5800
Shortest paths

mar 8/10 2022
shelat

What about Negative edge weights?

Mid-market rates as of 2016-03-31 17:40 UTC

Currency code ▲▼	Currency name ▲▼	Units per EUR	EUR per Unit
USD	US Dollar	1.1386632306	0.8782227907
EUR	Euro	1.0000000000	1.0000000000
GBP	British Pound	0.7921136388	1.2624451227
INR	Indian Rupee	75.3658843112	0.0132686030
AUD	Australian Dollar	1.4859561878	0.6729673514
CAD	Canadian Dollar	1.4796754127	0.6758238945
SGD	Singapore Dollar	1.5347639238	0.6515660060
CHF	Swiss Franc	1.0917416715	0.9159676012
MYR	Malaysian Ringgit	4.4140052400	0.2265516114
JPY	Japanese Yen	128.1388820287	0.0078040325
CNY	Chinese Yuan Renminbi	7.3411003512	0.1362193612
NZD	New Zealand Dollar	1.6484648003	0.6066250246
THB	Thai Baht	39.9627318192	0.0250233143
HUF	Hungarian Forint	313.9042436792	0.0031856849
AED	Emirati Dirham	4.1823100458	0.2391023117

XE Currency Table: AED - Emirati Dirham

Mid-market rates as of 2016-03-31 17:39 UTC

Currency code ▲▼	Currency name ▲▼	Units per AED	AED per Unit
USD	US Dollar	0.2722570106	3.6730000000
EUR	Euro	0.2391289974	4.1818433177
GBP	British Pound	0.1893997890	5.2798369266
INR	Indian Rupee	18.0207422309	0.0554916100
AUD	Australian Dollar	0.3552996418	2.8145257760
CAD	Canadian Dollar	0.3538334124	2.8261887234
SGD	Singapore Dollar	0.3669652245	2.7250538559
CHF	Swiss Franc	0.2610686193	3.8304105746
MYR	Malaysian Ringgit	1.0548325619	0.9480177576
JPY	Japanese Yen	30.6399242607	0.0326371564
CNY	Chinese Yuan Renminbi	1.7555154332	0.5696332719
NZD	New Zealand Dollar	0.3941937299	2.5368237088
THB	Thai Baht	9.5553789460	0.1046530970
HUF	Hungarian Forint	75.0637936939	0.0133220019
AED	Emirati Dirham	1.0000000000	1.0000000000

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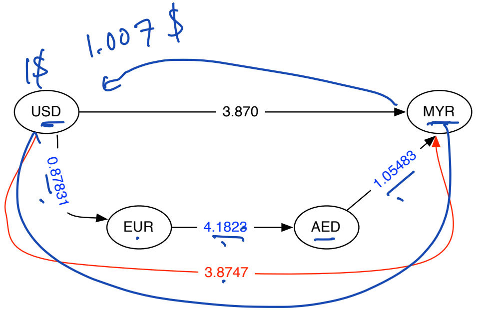
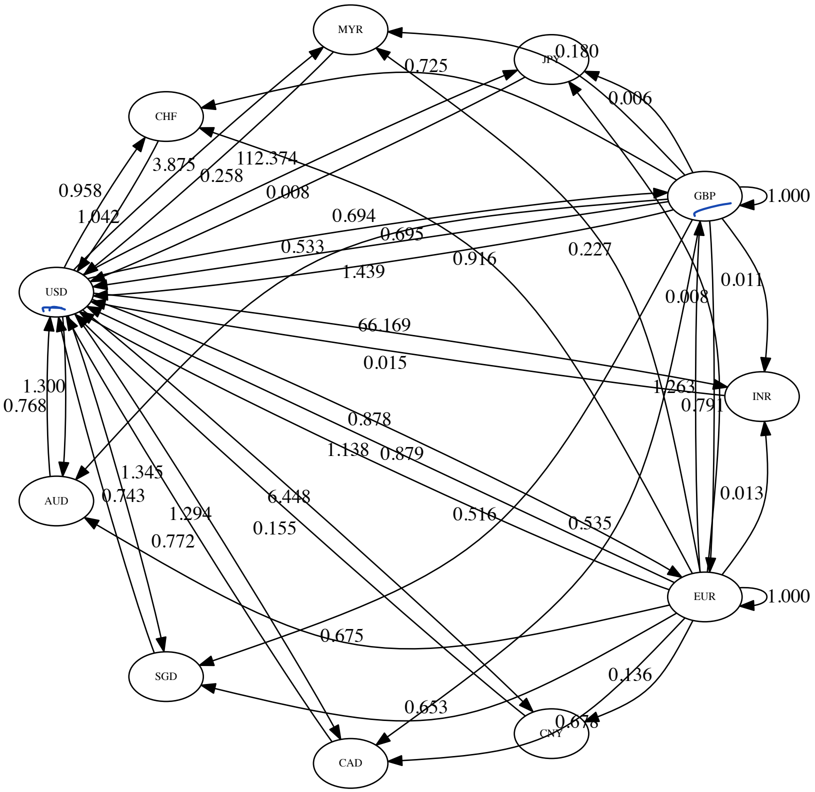
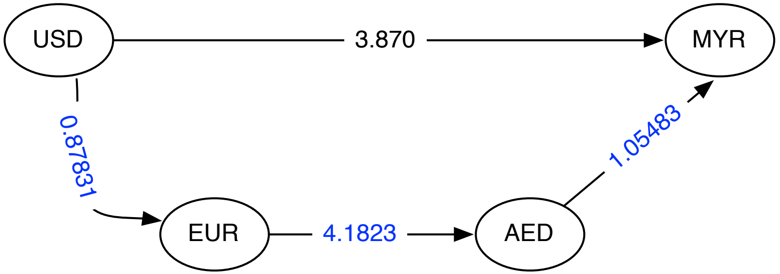
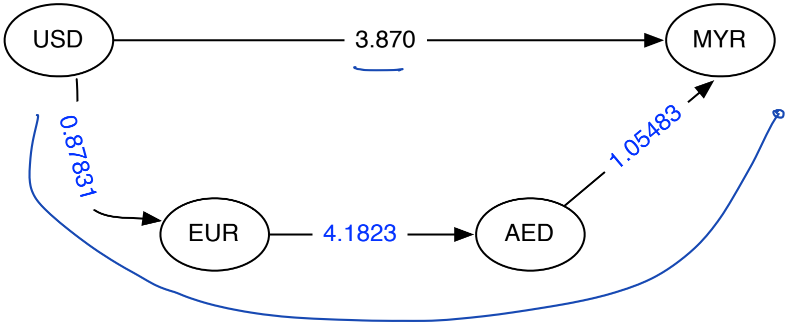




