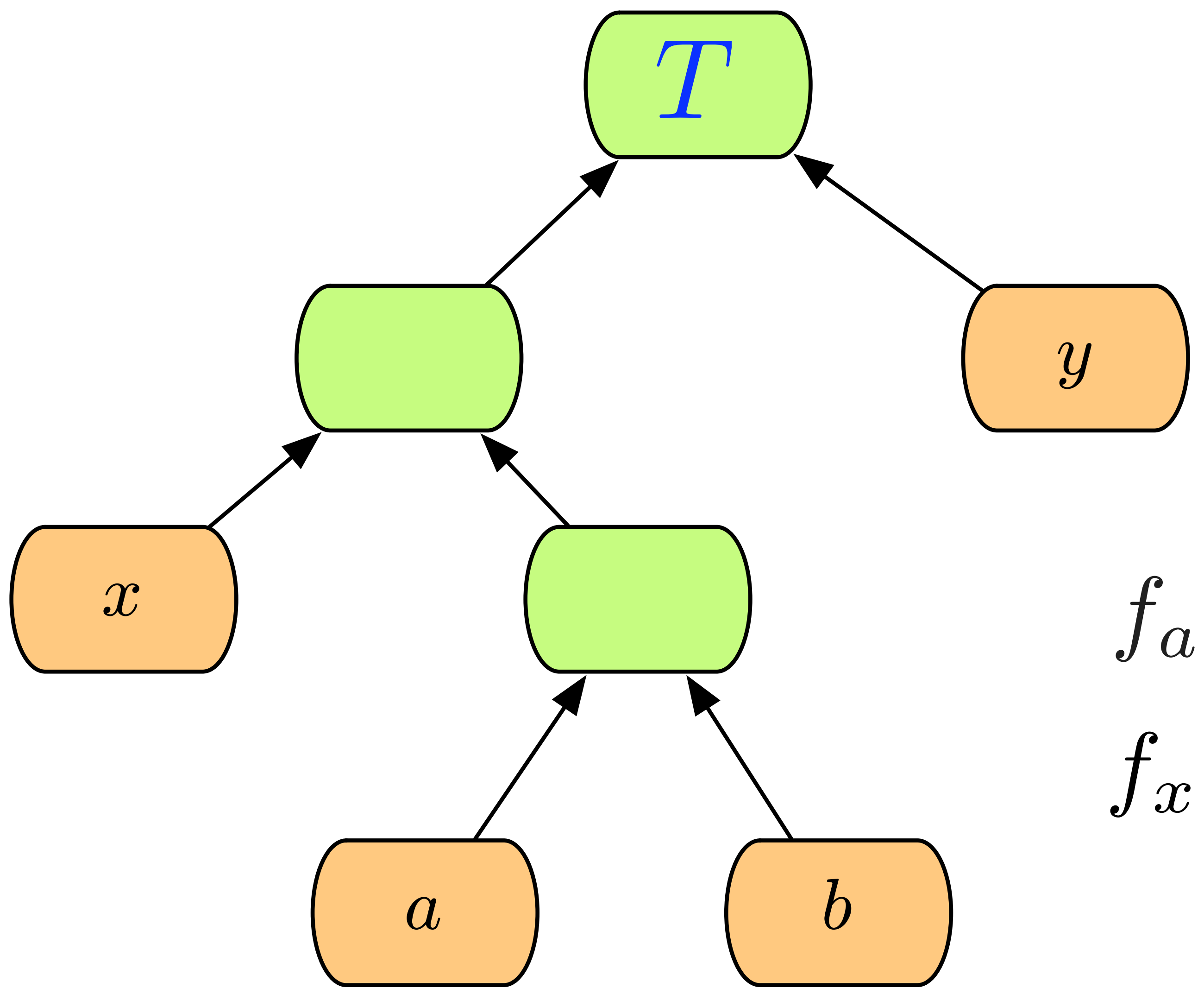


Suppose wlog that $f_x \leq f_a, f_y \leq f_b$

The first step is to exchange x with a to construct a new tree T'.

exchange argument

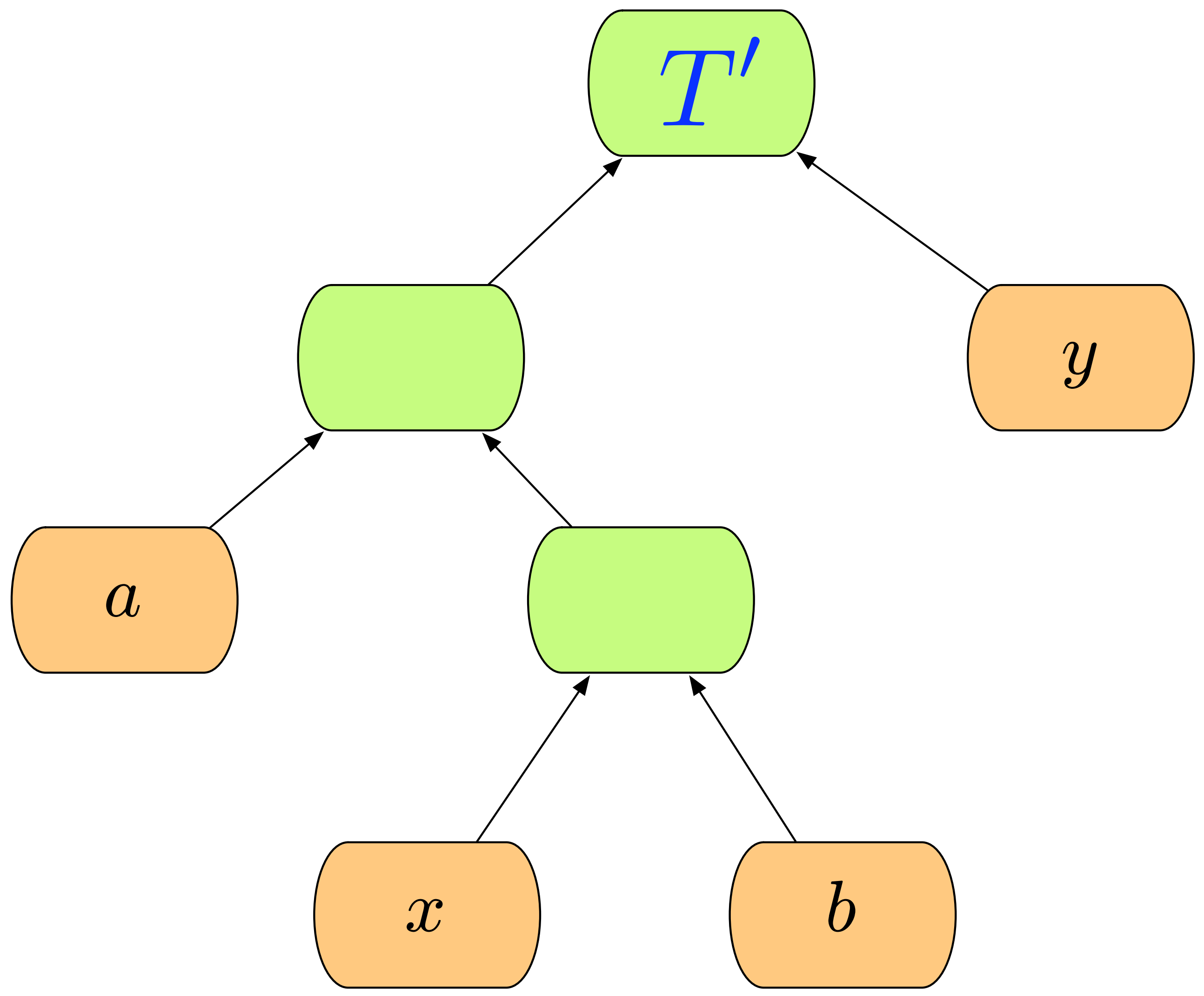
LEMMA: Let $x, y \in C$ be characters with smallest frequencies f_x, f_y. There exists an optimal prefix code T'' for C in which x, y are siblings. That is, the codes for x, y have the same length and only differ in the last bit.



