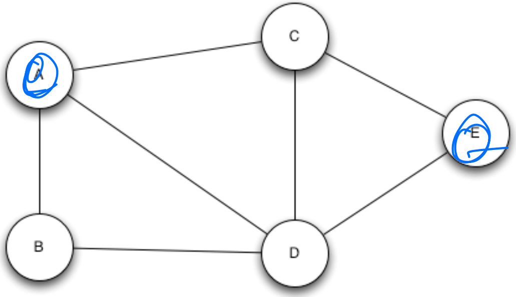
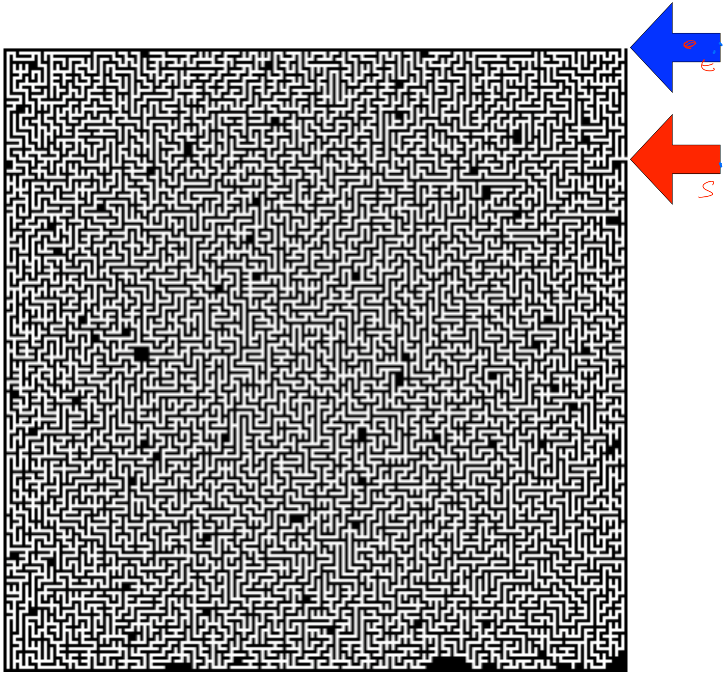
5800
Shortest paths

mar 8/10 2022
shelat

simple graph questions



WHAT IS THE LENGTH OF THE PATH FROM A TO E?


E
S

shortest path property

DEFINITION:

$\delta(s, v)$ = length of the shortest path from s using edges in G to v.

sum of the weights along the edges on the path

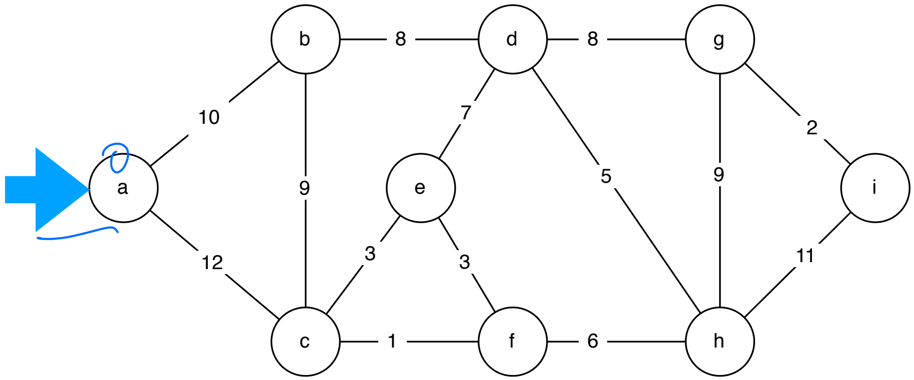
→ By convention, we set this to ∞ if there is no path from s to v.

