5800
Shortest paths

mar 8/10 2022
shelat

What about Negative edge weights?

Mid-market rates as of **2016-03-31 17:40 UTC**

Currency code ▲▼	Currency name ▲▼	Units per EUR	EUR per Unit
USD	US Dollar	1.1386632306	0.8782227907
EUR	Euro	1.0000000000	1.0000000000
GBP	British Pound	0.7921136388	1.2624451227
INR	Indian Rupee	75.3658843112	0.0132686030
AUD	Australian Dollar	1.4859561878	0.6729673514
CAD	Canadian Dollar	1.4796754127	0.6758238945
SGD	Singapore Dollar	1.5347639238	0.6515660060
CHF	Swiss Franc	1.0917416715	0.9159676012
MYR	Malaysian Ringgit	4.4140052400	0.2265516114
JPY	Japanese Yen	128.1388820287	0.0078040325
CNY	Chinese Yuan Renminbi	7.3411003512	0.1362193612
NZD	New Zealand Dollar	1.6484648003	0.6066250246
THB	Thai Baht	39.9627318192	0.0250233143
HUF	Hungarian Forint	313.9042436792	0.0031856849
AED	Emirati Dirham	4.1823100458	0.2391023117

XE Currency Table: AED - Emirati Dirham

Mid-market rates as of **2016-03-31 17:39 UTC**

Currency code ▲▼	Currency name ▲▼	Units per AED	AED per Unit
USD	US Dollar	0.2722570106	3.6730000000
EUR	Euro	0.2391289974	4.1818433177
GBP	British Pound	0.1893997890	5.2798369266
INR	Indian Rupee	18.0207422309	0.0554916100
AUD	Australian Dollar	0.3552996418	2.8145257760
CAD	Canadian Dollar	0.3538334124	2.8261887234
SGD	Singapore Dollar	0.3669652245	2.7250538559
CHF	Swiss Franc	0.2610686193	3.8304105746
MYR	Malaysian Ringgit	1.0548325619	0.9480177576
JPY	Japanese Yen	30.6399242607	0.0326371564
CNY	Chinese Yuan Renminbi	1.7555154332	0.5696332719
NZD	New Zealand Dollar	0.3941937299	2.5368237088
THB	Thai Baht	9.5553789460	0.1046530970
HUF	Hungarian Forint	75.0637936939	0.0133220019
AED	Emirati Dirham	1.0000000000	1.0000000000

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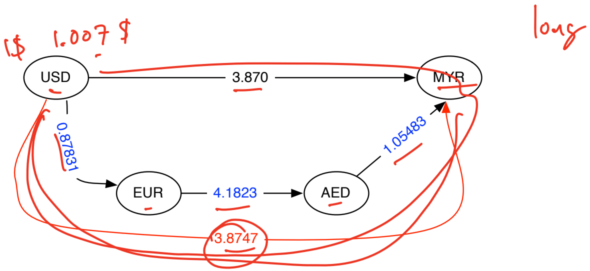
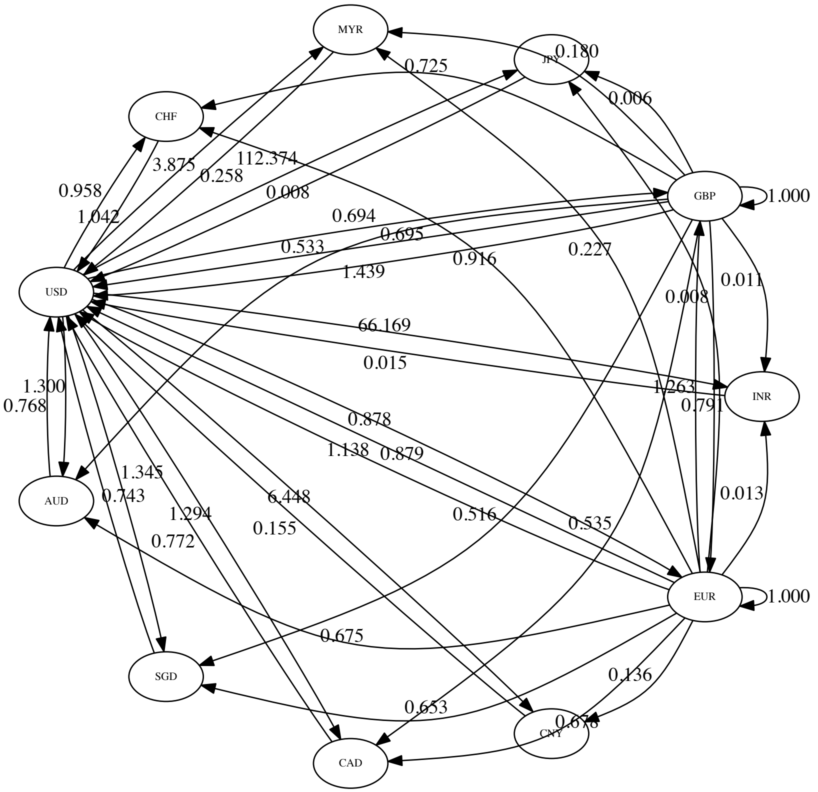
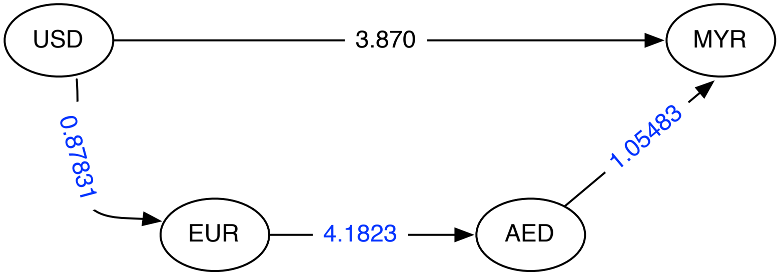
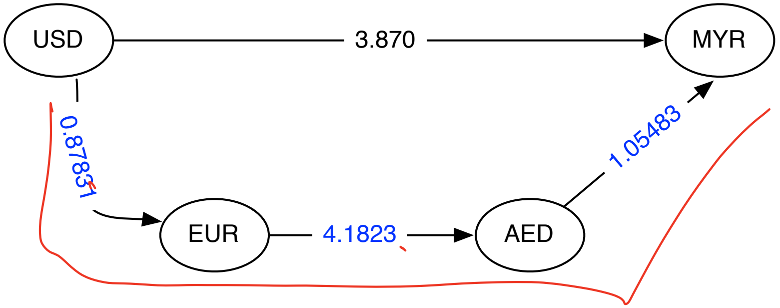