FIRST STEP

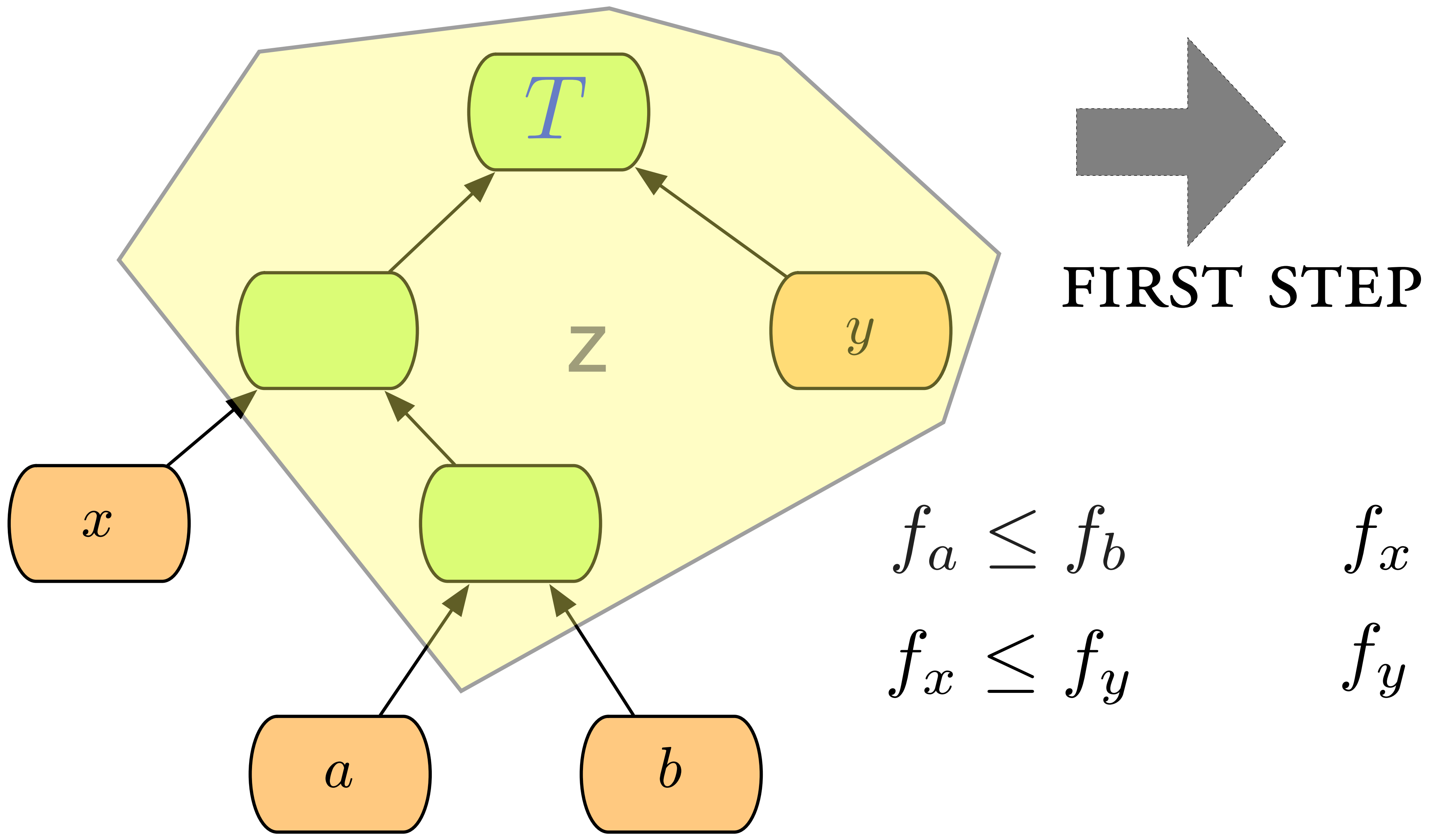
$f_a \leq f_b$ $\quad$ $f_x \leq f_a$

$f_x \leq f_y$ $\quad$ $f_y \leq f_b$



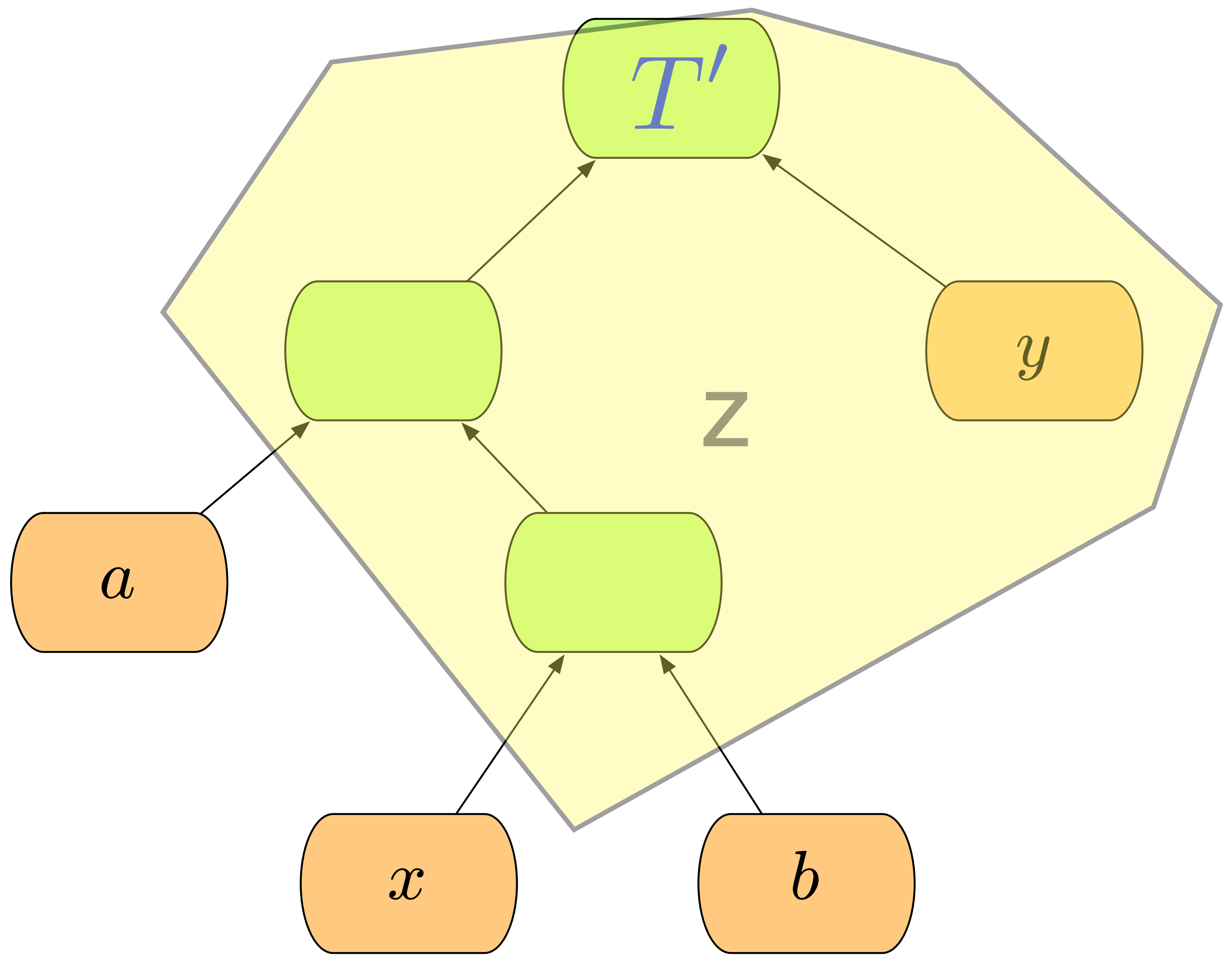
exchange argument

LEMMA: Let $x, y \in C$ be characters with smallest frequencies f_x, f_y. There exists an optimal prefix code T'' for C in which x, y are siblings. That is, the codes for x, y have the same length and only differ in the last bit.



$f_a \leq f_b$ $\quad$ $f_x \leq f_a$

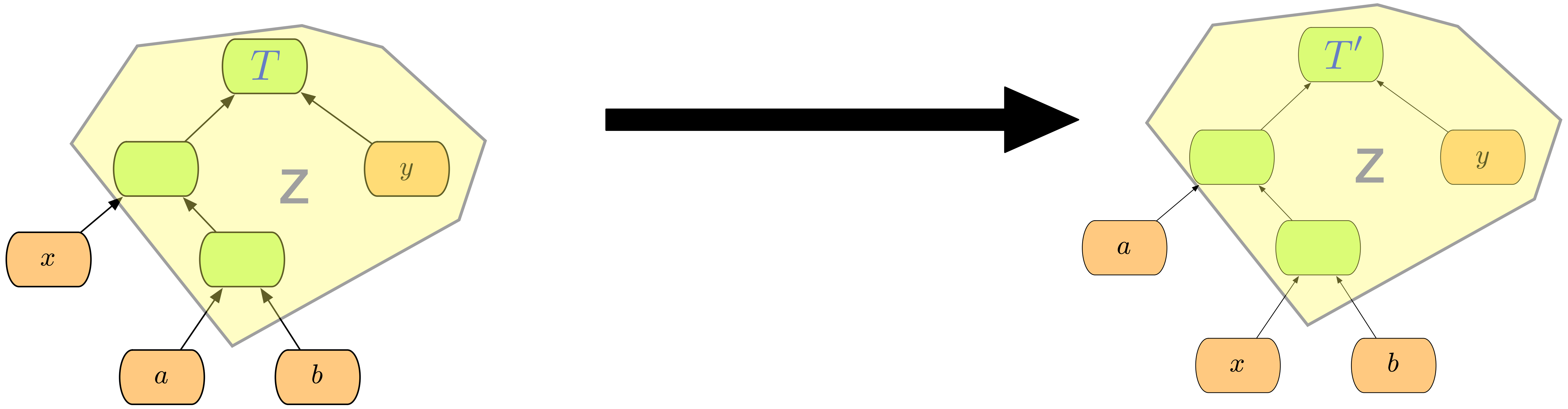
$f_x \leq f_y$ $\quad$ $f_y \leq f_b$



$$B(T) = Z + f_x \cdot \ell_x + f_a \cdot \ell_a$$

$$B(T') = Z + f_x \cdot \ell_a + f_a \cdot \ell_x$$

This tree is optimal.

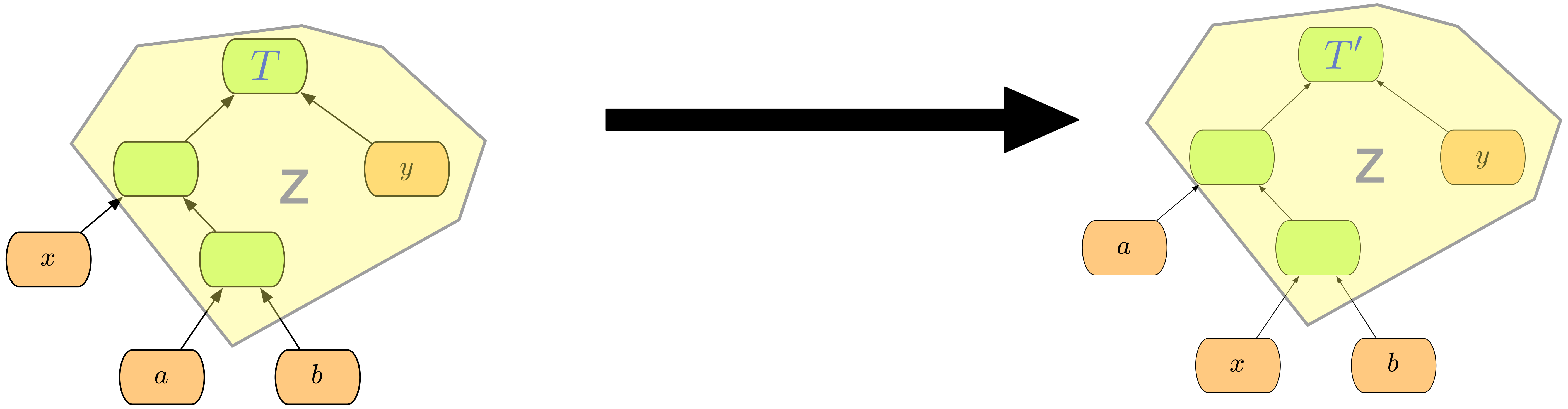


$$B(T) = Z + f_x \cdot \ell_x + f_a \cdot \ell_a$$

$$B(T') = Z + f_x \cdot \ell_a + f_a \cdot \ell_x$$

$$B(T) - B(T') =$$

This tree is optimal.