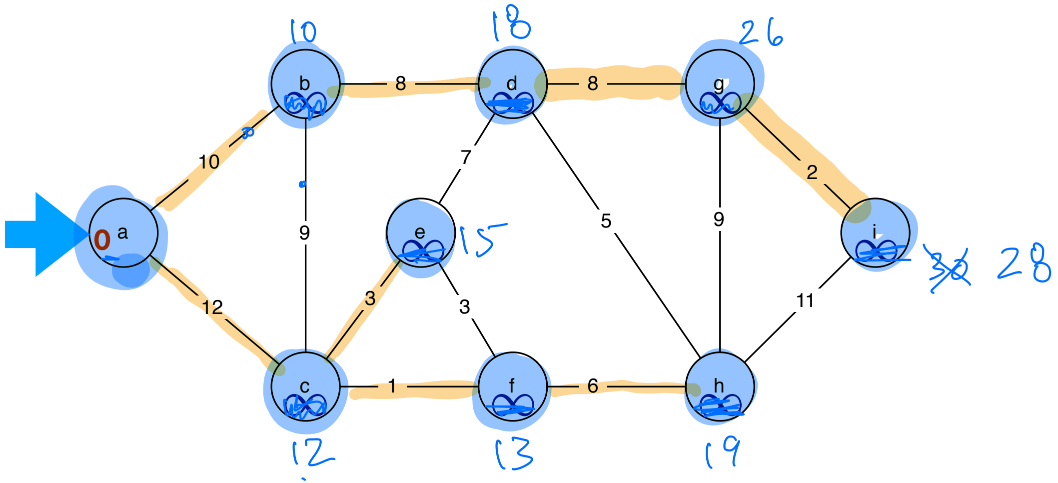
shortest path property

DEFINITION:

$\delta(s, v)$ Length of the shortest path from s to v, set to ∞ if there is no path from s to v

shortest paths from a





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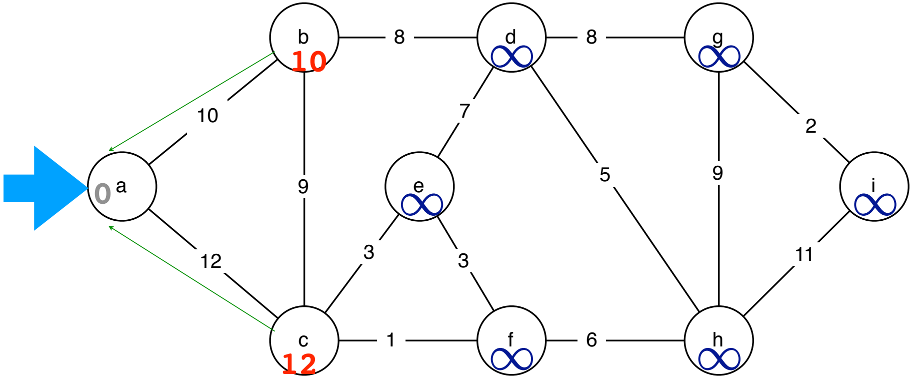

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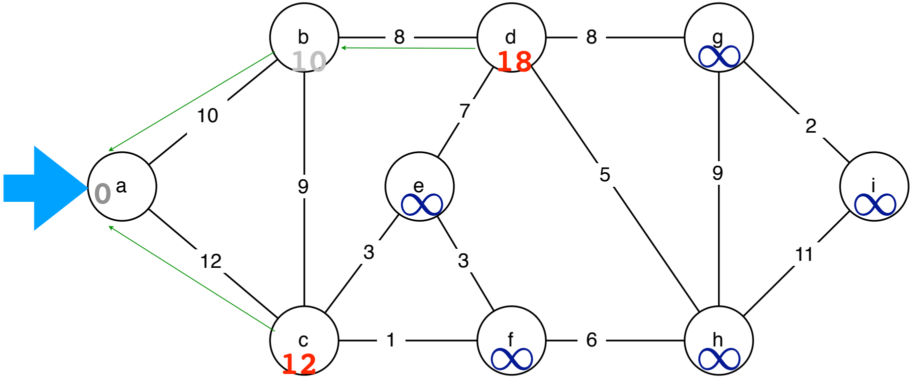

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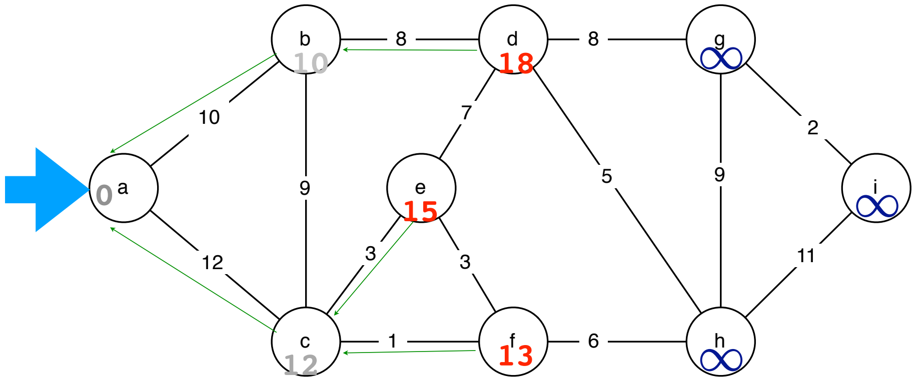