MYR
JPY
CHF
GBP
USD
INR
AUD
EUR
SGD
CNY
CAD
0.180
0.725
0.006
3.875
112.374
0.258
0.958
0.008
1.042
0.694
0.695
0.533
0.916
1.439
0.227
1.000
0.011
0.008
66.169
0.015
1.263
0.791
1.300
0.768
0.878
1.138
0.879
1.345
0.743
6.448
1.294
0.013
0.155
0.516
0.535
0.772
1.000
0.675
0.136
0.653
0.678

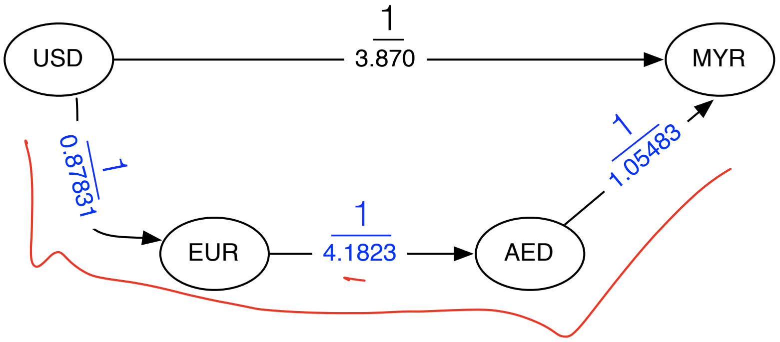


Trying to find Max weight (mult) Paths

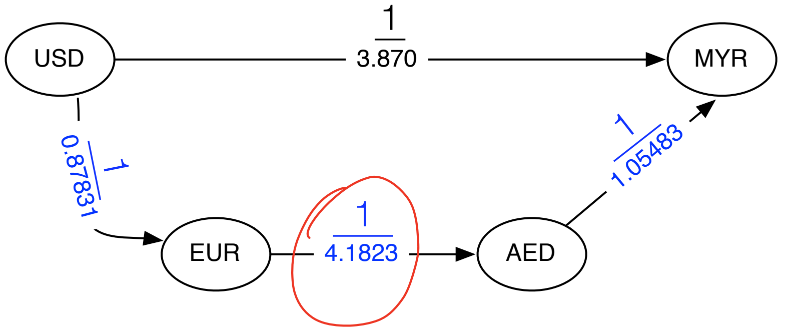


Trying to find Max weight (mult) Paths

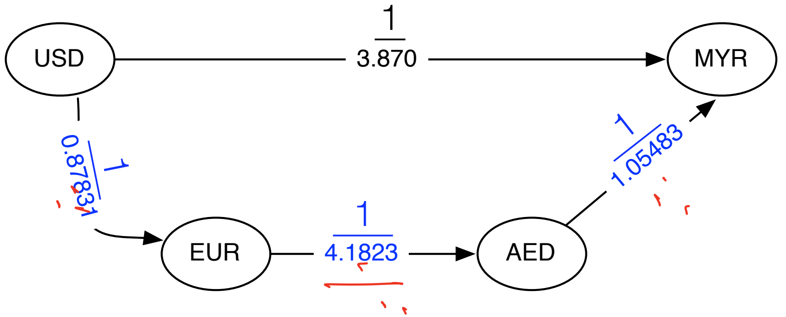
inverse of each weight



Trying to find MIN weight (mult) Paths

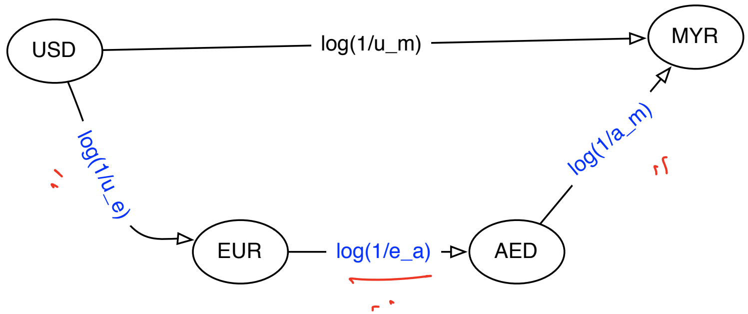


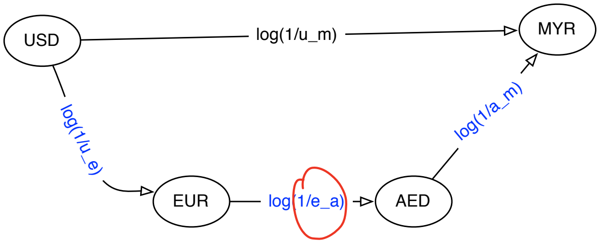
Trying to find MIN weight (mult) Paths



Trying to find MIN weight (mult) Paths

$\log(a \cdot b) = \log(a) + \log(b)$





$$\log\left(\frac{1}{4.1813}\right) =$$

$$\log(0.2677) =$$

$$-0.682$$

if $\frac{1}{e_a} < 1$,

then the edge weight will be negative

where does old argument break down

Recall this line in our proof of Dijkstra:

$$w(p) = d_y + w(y, u)$$

vi

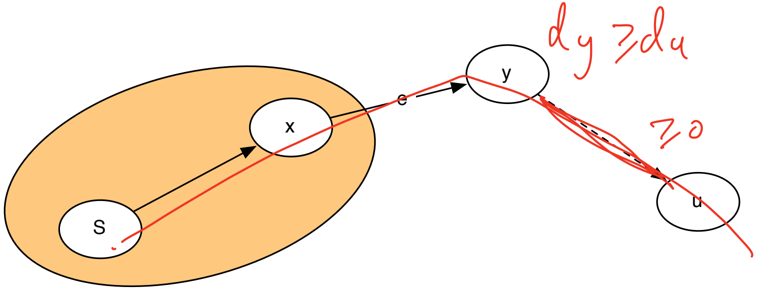
du

0

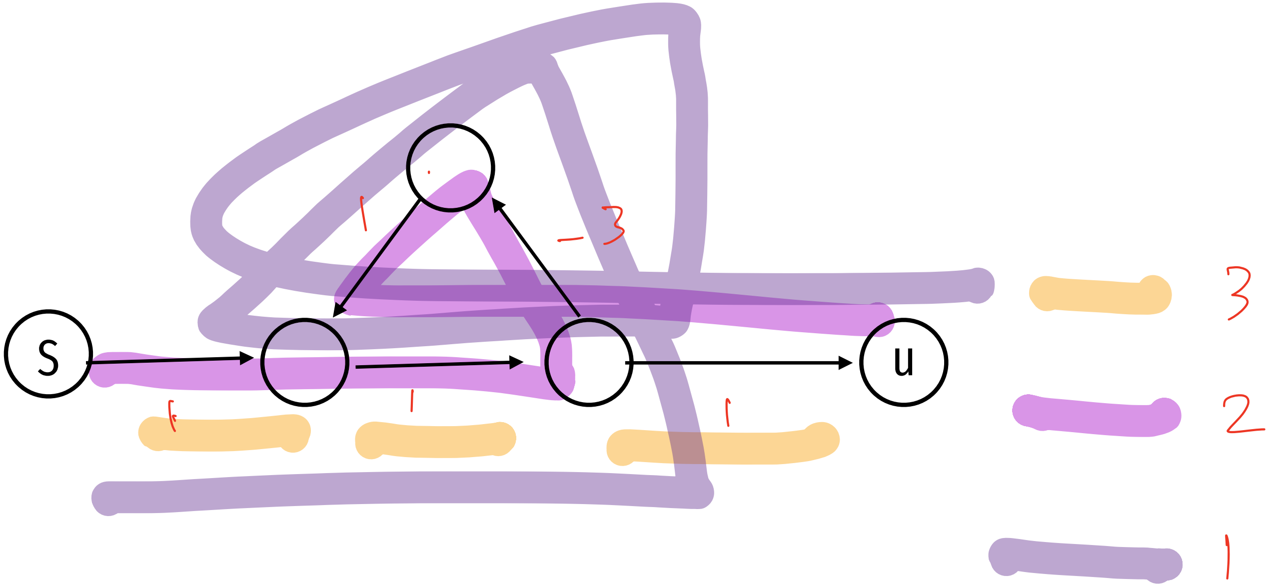
$\wedge$ i

we relied on this value being non-negative

where does old argument break down

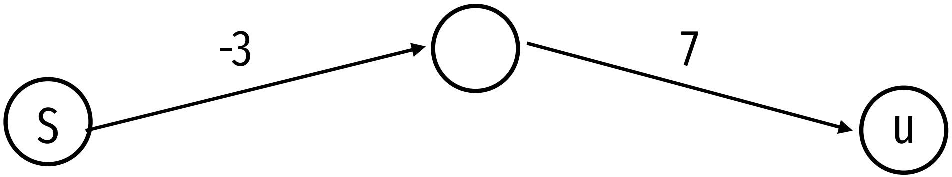


2nd problem: cycles



first ideas: Add to each edge

Idea: add a constant to each edge so that all weights are non-negative



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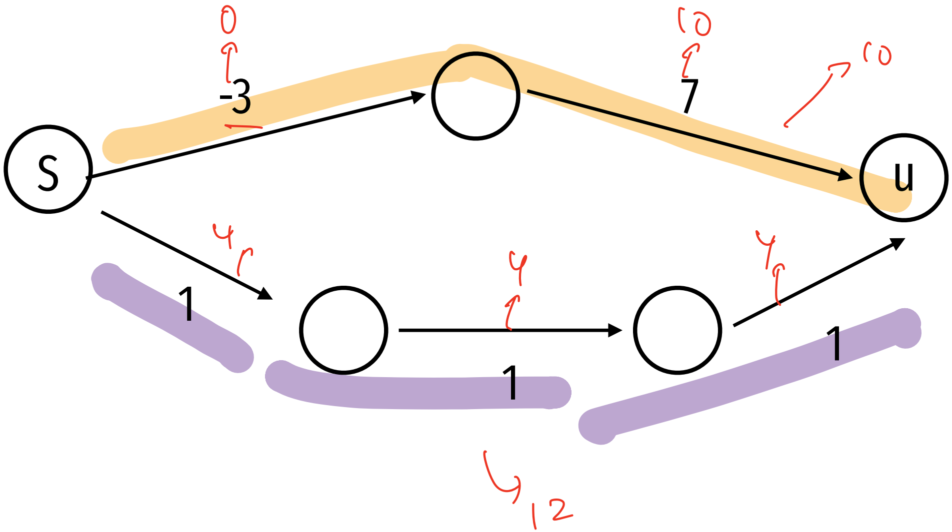
Idea: add a constant to each edge so that all weights are non-negative



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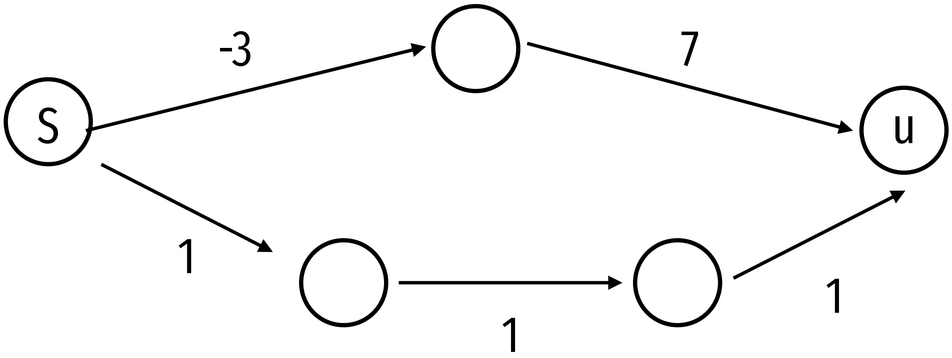


Problem: this can change the shortest paths of the graph.

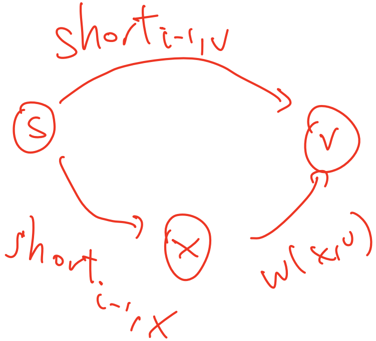
sssp(G,s) single source shortest path problem (BF)

$\textbf{SHORT}_{i,v}$ = length of the shortest path from s to v that uses at most i edges

$\text{short}_{0,v} = \begin{cases} 0 & \text{if } v = s \\ \infty & \text{for all other nodes.} \end{cases}$

$\text{short}_{i,v} = \min_{x \in V} \begin{cases} \text{short}_{i-1,v} \\ \text{short}_{i-1,x} + w(x,v) \end{cases}$

$x \in adj(v)$



sssp(G,s)

$\text{SHORT}_{i,v} =$ Length of the shortest path from s to v that uses at most i edges.