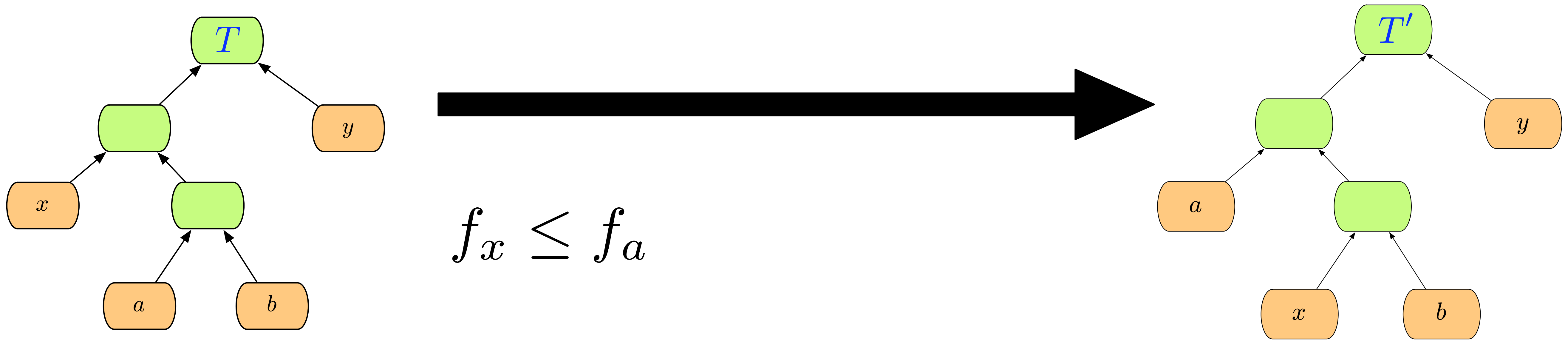


$$B(T) = Z + f_x \cdot \ell_x + f_a \cdot \ell_a \qquad B(T') = Z + f_x \cdot \ell_a + f_a \cdot \ell_x$$

$$\begin{aligned} B(T) - B(T') &= f_x\ell_x + f_a\ell_a - f_a\ell_x - f_x\ell_a \\ &= f_x(\ell_x - \ell_a) - f_a(\ell_x - \ell_a) \\ &= (f_x - f_a)(\ell_x - \ell_a) \end{aligned}$$

Both terms must be ≤ 0 because $f_x \leq f_a, \ell_x \leq \ell_a$

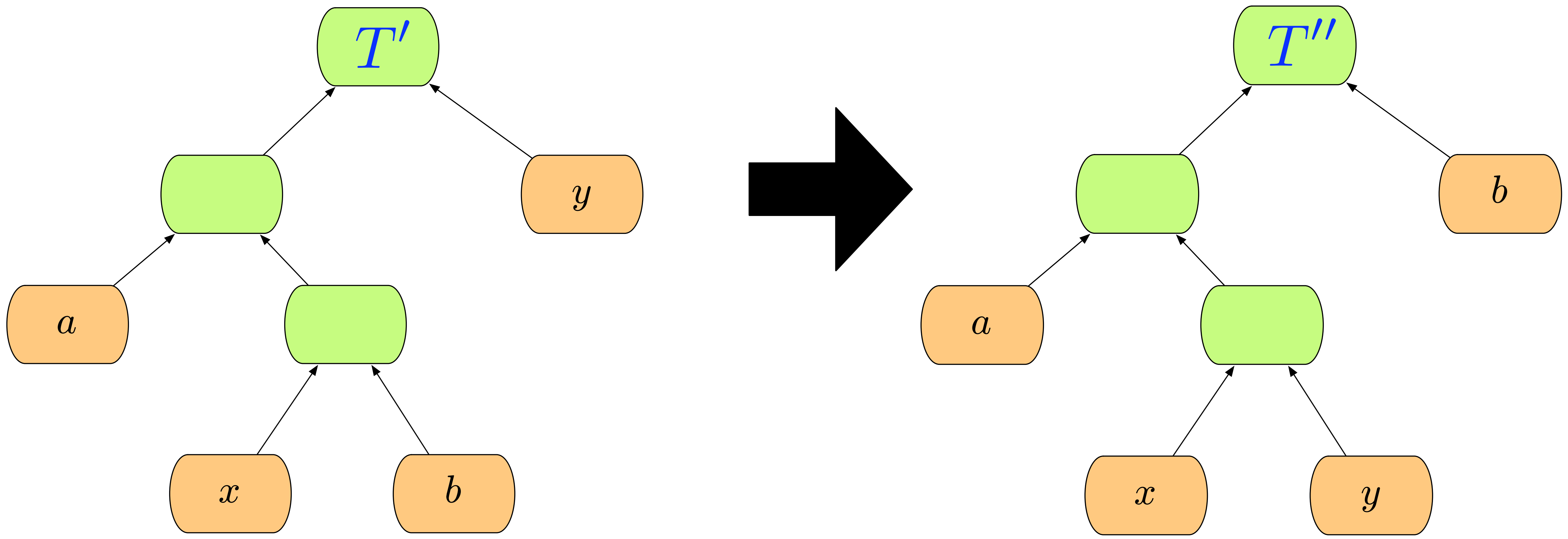
But since $B(T)$ is optimal, the product must be 0.



$$B(T) = \sum_c f_c \ell_c + f_x \ell_x + f_a \ell_a \qquad B(T') = \sum_c f_c \ell'_c + f_x \ell'_x + f_a \ell'_a$$

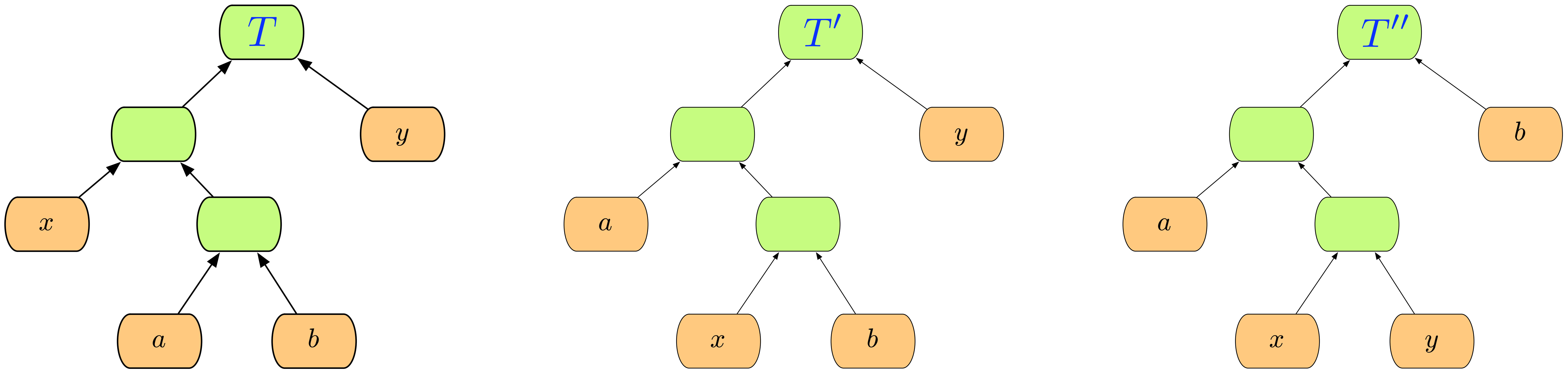
$$B(T) - B(T') = 0$$

exchange argument



We can apply the same argument to y, b.

$$B(T') - B(T'') = 0$$



T
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b
y
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a
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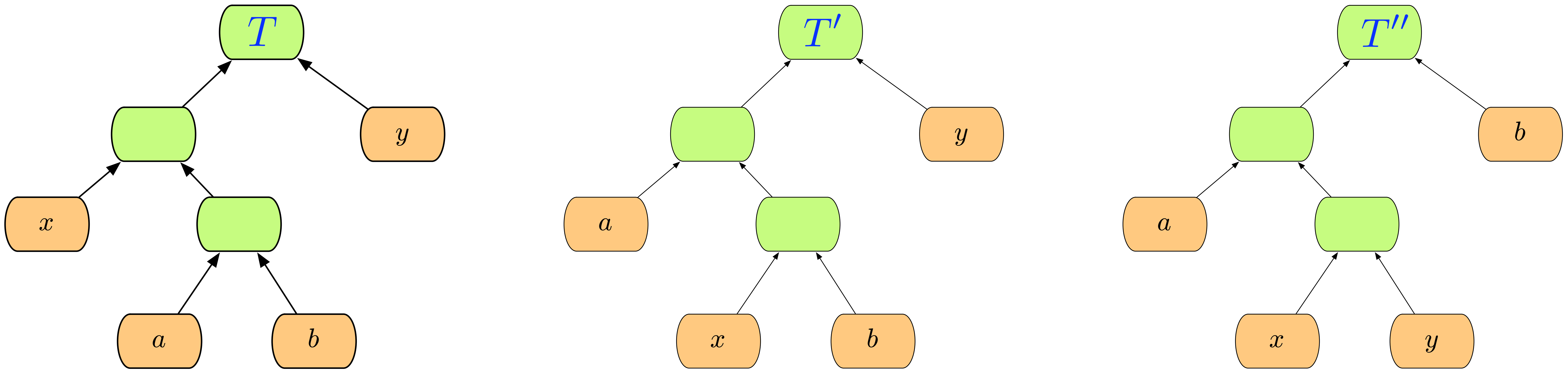
b
y
T′′
a
x
y
b