MYR
JPY
CHF
GBP
USD
INR
AUD
EUR
SGD
CNY
CAD
0.180
0.725
0.006
3.875
112.374
0.258
0.958
0.008
1.000
1.042
0.694
0.695
0.533
0.916
0.227
1.439
0.011
0.008
66.169
0.015
1.263
0.791
1.300
0.768
0.878
1.138
0.879
1.345
0.743
6.448
0.013
1.294
0.155
0.516
0.535
0.772
1.000
0.675
0.136
0.653
0.678

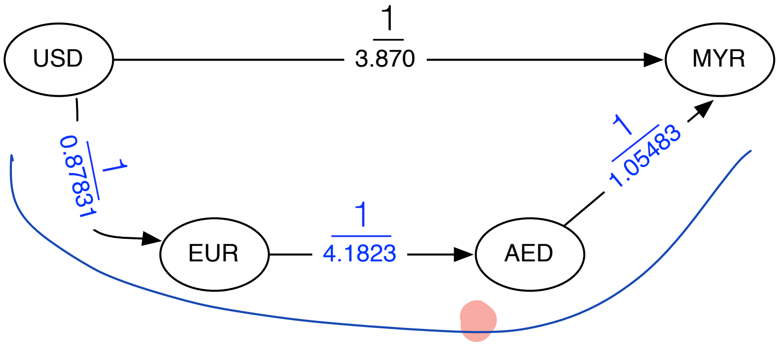


Trying to find
Max weight
(mult) Paths

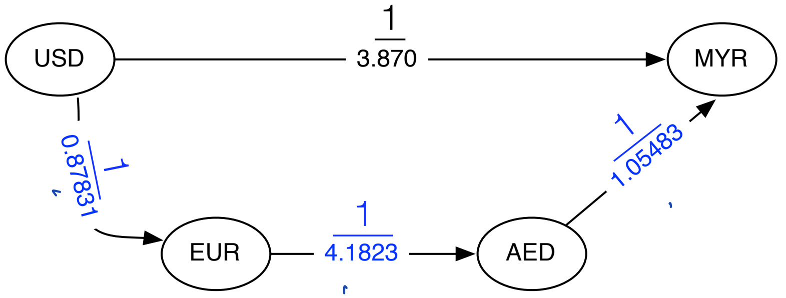


Trying to find
Max weight
(mult) Paths

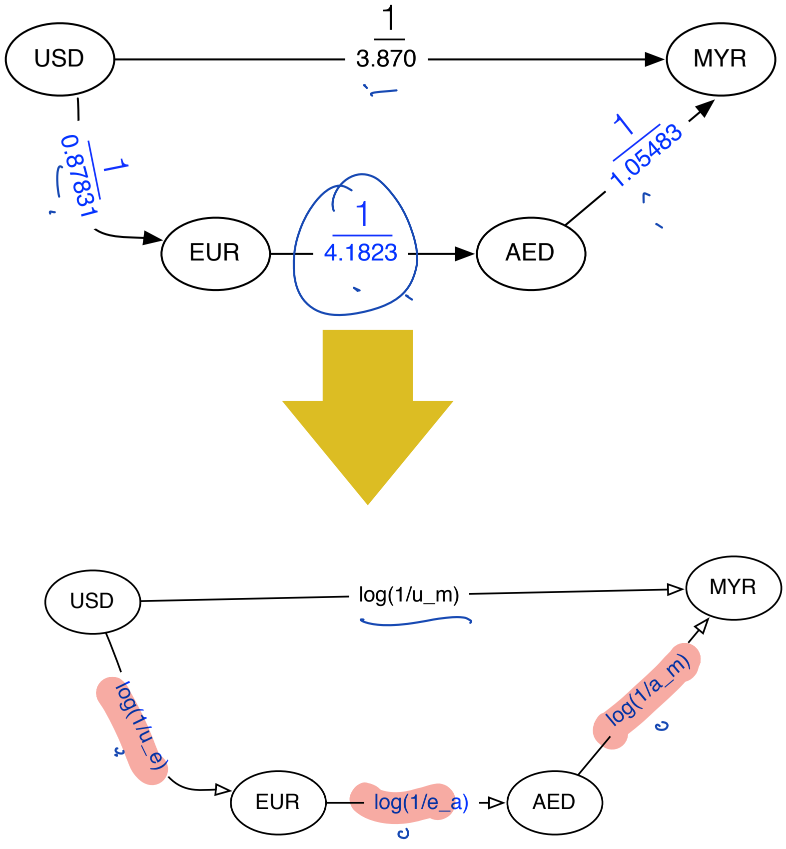
take inverses of
each edge
weight.