Two possibilities:

sssp(G,s)

$$\text{SHORT}_{i,v} = \begin{cases} \infty & i = 0 \\ 0 & v = s \\ \min_{x \in V} & \left\{ \begin{array}{l} \text{SHORT}_{i-1,v} \\ \text{SHORT}_{i-1,x} + w(x,v) \end{array} \right\} \end{cases}$$

$x \in adj(v)$

Either the shortest path from s to v
uses at most $i-1$ edges or
it uses at most $i-1$ edges to x and then uses edge $e=(x,v)$

max len of a simple path:

$|V|-1$

$\Rightarrow$ $\text{short}_{|V|-1,\,u}$ for all $u \in V$

max len of a simple path:

The max length of a path from s to v is $|V| - 1$.
Otherwise, the path contains a cycle.

BELLMAN-FORD(G,S)

1. $short_{0,u} = \infty$ for all $u \in V$
2. $short_{0,s} = 0$.
3. for $i = 1$ to $|V|-1$
4. for each $v \in V - \{s\}$
5. $$short_{i,v} = \min_{x \in adj(v)} \begin{cases} short_{i-1,v} \\ short_{i-1,x} + w(x,y) \end{cases}$$

BELLMAN-FORD(G, s)

$\text{SHORT}_{0,s} \leftarrow 0$

$\forall v \in V - \{s\}$, $\text{SHORT}_{0,v} \leftarrow \infty$

for $i = 1, \dots, V - 1$

do for each $v \in V - \{s\}$

do $\text{SHORT}_{i,v} = \min_{x \in Adj(v)} \left\{ \begin{array}{l} \text{SHORT}_{i-1,v} \\ w(x,v) + \text{SHORT}_{i-1,x} \end{array} \right\}$

iterating over all the edges in the graph

$\Theta(V \cdot E)$

BELLMAN-FORD(G, s)

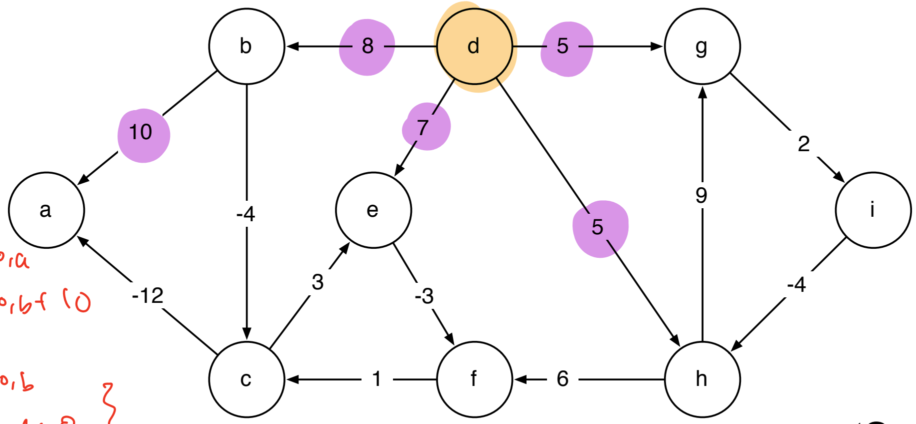
$\text{SHORT}_{0,s} \leftarrow 0$

$\forall v \in V - \{s\}$, $\text{SHORT}_{0,v} \leftarrow \infty$

for $i = 1, \ldots, V - 1$

  do for each $e = (x, y) \in E$

    do $\text{SHORT}_{i,y} = \min \begin{Bmatrix} \text{SHORT}_{i-1,y} \\ \text{SHORT}_{i,y} \\ w(x,y) + \text{SHORT}_{i-1,x} \end{Bmatrix}$



(b,a)

$short_{1,a} = \min \begin{cases} S_{0,a} \\ S_{0,b} + 10 \end{cases}$

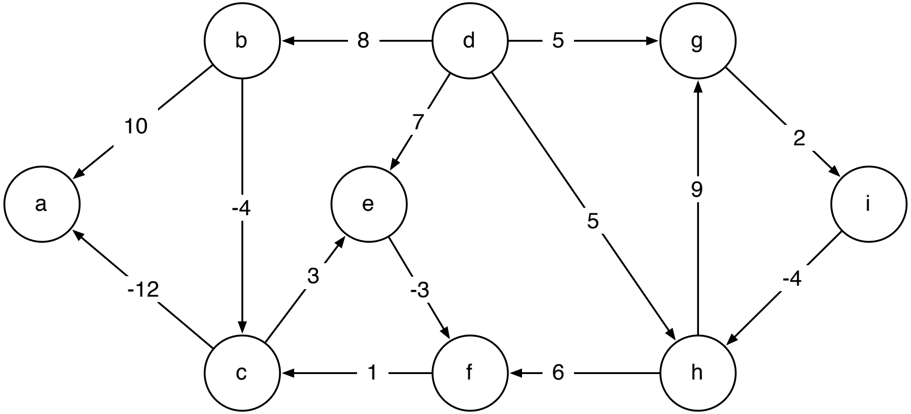
$short_{1,b} = \min \begin{cases} S_{0,b} \\ S_{0,d} + 8 \end{cases}$