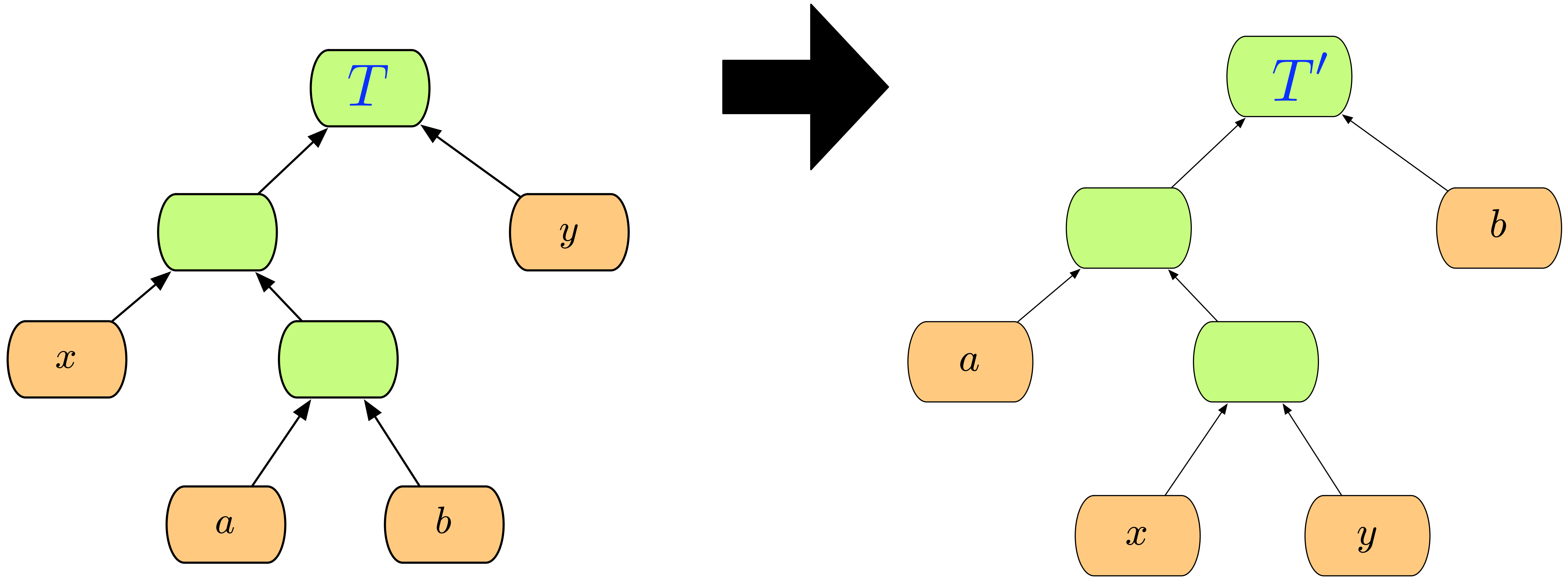


$B(T) - B(T') \geq 0$ $\quad$ $B(T') - B(T'') \geq 0$

T'' IS ALSO OPTIMAL

exchange argument

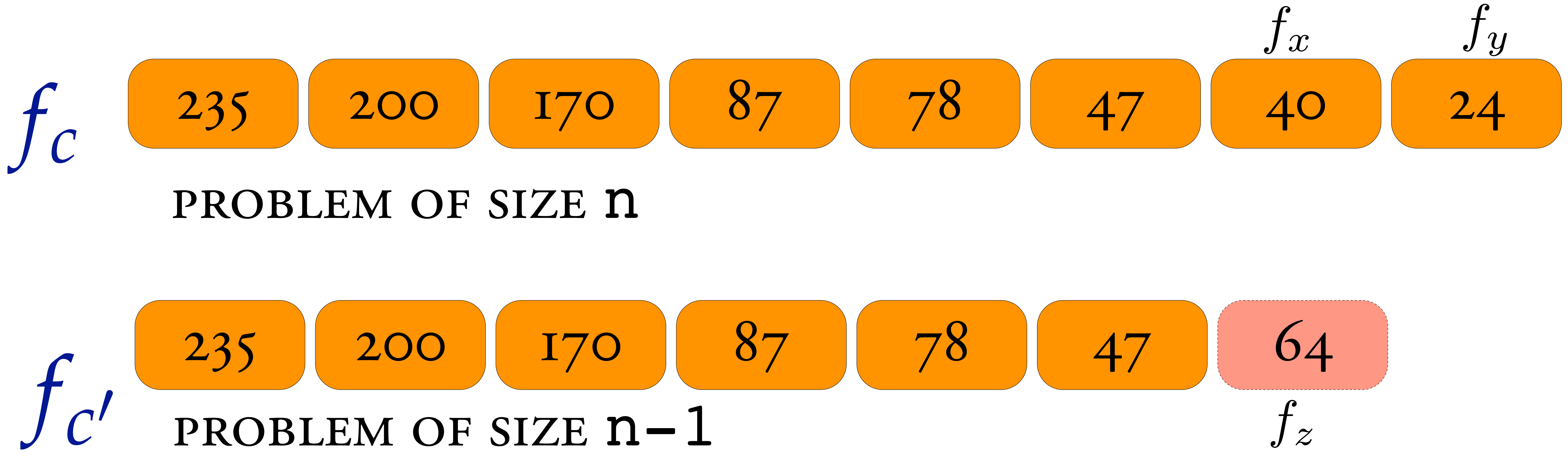
LEMMA: Let $x, y \in C$ be characters with smallest frequencies f_x, f_y. There exists an optimal prefix code T'' for C in which x, y are siblings. That is, the codes for x, y have the same length and only differ in the last bit.



optimal sub-structure

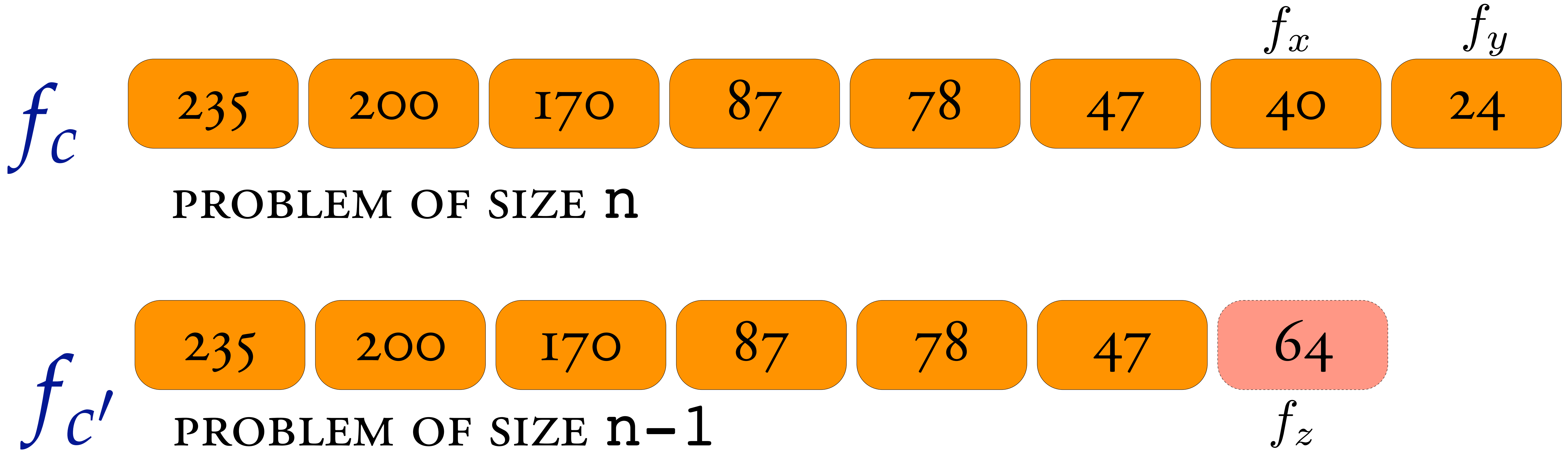
							f_x	f_y
f_c	235	200	170	87	78	47	40	24

optimal sub-structure



LEMMA:

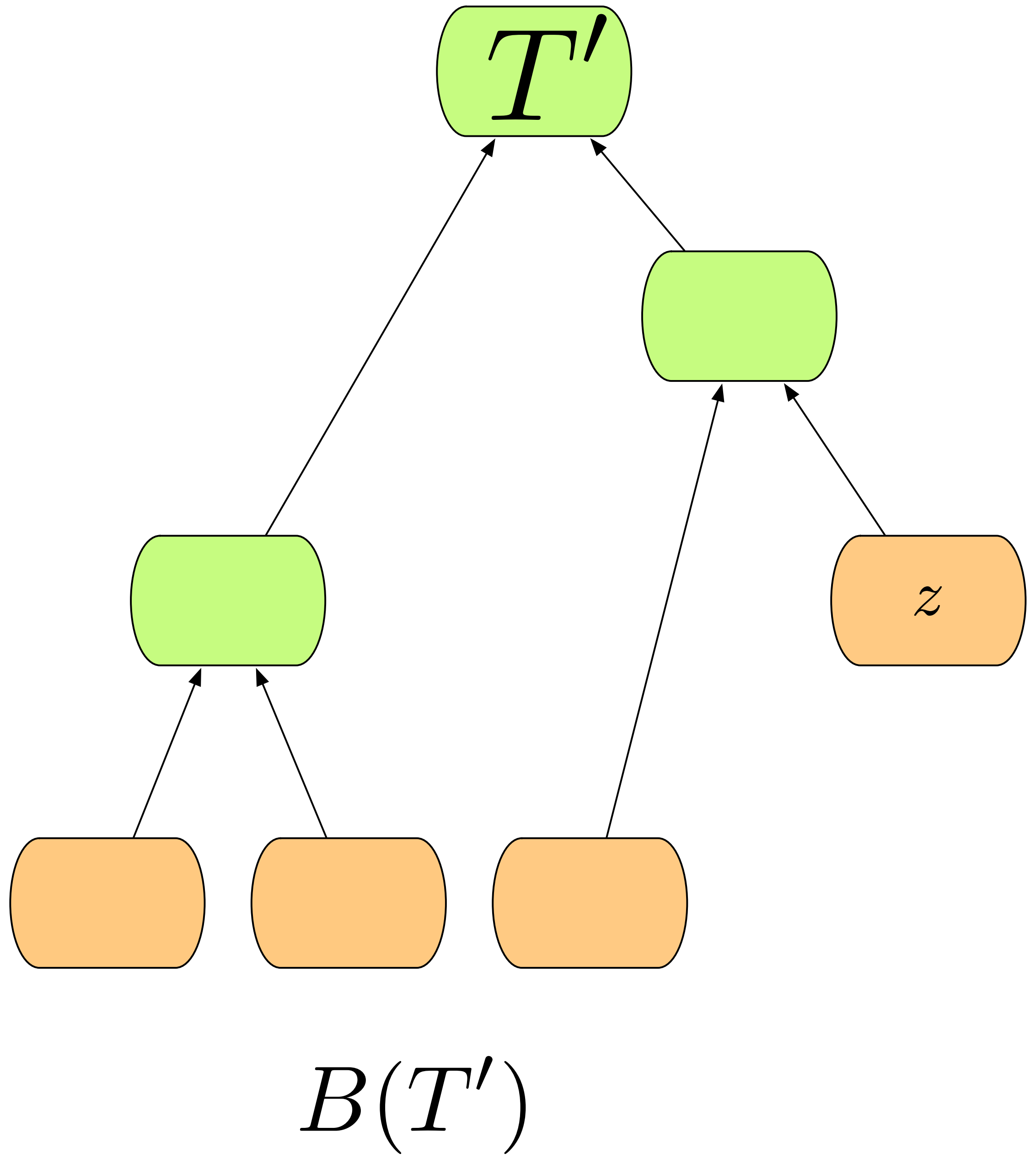
optimal sub-structure



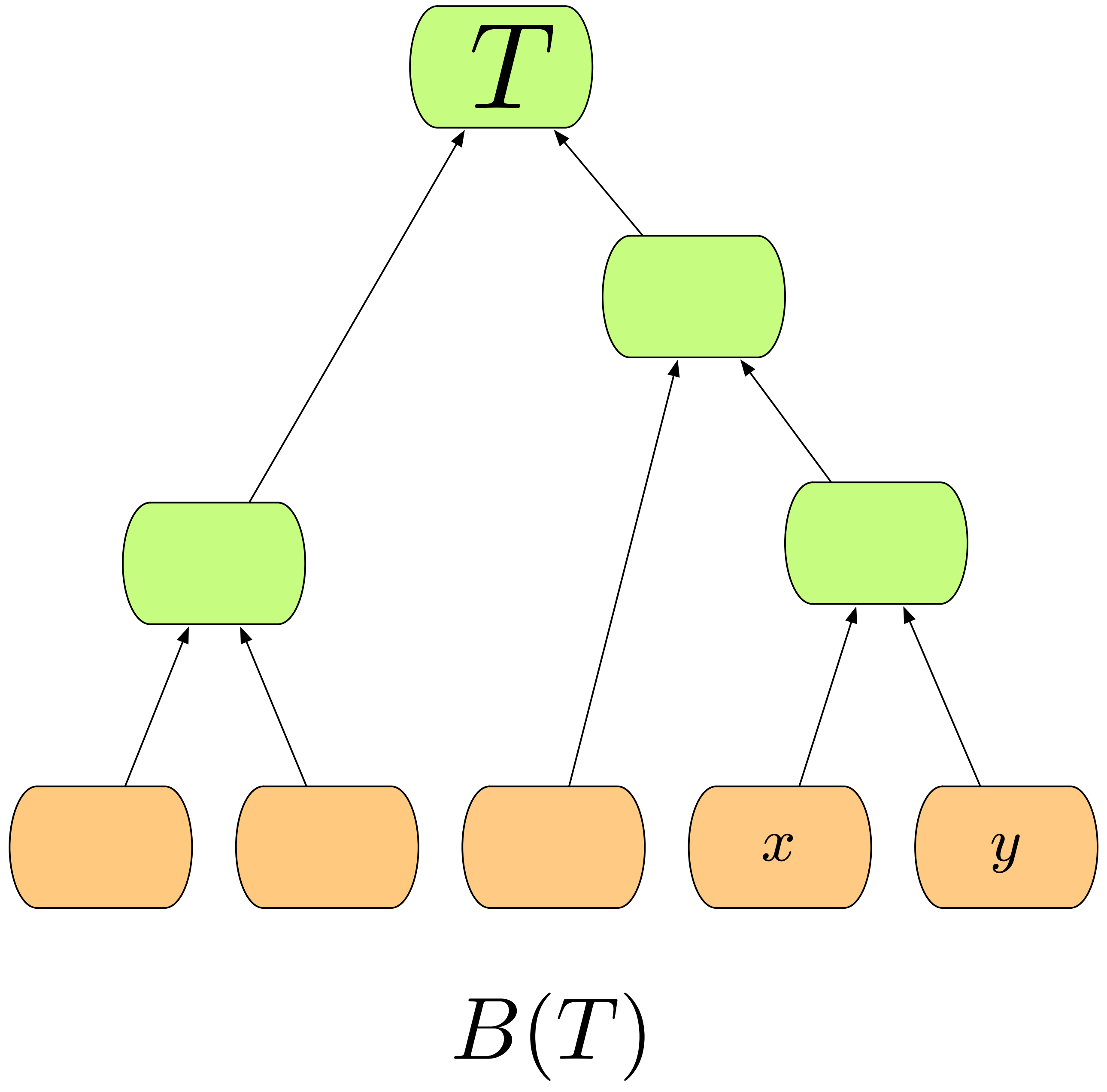
LEMMA: The optimal solution T for f_c consists of computing an optimal solution T' for $f_{c'}$ and replacing the node for z with an internal node having children x, y.