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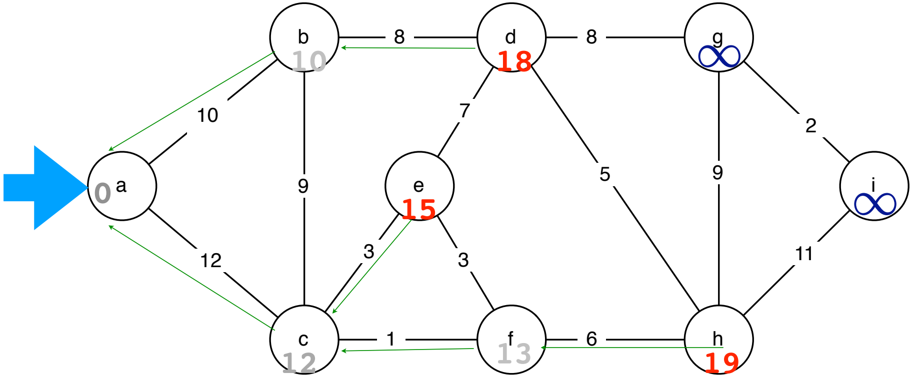

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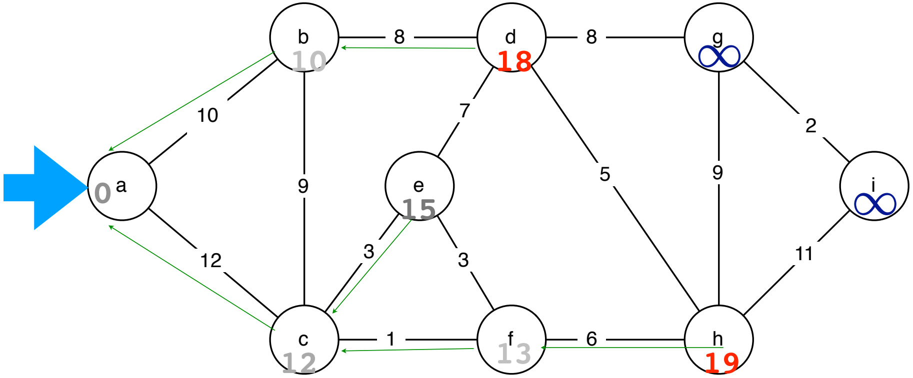

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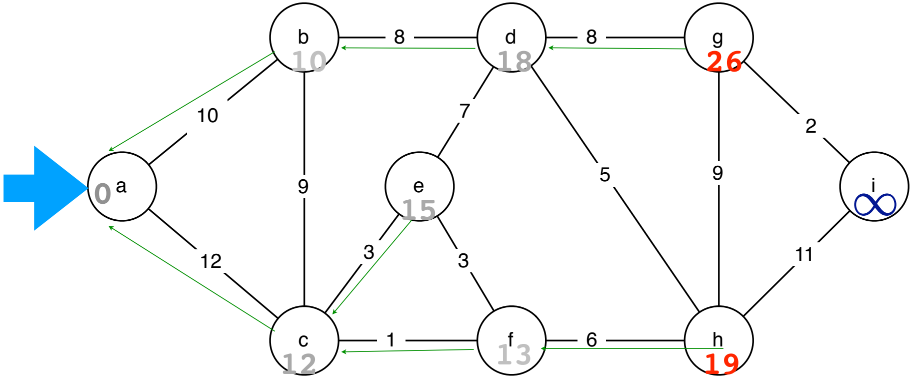
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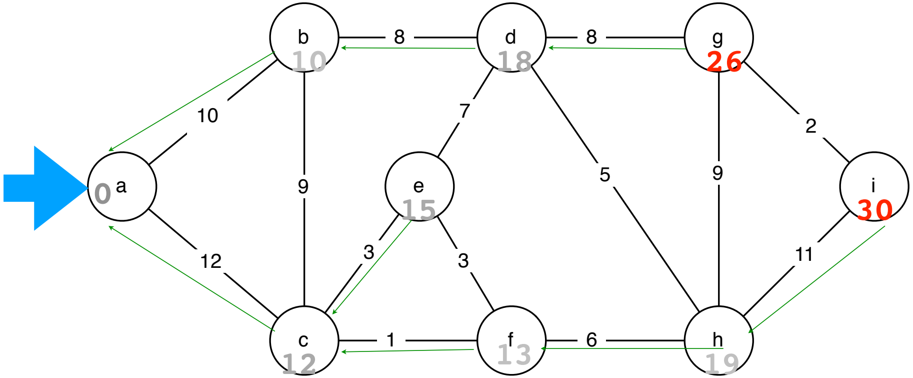