$$\text{SHORT}_{i,v} = \begin{cases} \infty & i = 0 \\ 0 & v = s \\ \min_{x \in V} & \left\{ \begin{array}{l} \text{SHORT}_{i-1,v} \\ \text{SHORT}_{i-1,x} + w(x,v) \end{array} \right\} \end{cases}$$

$\circ = \infty$

BF(G,d)

	0	1	2	3	4	5	6	7
A	∘	∘						
B	∘	8						
C	∘							
D	0							
E	∘	7						
F	∘							
G	∘	5						
H	∘	5						
I	∘							



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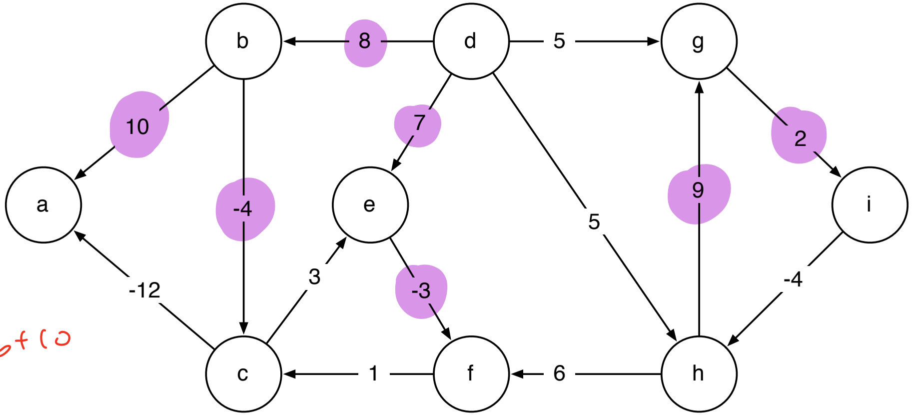
BF(G,d)

	0	1	2	3	4	5	6	7
A	∞							
B	∞							
C	∞							
D	0							
E	∞							
F	∞							
G	∞							
H	∞							
I	∞							

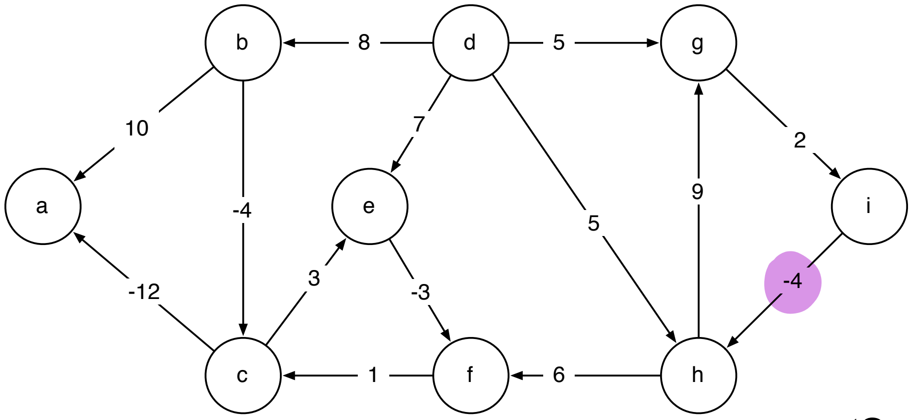


$S_{2,a} = \min \begin{cases} \infty \\ S_{1,b} + 10 \end{cases}$

$$\text{SHORT}_{i,v} = \begin{cases} \infty & i = 0 \\ 0 & v = s \\ \min_{x \in V} & \left\{ \begin{array}{l} \text{SHORT}_{i-1,v} \\ \text{SHORT}_{i-1,x} + w(x,v) \end{array} \right\} \end{cases}$$

BF(G,d)

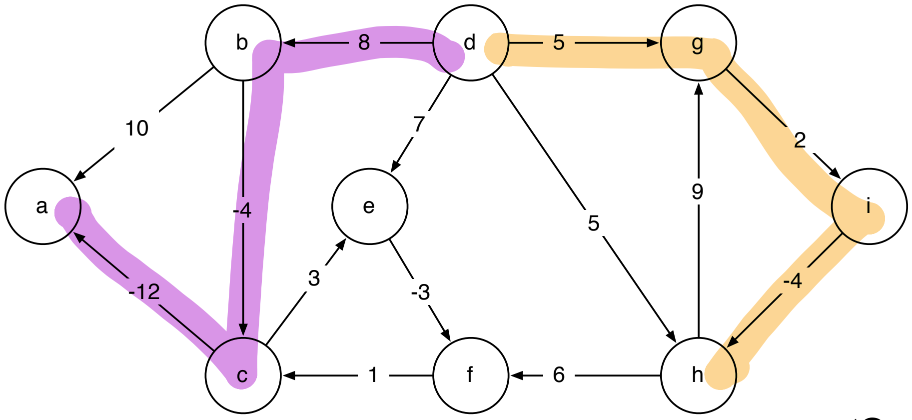
	0	1	2	3	4	5	6	7
A	∞		18					
B	∞	8	8					
C	∞		4					
D	0	0	0					
E	∞	7	7					
F	∞		4					
G	∞	5	5					
H	∞	5	5					
I	∞		7					