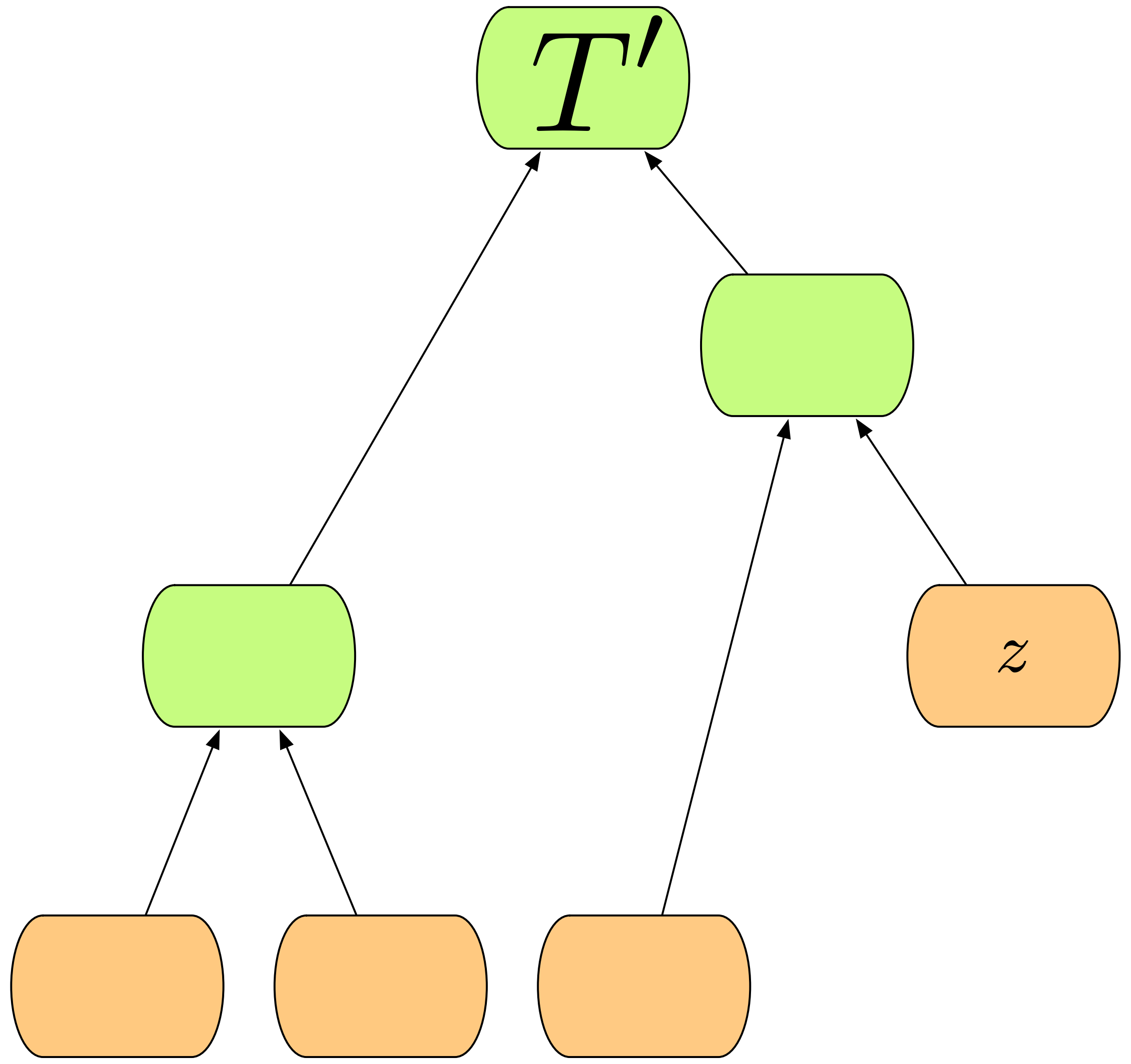
Let T' be an optimal solution for $f_{c'}$ of size n-1.

Our lemma suggests constructing T by replacing z with {x,y} leaves.