Trying to find
MIN weight
(mult) Paths

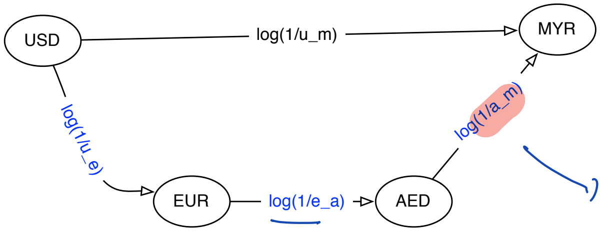


Trying to find MIN weight (mult) Paths



Trying to find MIN weight (mult) Paths

MIN weight paths with additive weights



could be $\frac{1}{x} < 1$

$$\log\left(\frac{1}{4.1823}\right)$$

$$= \log_{10}(0.2391\ldots)$$

$$= -0.62\ldots$$

where does old argument break down

Recall this line in our proof of Dijkstra:

$$w(p) \geq d_y + w(y, u)$$

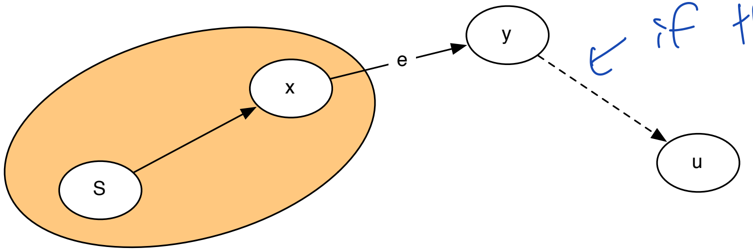
in previous graphs,

we concluded that

$w(y,u) \geq 0$

$\Rightarrow w(p) \geq d_u$

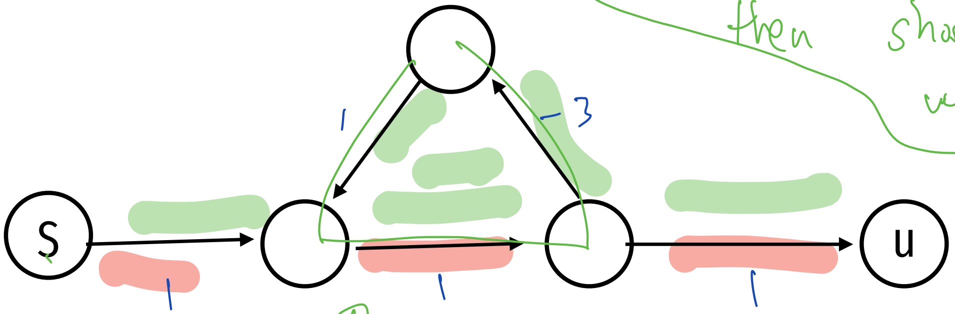
where does old argument break down



if this weight was negative, then we cannot be certain that some other future path does not have a shorter route to u.

2nd problem: cycles

if G has negative weight cycles, then shortest paths are not well defined.