$$\text{SHORT}_{i,v} = \begin{cases} \infty & i = 0 \\ 0 & v = s \\ \min_{x \in V} & \left\{ \begin{array}{l} \text{SHORT}_{i-1,v} \\ \text{SHORT}_{i-1,x} + w(x,v) \end{array} \right\} \end{cases}$$

BF(G,d)

	0	1	2	3	4	5	6	7
A	∞		18					
B	∞	8	8					
C	∞		4					
D	0	0	0					
E	∞	7	7					
F	∞		4					
G	∞	5	5					
H	∞	5	5	3				
I	∞		7					



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BF(G,d)

	0	1	2	3	4	5	6	7
A	∞		18	-8				
B	∞	8	8	8				
C	∞		4	4				
D	0	0	0	0				
E	∞	7	7	7				
F	∞		4	4				
G	∞	5	5	5				
H	∞	5	5	3				
I	∞		7	7				

V^2

optimization

BELLMAN-FORD(G, s)

$\text{SHORT}_{0,s} \leftarrow 0$
$\forall v \in V - \{s\}$, $\text{SHORT}_{0,v} \leftarrow \infty$
for $i = 1, \ldots, V - 1$
    do for each $e = (x, y) \in E$
        do $\text{SHORT}_{i,y} = \min \left\{ \begin{array}{l} \text{SHORT}_{i-1,y} \\ \text{SHORT}_{i,y} \\ w(x,y) + \text{SHORT}_{i-1,x} \end{array} \right\}$

BELLMAN-FORD(G, s)

$d_s \leftarrow 0$
$\forall v \in V - \{s\}$, $d_v \leftarrow \infty$
for $i = 1, \ldots, V - 1$
    do for each $e = (x, y) \in E$
        do $d_y \leftarrow \min\{\, d_y, w(x,y) + d_x \,\}$

running time

BELLMAN-FORD(G, s)

$d_s \leftarrow 0$

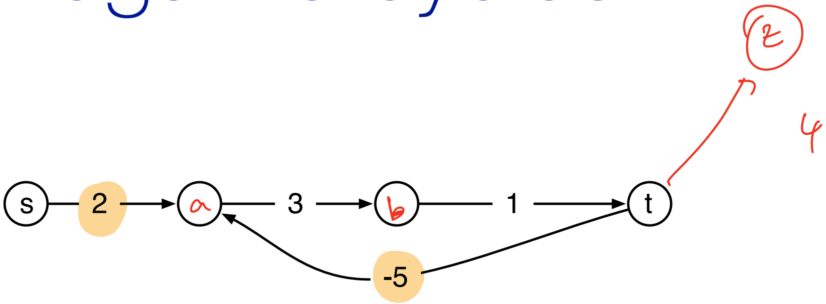
$\forall v \in V - \{s\},\ d_v \leftarrow \infty$

for $i = 1, \ldots, V - 1$

  do for each $e = (x, y) \in E$

    do $d_y \leftarrow \min\{\ d_y, w(x, y) + d_x\ \}$

negative cycles?



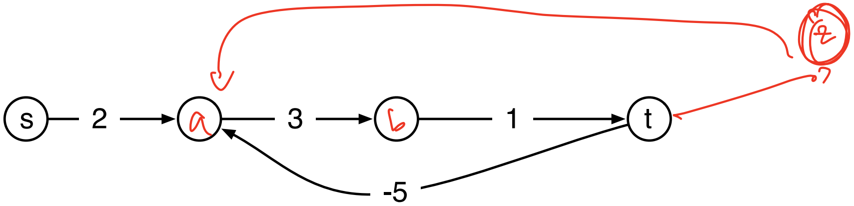
	1	2	3	4
S	0	0	0	
A	2	2	2	1
B	·	5	5	
T	·	·	6	

→ min $\{2,\ 6 + -5\}$

the d_v decreases for any v on the V^{th} iteration

$\iff$ there exists a negative weight cycle

negative cycles?