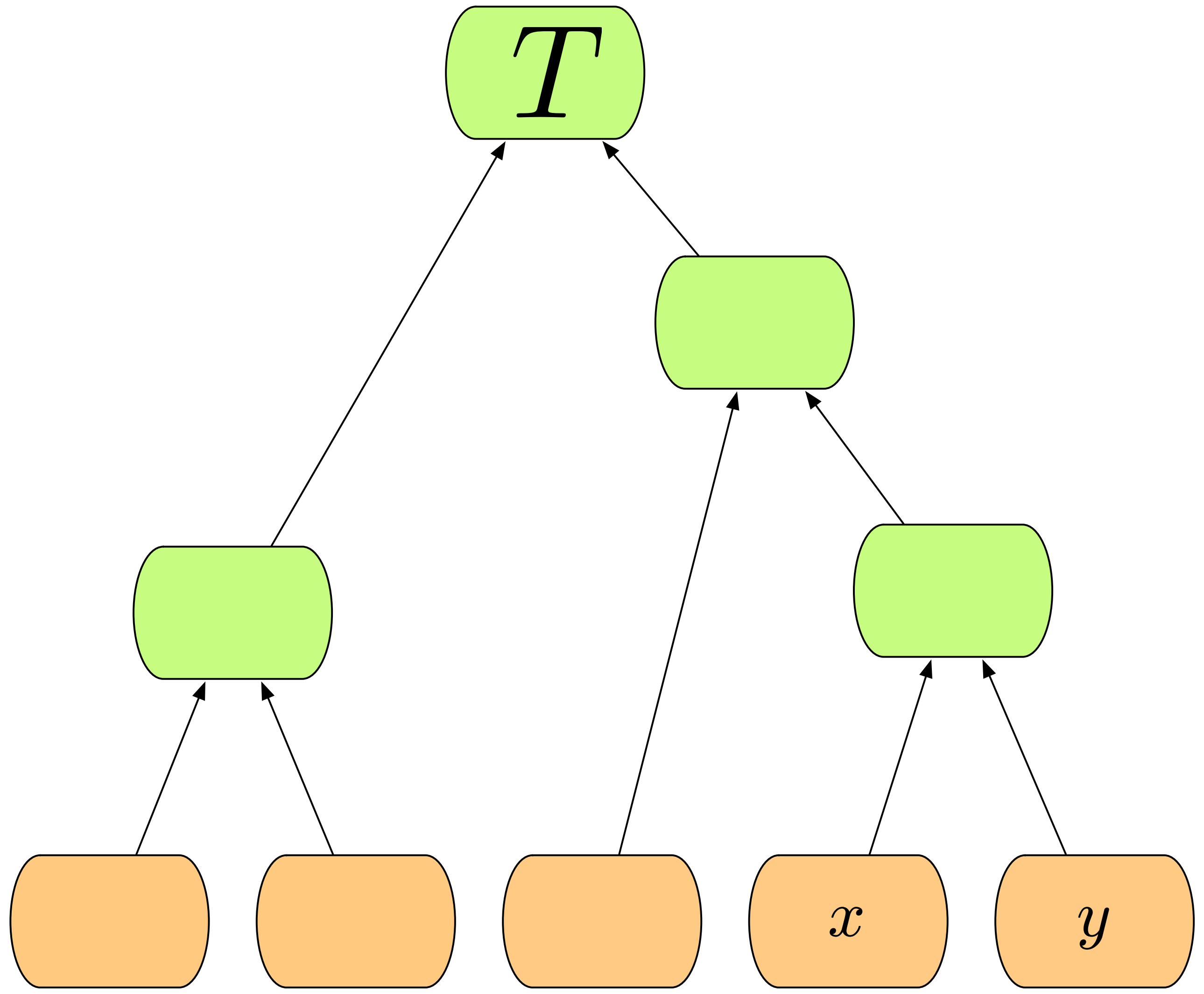
Lets analyze $B(T)$





$B(T')$

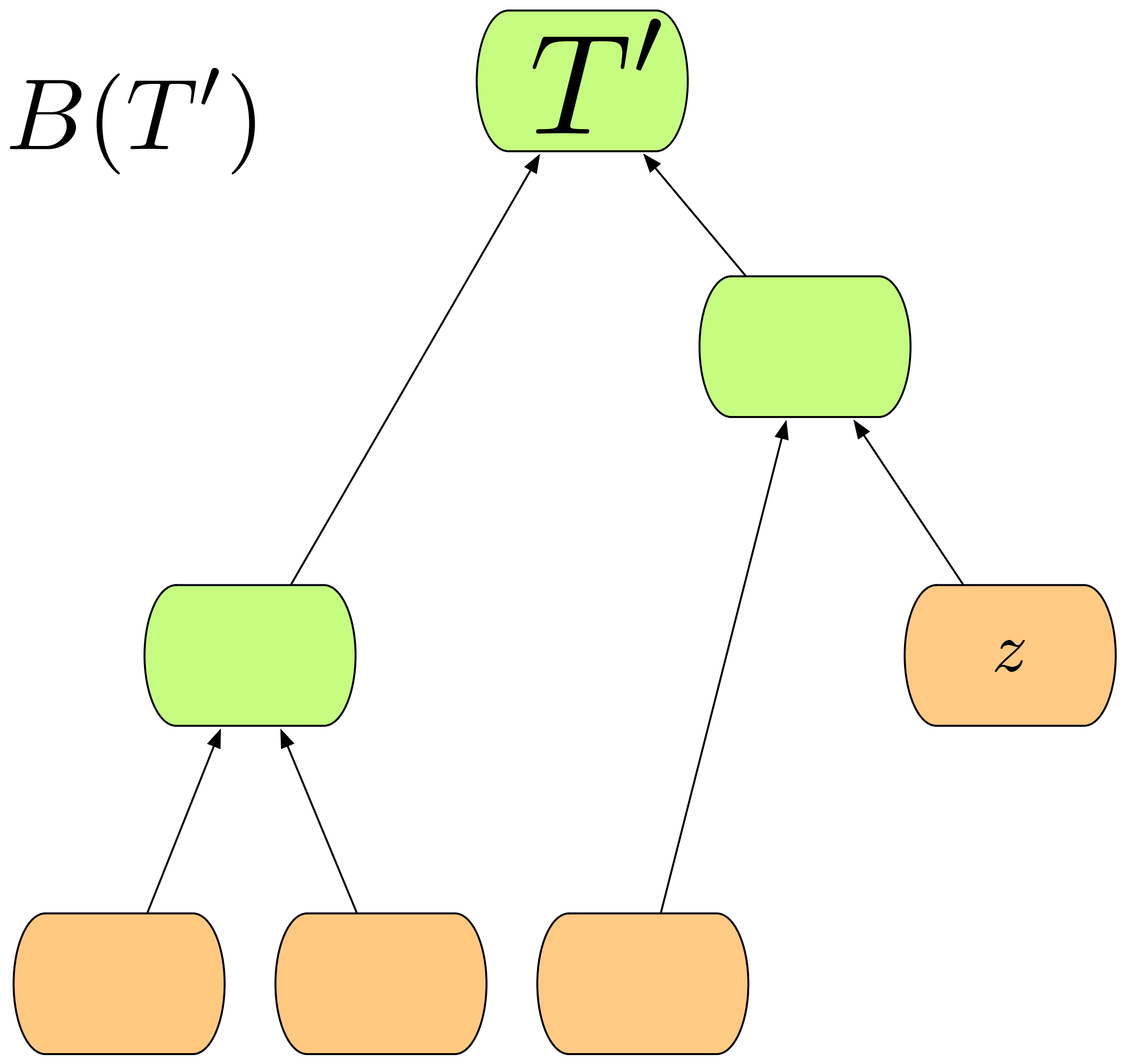
Lets analyze $B(T)$

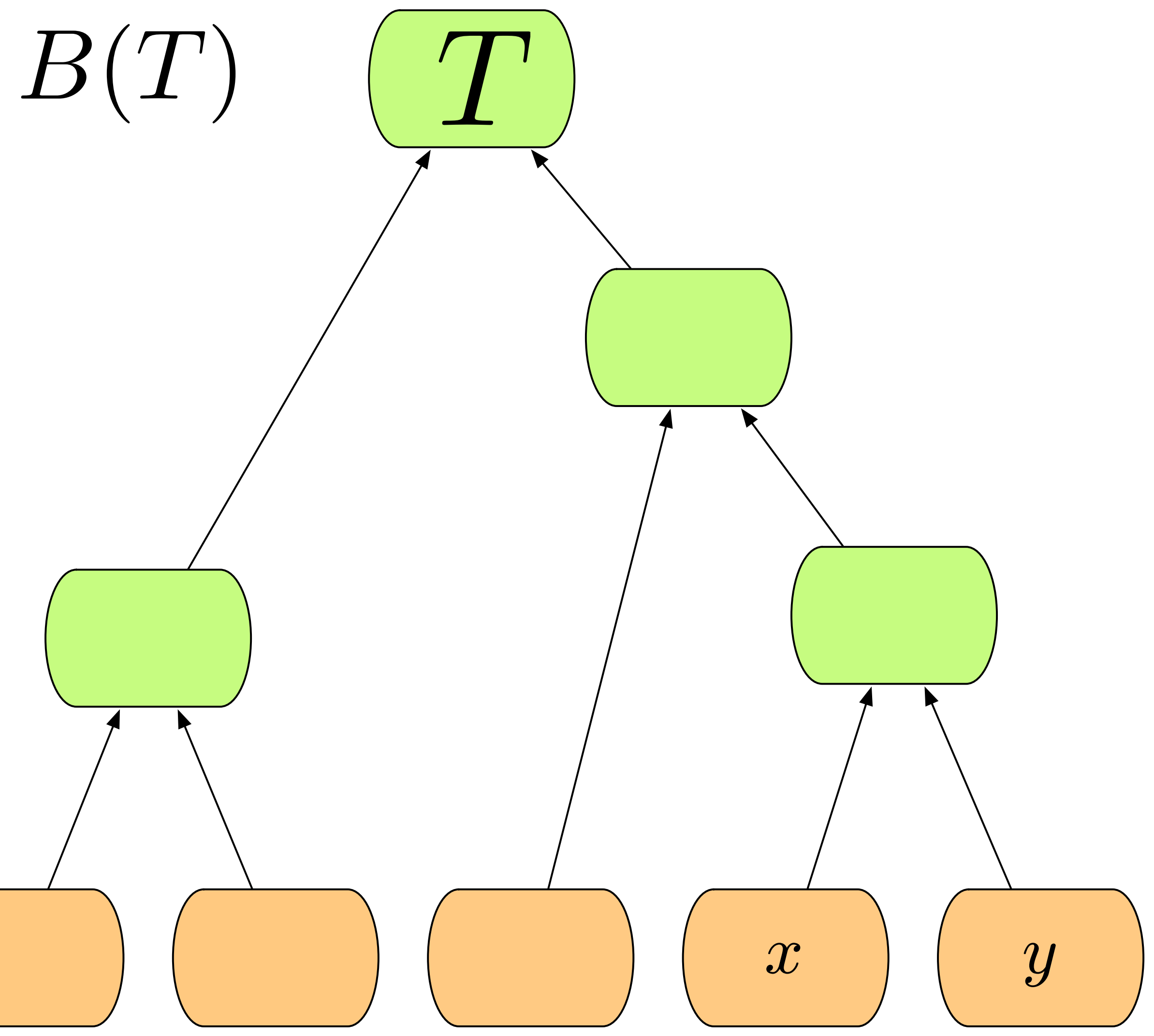


$B(T)$

$$B(T) = B(T') - f_z \ell_z + (\ell_z + 1)(f_x + f_y)$$

$$= B(T') + f_x + f_y$$





Rearranging, we get

$$B(T') = B(T) - f_x - f_y$$

Suppose T is not optimal

What does that mean?

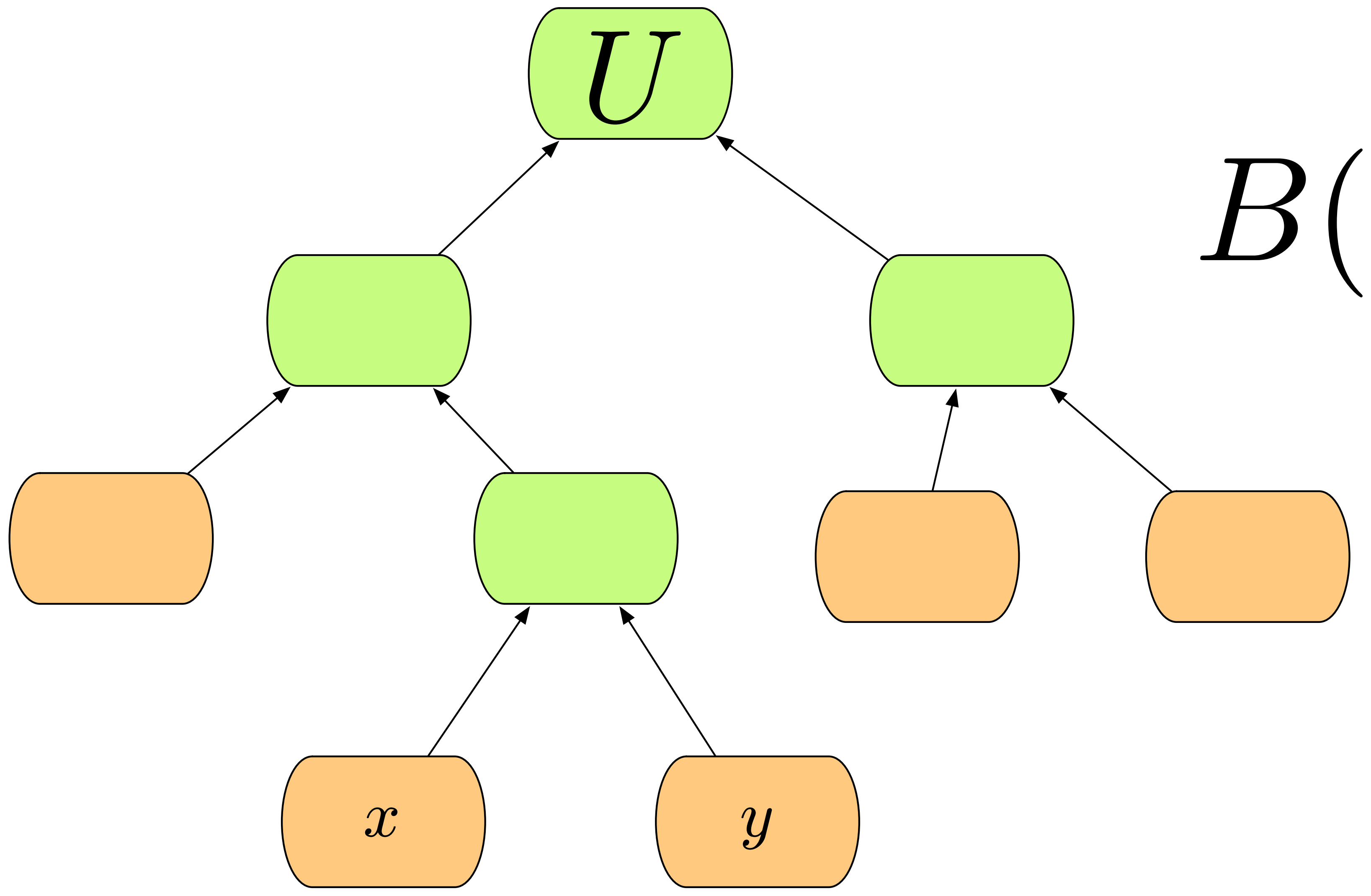
Suppose T is not optimal

What does that mean?

There exists another tree U such that $B(U) < B(T)$.