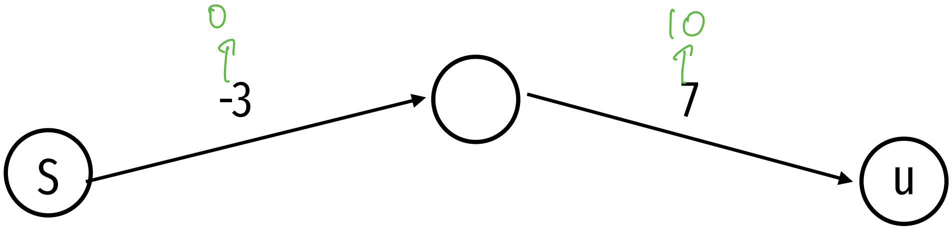
cycle of weight = −1

wt = 3

wt = 1+1+-3+1 +1+1 = 2

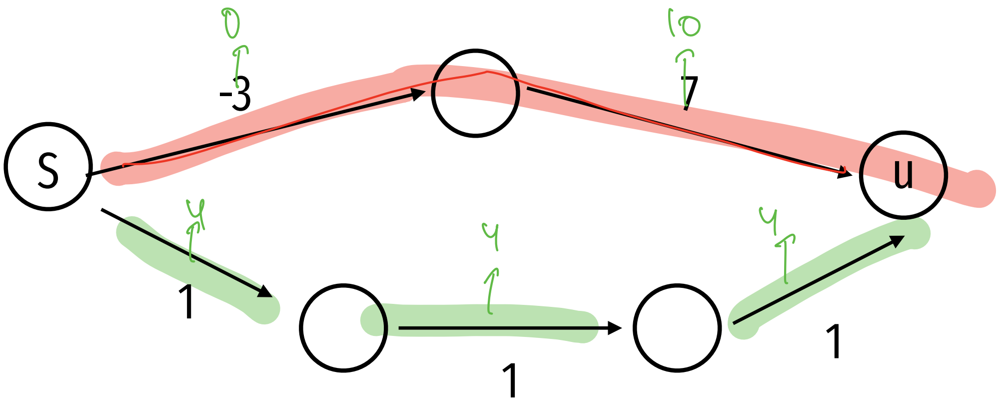
first ideas: Add to each edge

Idea: add a constant to each edge so that all weights are non-negative



first ideas: Add to each edge

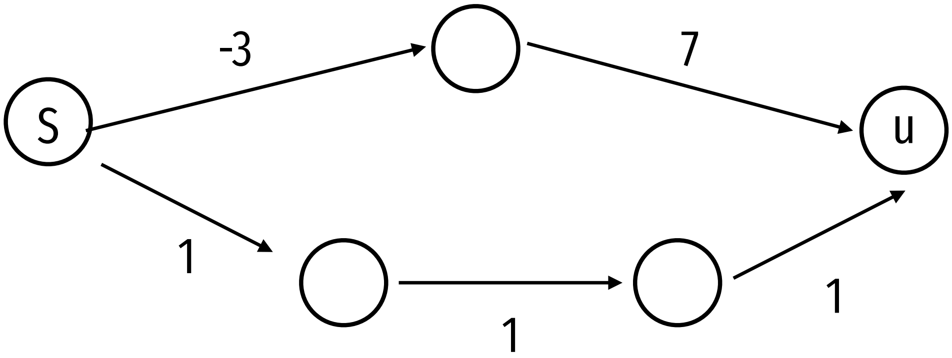
Idea: add a constant to each edge so that all weights are non-negative



In the new graph, the shortest path changes to become the upper path.
So this idea does not preserve shortest paths in G.

first ideas: Add to each edge

Idea: add a constant to each edge so that all weights are non-negative



Problem: this can change the shortest paths of the graph.

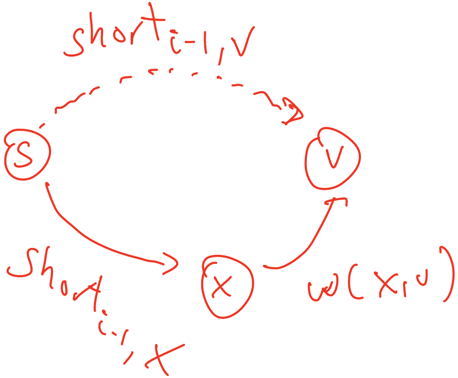
$$\text{sssp}(G,s) =$$ single source (s) shortest paths (with neg edge weights)

$$\text{SHORT}_{i,v} =$$ length of the shortest path from s to v that uses at most i edges

$$\text{short}_{0,v} = \begin{cases} 0 & \text{if } v = s \\ \text{else } \infty \end{cases}$$

sssp(G,s)

$\text{SHORT}_{i,v} =$ Length of the shortest path from s to v that uses at most i edges.



Two possibilities:

(1) use i-1 edges to get from s to v

(2) use i-1 edges to get to some intermediate node x and then use the edge (x,v)

sssp(G,s)

$$\text{SHORT}_{i,v} = \begin{cases} \infty & i = 0 \\ 0 & v = s \\ \min_{x \in V} & \left\{ \begin{array}{l} \text{SHORT}_{i-1,v} \\ \text{SHORT}_{i-1,x} + w(x,v) \end{array} \right\} \end{cases}$$

Either the shortest path from s to v
uses at most $i - 1$ edges or
it uses at most $i - 1$ edges to x and then uses edge $e = (x, v)$

max len of a simple path:

is $|V|-1$.

and so we only need to consider

short$_{|V|-1, u}$ for every $u \in V$.

max len of a simple path:

The max length of a path from s to v is $|V| - 1$.
Otherwise, the path contains a cycle.

BELLMAN-FORD(G,s)

1. $short_{0,v} = \infty$ for all $v \in V$
2. $short_{0,s} = 0$.
3. for $i=1$ to $|V|-1$
4. for each node $v \in V - \{s\}$
5. $short_{i,v} = \min_{x \in Adj(v)} \begin{cases} short_{i-1,v} \\ short_{i-1,x} + w(x,v) \end{cases}$

BELLMAN-FORD(G, s)

$\text{SHORT}_{0,s} \leftarrow 0$

$\forall v \in V - \{s\}$, $\text{SHORT}_{0,v} \leftarrow \infty$

for $i = 1, \ldots, V - 1$

  do for each $v \in V - \{s\}$

    do $\text{SHORT}_{i,v} = \min_{x \in Adj(v)} \begin{cases} \text{SHORT}_{i-1,v} \\ w(x,v) + \text{SHORT}_{i-1,x} \end{cases}$

equivalent loops

- we can rewrite this loop to instead loop over ~~every~~ edges of the graph: for each edge $e = (x,y)$

$$\text{short}_{i,y} = \min \begin{cases} \text{short}_{i-1,y} \\ \text{short}_{i-1,x} + w(x,y) \\ \text{short}_{i,y} \end{cases}$$

- what is the run time of this algorithm??