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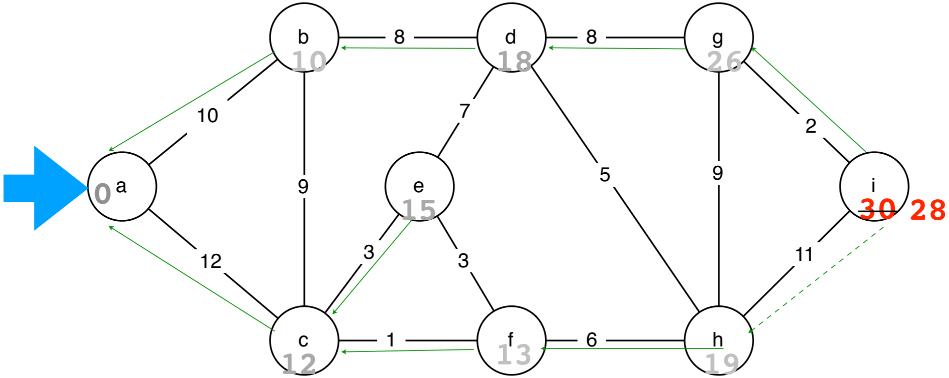

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Algorithm

Dijkstra (G, w, s) $\quad s \in V$.

① INITIALIZE $d_v = \infty$ for $v \in S$, $d_s \leftarrow 0$.

② Add all nodes to Q

③ while Q is not empty

$u \leftarrow$ extractmin (Q)

for each neighbor v of u: $\quad$ (i.e. $\exists\, e = (u,v) \in E$)

if $d_u + w(u,v) < d_v$

$d_v = d_u + w(u,v)$

decreaseKey (v, d_v)

```
DIJKSTRA(G = (V, E), s)
for all v ∈ V
        do d_u ← ∞
           π_u ← NIL
d_s ← 0
Q ← MAKEQUEUE(V)     ▷ use d_u as key
while Q ≠ ∅
        do u ← EXTRACTMIN(Q)
           for each v ∈ Adj(u)
               do if d_v > d_u + w(u, v)
                     then d_v ← d_u + w(u, v)
                          π_v ← u
                          DECREASEKEY(Q, v)
```

Very similar structure

DIJKSTRA$(G = (V, E), s)$

for all $v \in V$
  do $d_u \leftarrow \infty$
    $\pi_u \leftarrow$ NIL
$d_s \leftarrow 0$
$Q \leftarrow$ MAKEQUEUE(V) $\triangleright$ use d_u as key
while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACTMIN(Q)
    for each $v \in Adj(u)$
      do if $d_v > d_u + w(u, v)$
        then $d_v \leftarrow d_u + w(u, v)$
          $\pi_v \leftarrow u$
          DECREASEKEY(Q, v)

PRIM$(G = (V, E))$

$Q \leftarrow \emptyset$ $\triangleright$ Q is a Priority Queue
Initialize each $v \in V$ with key $k_v \leftarrow \infty$, $\pi_v \leftarrow$ NIL
Pick a starting node r and set $k_r \leftarrow 0$
Insert all nodes into Q with key k_v.
while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    for each $v \in Adj(u)$
      do if $v \in Q$ and $w(u, v) < k_v$
        then $\pi_v \leftarrow u$
          DECREASE-KEY$(Q, v, w(u, v$

$\Theta(E \log(V))$ or $\Theta(E + V \cdot \log V)$

running time

```
DIJKSTRA(G = (V, E), s)
for all v ∈ V
        do d_u ← ∞
           π_u ← NIL
d_s ← 0
Q ← MAKEQUEUE(V)        ▷ use d_u as key
while Q ≠ ∅
        do u ← EXTRACTMIN(Q)
           for each v ∈ Adj(u)
                do if d_v > d_u + w(u, v)
                      then d_v ← d_u + w(u, v)
                           π_v ← u
                           DECREASEKEY(Q, v)
```

The running time is $\Theta(E \log V)$ because each DecreaseKey operation is called at most once on each edge.

why does Dijkstra work?