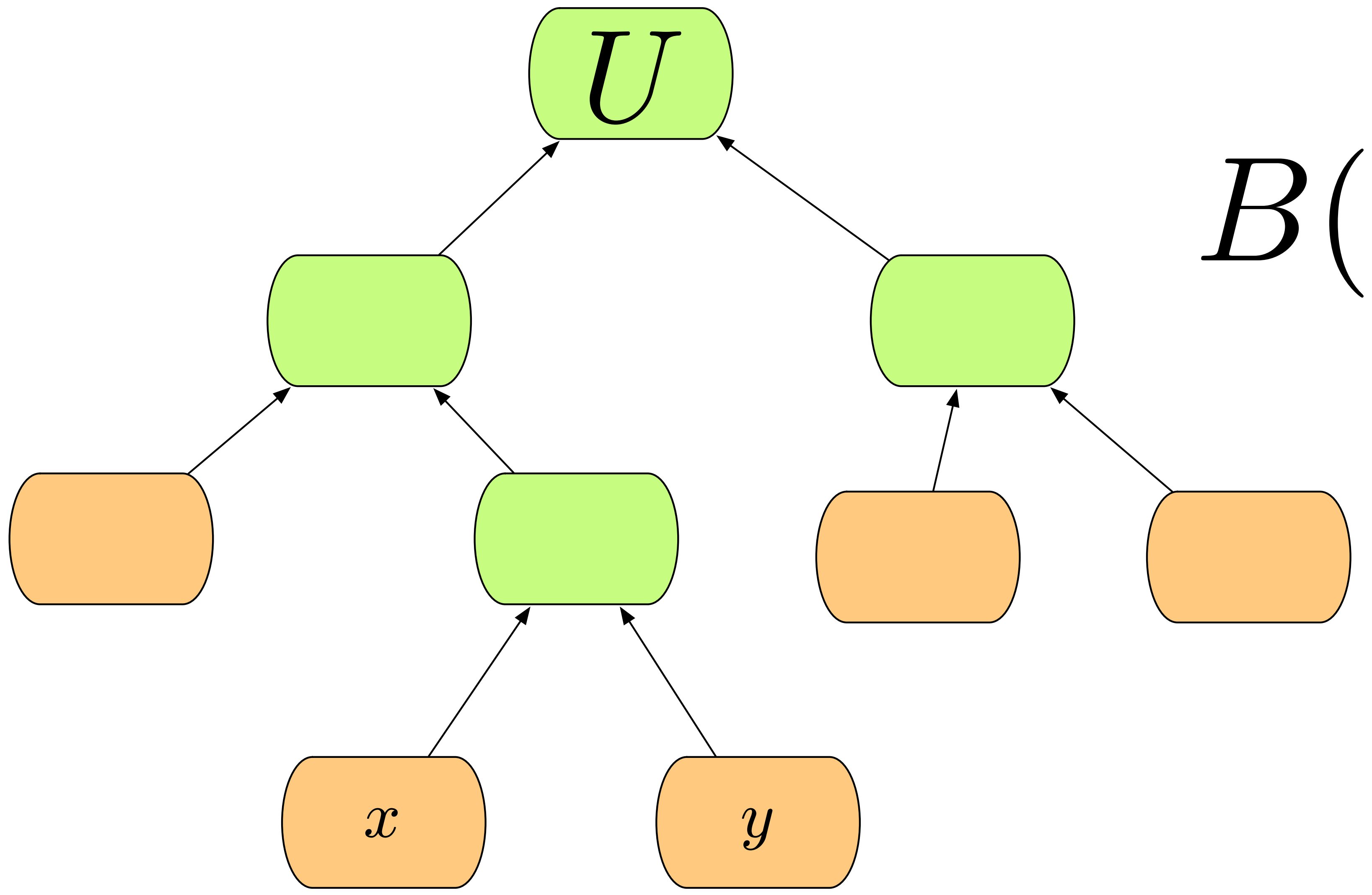
Moreover, by the exchange lemma, there exists a U' such that x,y are siblings.