$\Theta(V \cdot E)$

BELLMAN-FORD(G, s)

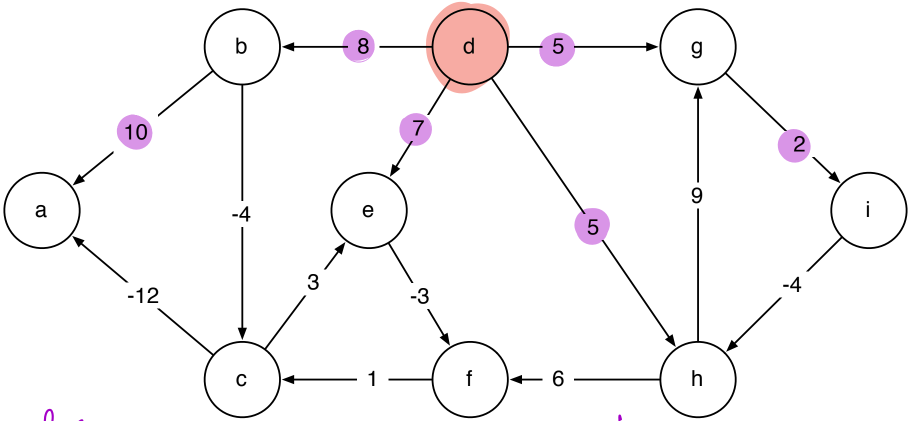
1 $\text{SHORT}_{0,s} \leftarrow 0$

2 $\forall v \in V - \{s\}$, $\text{SHORT}_{0,v} \leftarrow \infty$

3 **for** $i = 1, \ldots, V-1$ V

4   **do for** each $e = (x, y) \in E$

5   **do** $\text{SHORT}_{i,y} = \min \left\{ \begin{array}{l} \text{SHORT}_{i-1,y} \\ \text{SHORT}_{i,y} \\ w(x,y) + \text{SHORT}_{i-1,x} \end{array} \right\}$



loop over each edge,

$$\text{SHORT}_{i,v} = \begin{cases} \infty & i = 0 \\ 0 & v = s \\ \min_{x \in V} & \left\{ \begin{array}{l} \text{SHORT}_{i-1,v} \\ \text{SHORT}_{i-1,x} + w(x,v) \end{array} \right\} \end{cases}$$

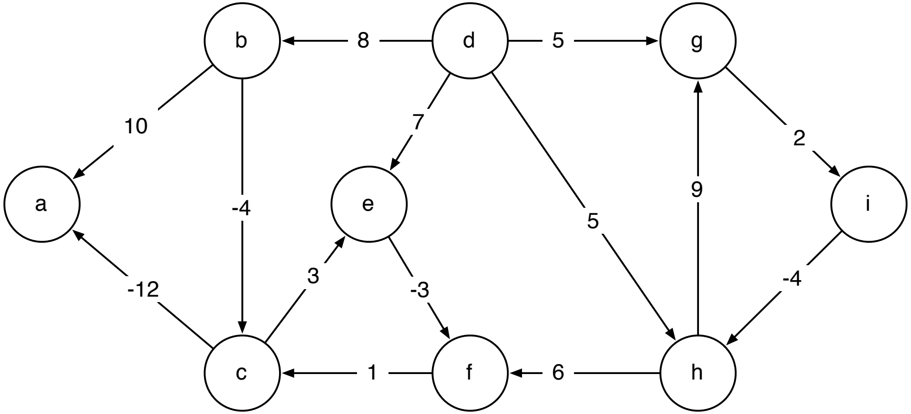
(d,b)

$$\text{short}_{1,b} = \min \begin{cases} \text{short}_{0,b} \\ \text{short}_{0,d} + 8 \end{cases}$$

BF(G,d)

$\text{short}_{i,v}$

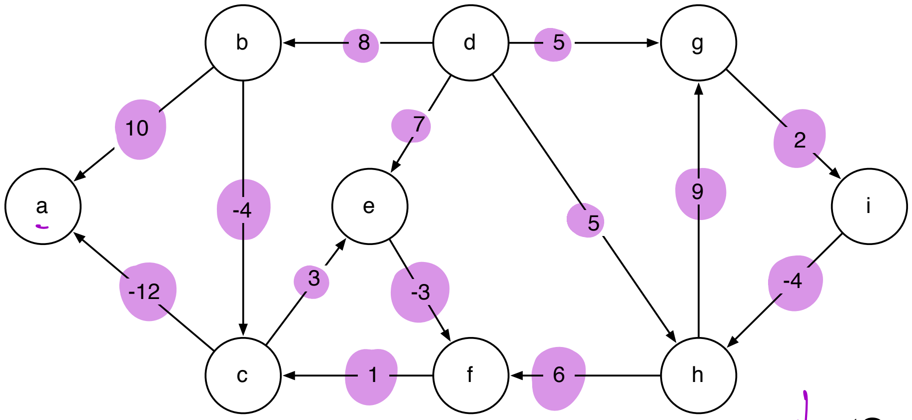
	0	1	2	3	4	5	6	7
A	∞	∞						
B	∞	8						
C	∞	-						
D	0	-						
E	∞	7						
F	∞	-						
G	∞	5						
H	∞	5						
I	∞	-						



$$\textsc{short}_{i,v} = \begin{cases} \infty & i = 0 \\ 0 & v = s \\ \min_{x \in V} & \left\{ \begin{array}{l} \textsc{short}_{i-1,v} \\ \textsc{short}_{i-1,x} + w(x,v) \end{array} \right\} \end{cases}$$