TRIANGLE INEQUALITY:

sp from s to v   sp from s to u.

$$\forall (u,v) \in E,\ \delta(s,v) \le \delta(s,u) + w(u,v)$$

Def of the shortest path

UPPER BOUND: $d_v \ge \delta(s,v)$

d_v begins at ∞. So this is true at the start.
AND, we only update d_v in the algorithm in line 10
if we have identified a node u s.t. $d_u + w(u,v)$
is smaller.

Theorem

Given any weighted directed graph $G = (V, E)$ with non-negative weights, and a source ~~e~~ s, Dijkstra(G, s) terminates with $d_v = \delta(s, v)$ for all $v \in V$.

Proof: Let S be the set of nodes not in Q.

At line 5, S is empty.

Property I: for all $v \in S$, $d_v = \delta(s, v)$.

At line 5, this property holds.

Theorem

Given any weighted directed graph $G = (V, E)$ with non-negative weights, and a source e, Dijkstra(G, s) terminates with $d_u = \delta(s, v)$ for all $v \in V$.

Let S be the set of elements not in Q.
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Theorem

Given any weighted directed graph $G = (V, E)$ with non-negative weights, and a source e, Dijkstra(G, s) terminates with $d_u = \delta(s, v)$ for all $v \in V$.

Let S be the set of elements not in Q.
At the beginning, this set is empty.

Property 1: For all $v \in S, d_v = \delta(s, v)$.

Proof (cont)

Suppose this property holds for the first i iterations of the loop.

Now consider the $(i+1)^{st}$ iteration of loop at line 6.

In line 7, the node u is removed from the Q.

$\Rightarrow$ u is added to S!

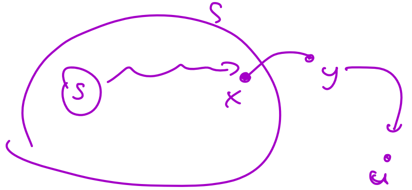
FACT 1. At this point $d_u = d_z + w(z,u)$ for some node in V. Since $z \in S$ at this point, by the inductive hypothesis, $d_u = \delta(s,z) + w(z,u)$.

Consider any path p from $s \leadsto u$.

Let $e = (x,y)$ be the first edge on the path p to cross the cut $(S, V-S)$.

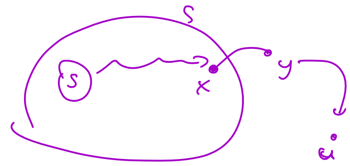
$$w(p) = w(s \leadsto x) + w(x,y) + w(y,u).$$
$$\delta(s,x) + w(x,y)$$

Since $x \in S$;



$w(p) = w(s \rightsquigarrow x) + w(x,y) + w(y,u)$. Since $x \in S$;

(by I.H) $\to \underline{\delta(s,x) + w(x,y)}$



Because $x \in S$, then $d_y \leq \delta(s,x) + w(x,y)$

This implies

$$w(p) \geq d_y + \delta(y,u)$$

Because we used extractmin in line 7, we know

$$d_u \leq d_y$$

$$\Rightarrow \quad w(p) \geq d_u + \delta(y,u)$$

But $\delta(y,u) \geq 0$, and therefore we know that every path from (s,u) has weight at least d_u.

Thus, $d_u = \delta(s,u)$.

Proof (cont) Suppose this property holds for the first i iterations of the loop.

Let u be the node extracted on line 7. By lines, 9,10,11, it follows that $d_u = d_z + w(z,u)$ for some node $z \in S$. By the hypothesis, $d_u = \delta(s,z) + w(z,u)$.

Consider any path p from s to u. Let $e = (x,y)$ be the first edge on path p that crosses cut $(S, V - S)$. By lines 9,10,11 and the inductive hypothesis, $d_y \leq \delta(s,x) + w(x,y)$. We now analyze the weight of path p:

$$\begin{aligned} w(p) &= w(s \leadsto x) + w(x,y) + w(y \leadsto u) \\ &\geq \delta(s,x) + w(x,y) + \delta(y,u) \end{aligned}$$

Substituting from above, we have that

$$w(p) \geq d_y + \delta(y,u)$$

However, by line 6, $d_u \leq d_y$, and so

$$w(p) \geq d_u + \delta(y,u)$$

Since all edges have non-negative weight, $\delta(y,u) \geq 0$ and so

$$w(p) \geq d_u$$

which implies that $d_u = \delta(s,u)$.