Suppose T is not optimal



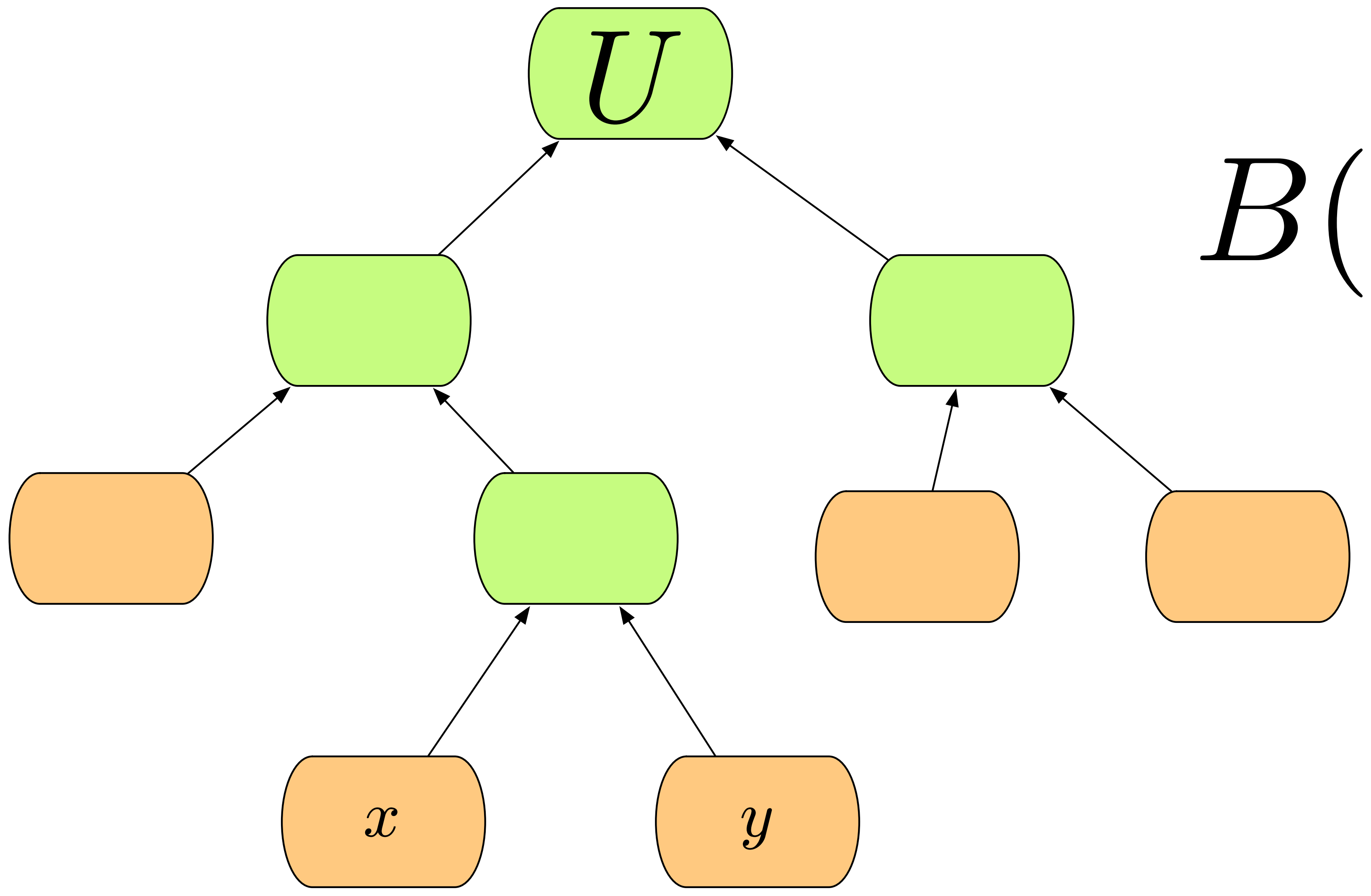
$$B(U) < B(T)$$

Suppose T is not optimal



$$B(U) < B(T) = B(T') + f_x + f_y$$

Suppose T is not optimal

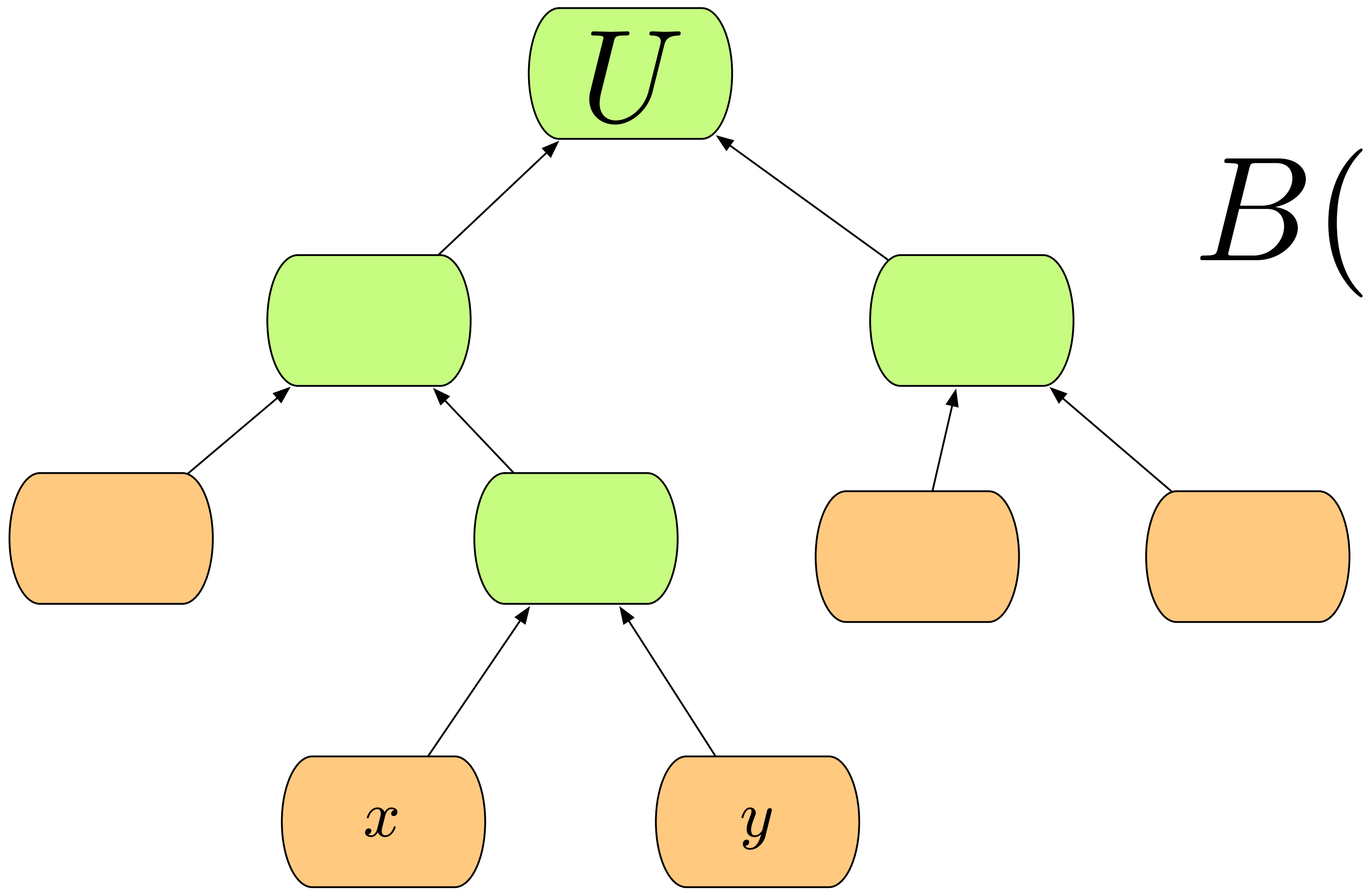


$$B(U) < B(T) = B(T') + f_x + f_y$$

This implies

$$B(U) - f_x - f_y < B(T')$$

Suppose T is not optimal



$$B(U) < B(T) = B(T') + f_x + f_y$$

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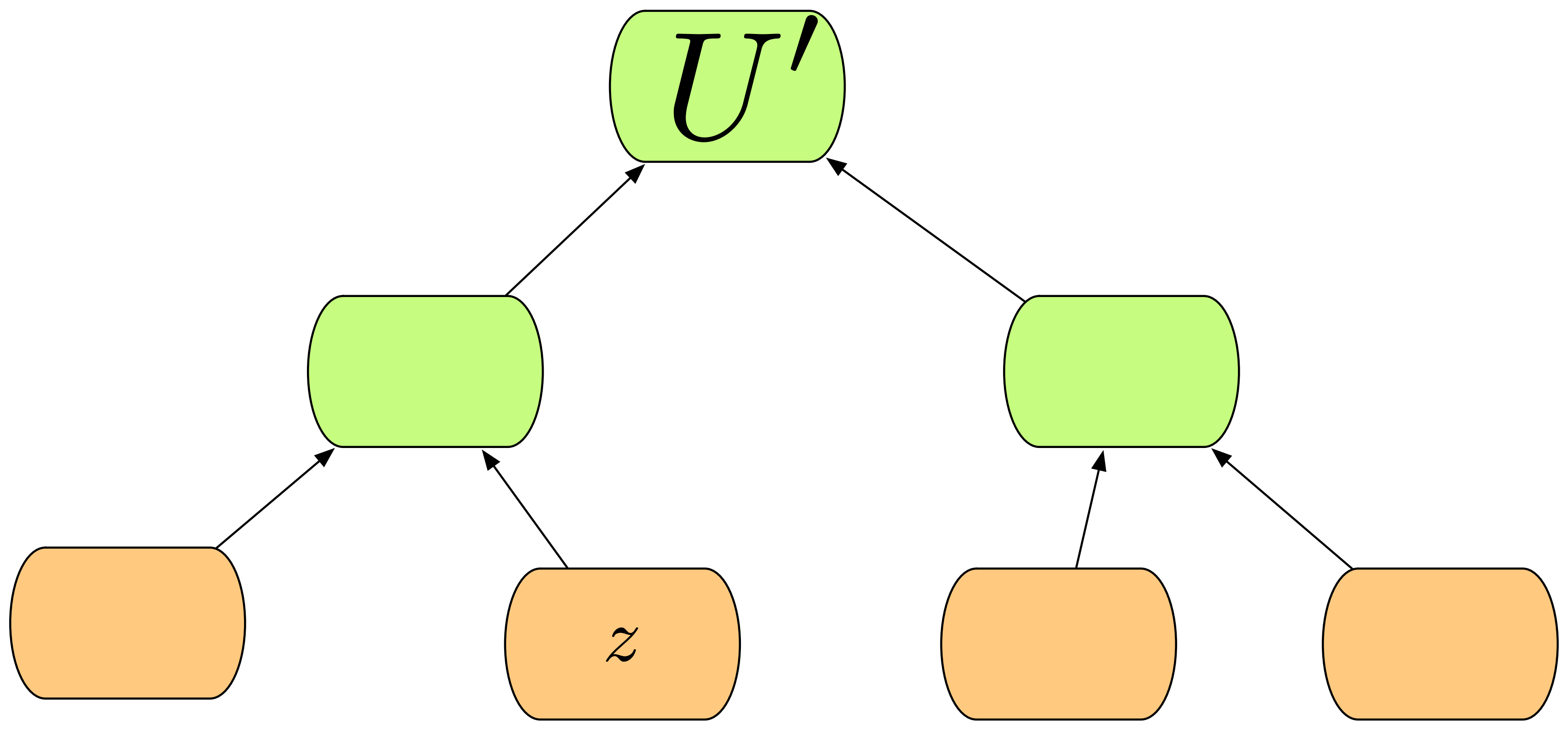
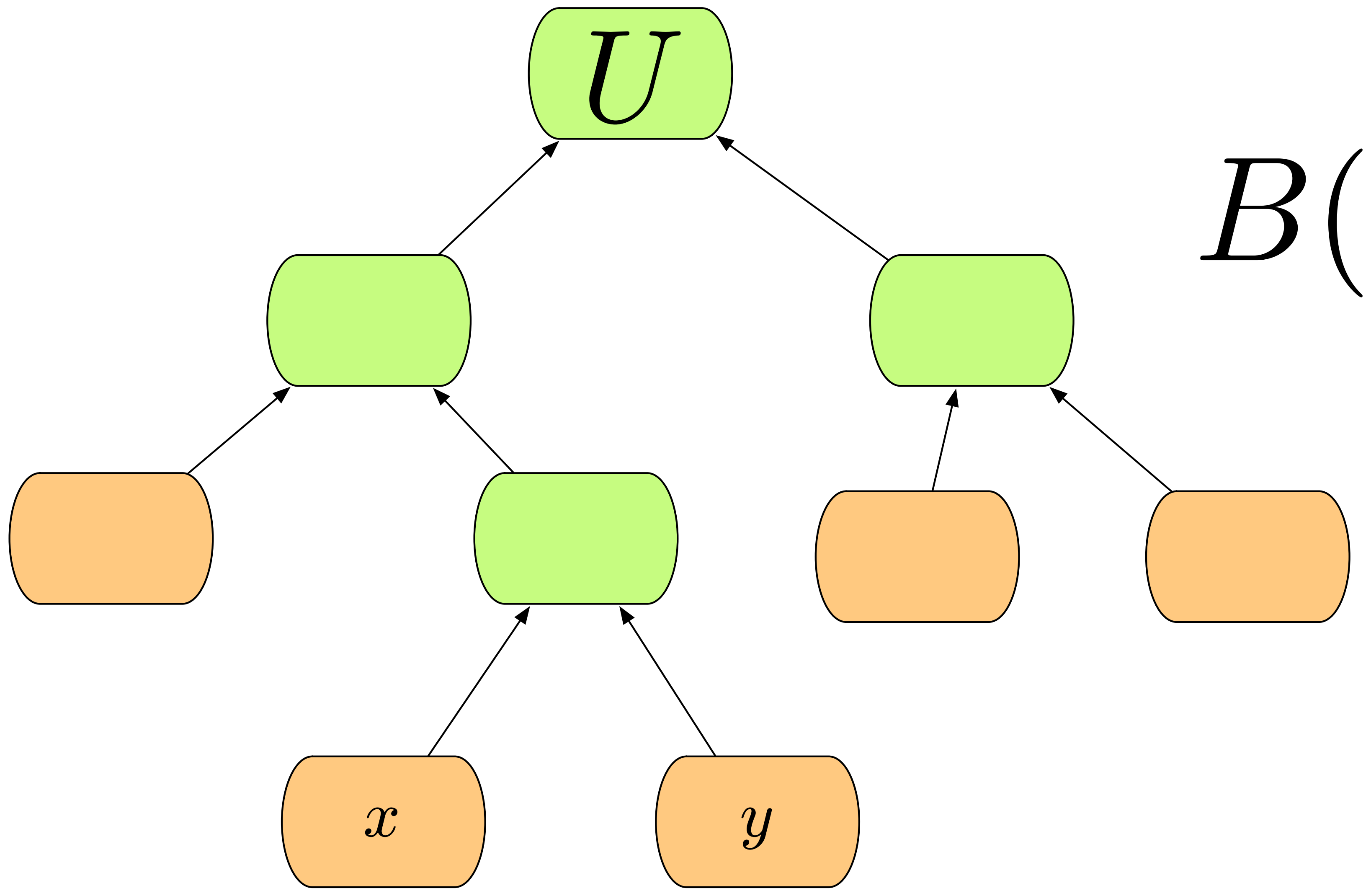
$$B(U') < B(T')$$



Suppose T is not optimal

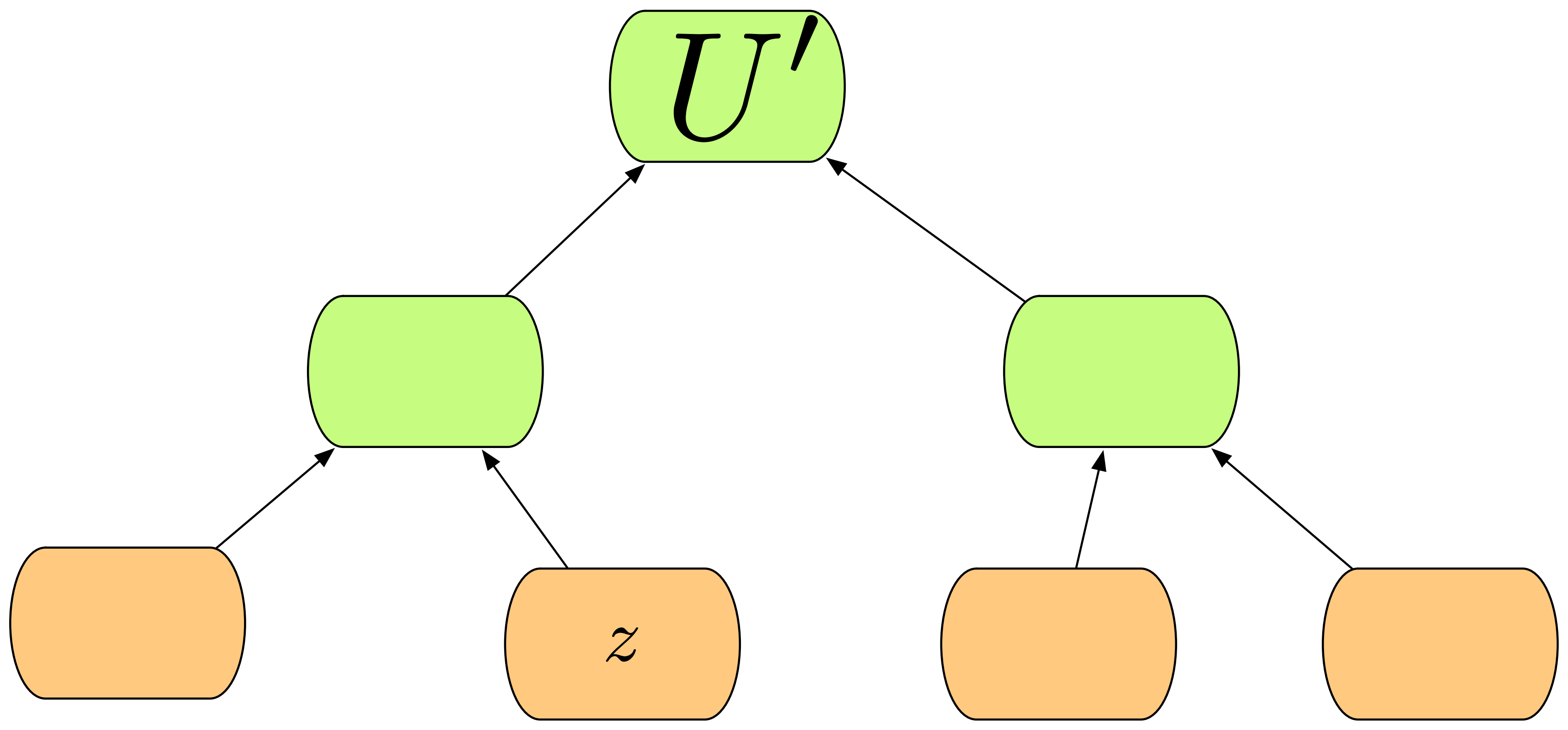


$$B(U) < B(T) = B(T') + f_x + f_y$$

This implies

$$B(U) - f_x - f_y < B(T')$$

$$B(U') < B(T')$$



Which means that T' was not optimal!
This is a contradiction, which means that our supposition (T not optimal) must be wrong.