BF(G,d)

	0	1	2	3	4	5	6	7
A	∞							
B	∞							
C	∞							
D	0							
E	∞							
F	∞							
G	∞							
H	∞							
I	∞							

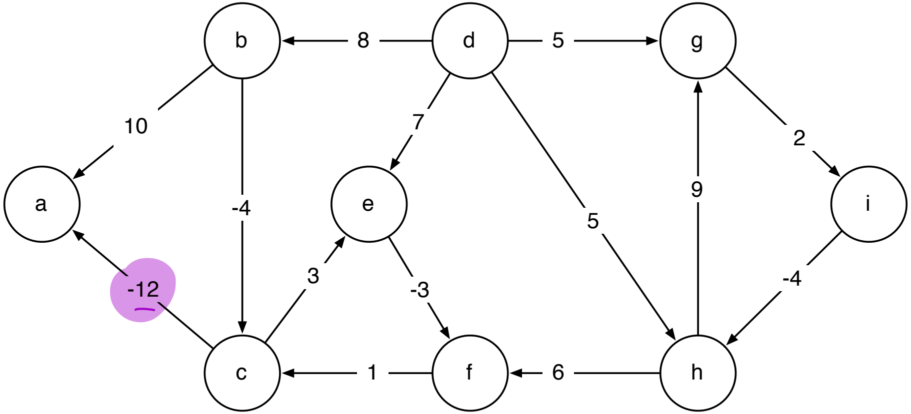


$\text{short}_{2,v}$

$$\text{SHORT}_{i,v} = \begin{cases} \infty & i = 0 \\ 0 & v = s \\ \min_{x \in V} & \left\{ \begin{array}{l} \text{SHORT}_{i-1,v} \\ \text{SHORT}_{i-1,x} + w(x,v) \end{array} \right\} \end{cases}$$

BF(G,d)

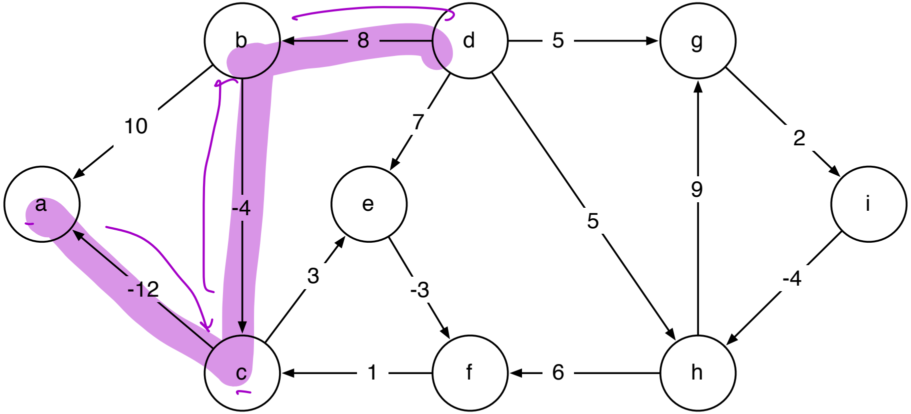
	0	1	2	3	4	5	6	7
A	∞		18					
B	∞	8	8					
C	∞		4					
D	0	0	0					
E	∞	7	7					
F	∞		4					
G	∞	5	5					
H	∞	5	5					
I	∞		7					



$$\mathrm{SHORT}_{i,v} = \begin{cases} \infty & i = 0 \\ 0 & v = s \\ \min_{x \in V} & \left\{ \begin{array}{l} \mathrm{SHORT}_{i-1,v} \\ \mathrm{SHORT}_{i-1,x} + w(x,v) \end{array} \right\} \end{cases}$$

BF(G,d)

	0	1	2	3	4	5	6	7
A	∞		18	-8				
B	∞	8	8					
C	∞		4					
D	0	0	0					
E	∞	7	7					
F	∞		4					
G	∞	5	5					
H	∞	5	5					
I	∞		7					



v^2

$$\textsc{short}_{i,v} = \begin{cases} \infty & i = 0 \\ 0 & v = s \\ \min_{x \in V} & \left\{ \begin{array}{l} \textsc{short}_{i-1,v} \\ \textsc{short}_{i-1,x} + w(x,v) \end{array} \right\} \end{cases}$$

BF(G,d)

	0	1	2	3	4	5	6	7
A	∞		18	-8				
B	∞	8	8	8				
C	∞		4	4				
D	0	0	0	0				
E	∞	7	7	7				
F	∞		4	4				
G	∞	5	5	5				
H	∞	5	5	3				
I	∞		7	7				

optimization

BELLMAN-FORD(G, s)

$\text{SHORT}_{0,s} \leftarrow 0$

$\forall v \in V - \{s\}$, $\text{SHORT}_{0,v} \leftarrow \infty$

for $i = 1, \ldots, V - 1$

  do for each $e = (x, y) \in E$

    do $\text{SHORT}_{i,y} = \min \left\{ \begin{array}{l} \text{SHORT}_{i-1,y} \\ \text{SHORT}_{i,y} \\ w(x,y) + \text{SHORT}_{i-1,x} \end{array} \right\}$

BELLMAN-FORD(G, s)

$d_s \leftarrow 0$

$\forall v \in V - \{s\}$, $d_v \leftarrow \infty$

for $i = 1, \ldots, V - 1$

  do for each $e = (x, y) \in E$

    do $d_y \leftarrow \min\{\ d_y, w(x,y) + d_x\ \}$

$\Theta(E \cdot V)$ time and $\Theta(V)$ space.

running time

BELLMAN-FORD(G, s)

$d_s \leftarrow 0$

$\forall v \in V - \{s\},\ d_v \leftarrow \infty$

for $i = 1, \ldots, V - 1$

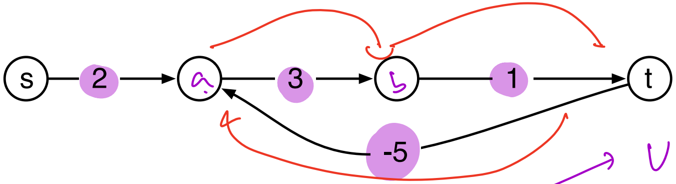
  do for each $e = (x, y) \in E$

    do $d_y \leftarrow \min\{\ d_y, w(x, y) + d_x\ \}$

negative cycles?

if G contains a negative cycle, then some distance value will decrease on the Vth iteration of the BF algorithm..

$3+1+(-5) = \underline{(-1)}$ negative cycle.



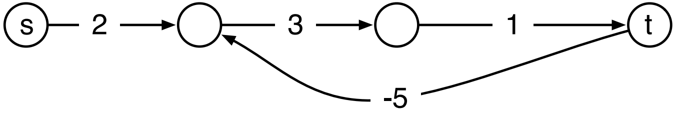
→ V th step

	1	2	3	4
S	0	0	0	0
A	2	2	2	2
B	.	5	5	5
T	.	.	6	6

$1 \leftarrow \min\begin{pmatrix} 2, \\ 6-5=1 \end{pmatrix}$

∞ if there is no value listed.

negative cycles?



S	0	0	0	0
A	2	2	2	1
B		5	5	5
T			6	6

if some value decreases on the Vth step then $\exists$ a negative cycle.

To show: if $\exists$ a negative cycle, some value decreases.

applications of BF

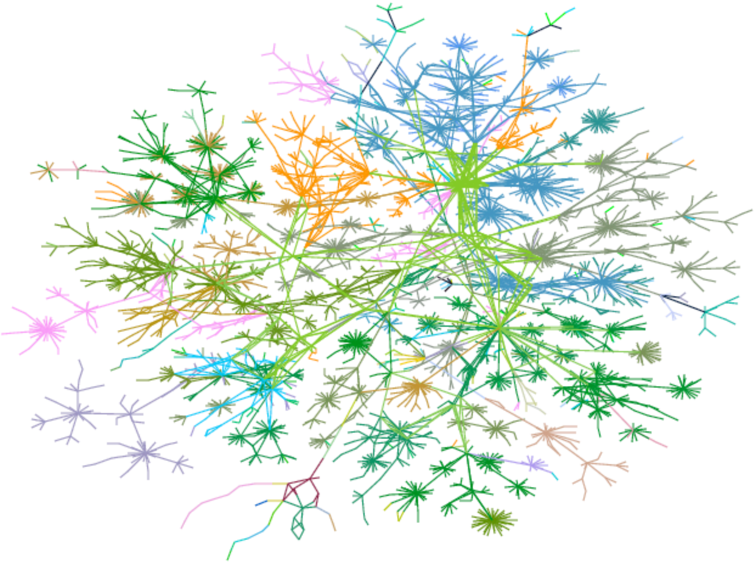
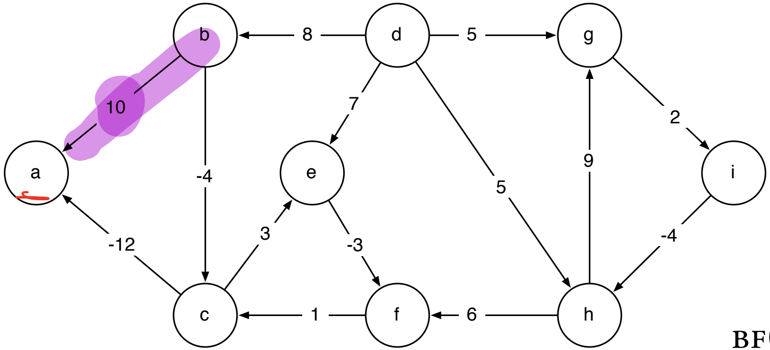


image: cheswick et al

Figure 3: Lucent's intranet as of 1 October 1999.



WHAT HAPPENS WHEN B CHANGES...

BF(G,d)

	0	1	2	3	4	5	6	7
A	∞							
B	∞	8						
C	∞							
D	0	0						
E	∞	7						
F	∞							
G	∞	5						
H	∞	5						
I	∞							

DISTANCE VECTOR

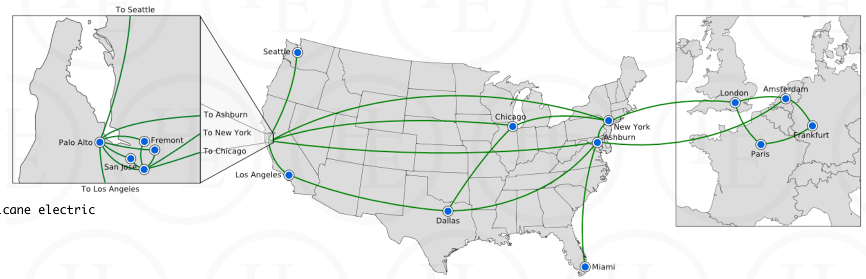
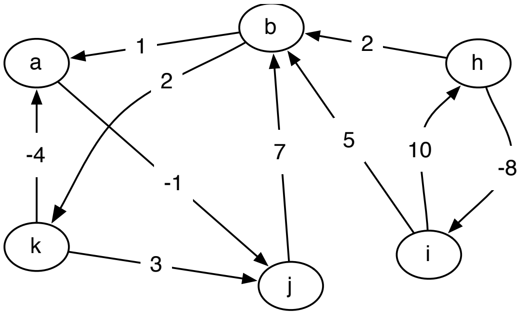


image: hurricane electric

⇒ BGP uses a distance vector (ie, BF inspired) algorithm to maintain shortest paths.

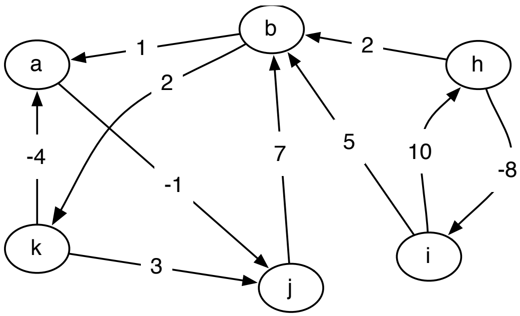
ALL-PAIRS SHORTEST PATH



First approach: run BF from each node.
What is the running time? $V \cdot \underline{EV} = \Theta(EV^2)$ $O(V^4)$

$O(V^2)$

ALL-PAIRS SHORTEST PATH



First approach: run BF from each node.
What is the running time?

$$O(EV^2)$$

NEW APPROACH TO ALL-PAIRS

$\text{ASHORT}_{i,j,k} =$ length of the shortest path between nodes i, j that only use intermediate nodes 1...k.

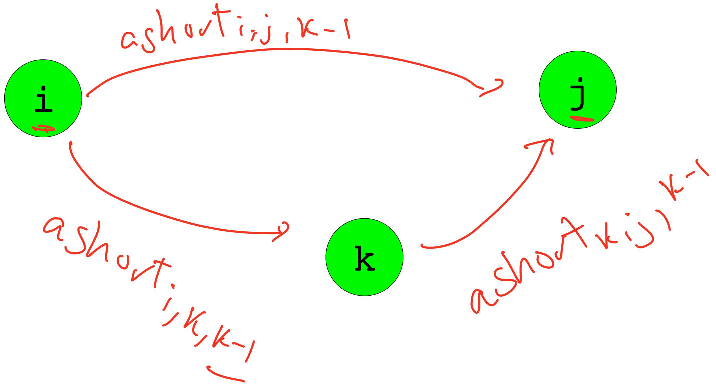
NEW APPROACH TO ALL-PAIRS

$\mathrm{ASHORT}_{i,j,k} =$ Length of the shortest path from i to j that only traverses nodes $1, \ldots, k$.

$1 \ldots k-1$

$$\text{ASHORT}_{i,j,k} = \min \begin{cases} w(i,j) & \text{if } k = 0 \quad \text{(base case)} \\ ashort_{i,j,k-1} \\ ashort_{i,k,k-1} + ashort_{k,j,k-1} \end{cases}$$

Q: how does node k help in getting from i to j?



using the first $k-1$ intermediate nodes

$ashort_{i,j,|V|}$

$$\mathrm{ASHORT}_{\mathtt{i},\mathtt{j},\mathtt{k}} =$$

$$\text{ASHORT}_{i,j,k} = \left\{ \begin{array}{ll} w_{i,j} & k = 0 \\ \min \left\{ \begin{array}{l} \text{ASHORT}_{i,j,k-1} \\ \text{ASHORT}_{i,k,k-1} + \text{ASHORT}_{k,j,k-1} \end{array} \right. & k \geq 1 \end{array} \right\}$$

FLOYD-WARSHALL(G,W)

INITIALIZE $ashort_{i,j,0}$

for $k = 1$ to $|V|$ $\qquad \Theta(V^3)$

for $i = 1$ to $|V|$

for $j = 1$ to $|V|$

$ashort_{i,j,k} = \min \{$ equation from previous slide