S	0	0	0	0
A	2	2	2	1
B		5	5	5
T			6	6

applications of BF

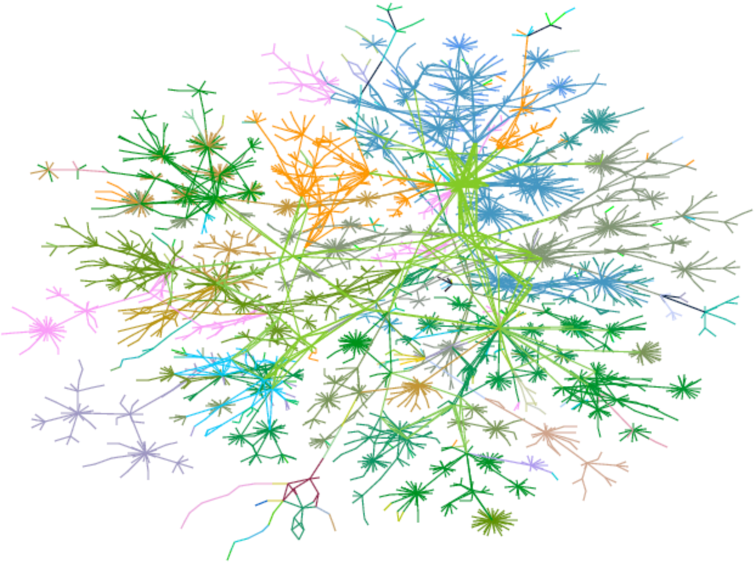
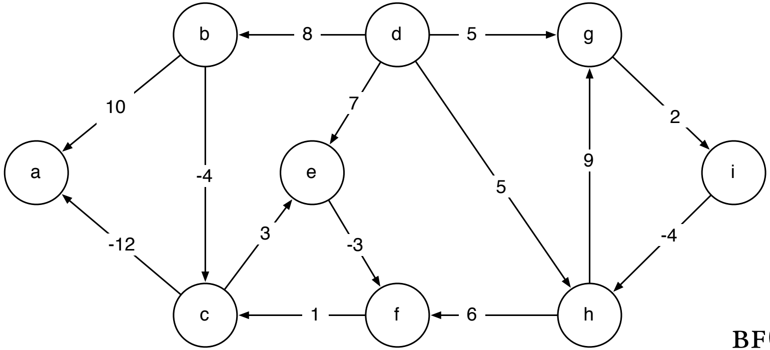


image: cheswick et al
Figure 3: Lucent's intranet as of 1 October 1999.



WHAT HAPPENS WHEN B CHANGES...

BF(G,d)

	0	1	2	3	4	5	6	7
A	∞							
B	∞	8						
C	∞							
D	0	0						
E	∞	7						
F	∞							
G	∞	5						
H	∞	5						
I	∞							

DISTANCE VECTOR

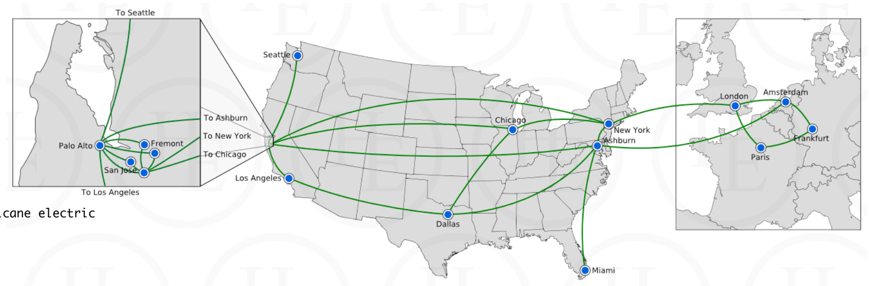
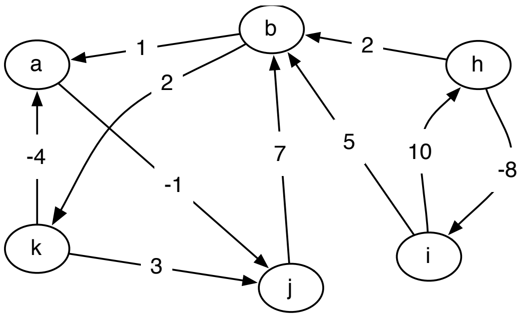


image: hurricane electric

BGP routing algorithm.

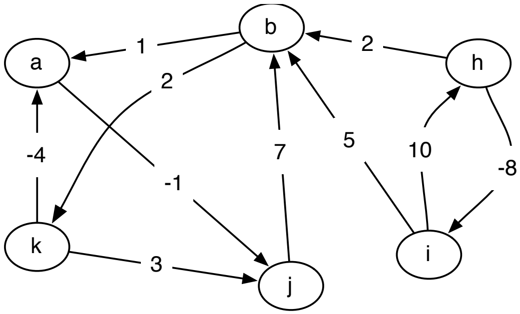
ALL-PAIRS SHORTEST PATH



First approach: run BF from each node.
What is the running time?

$V \cdot \Theta(E \cdot V) \rightarrow \Theta(EV^2)$

ALL-PAIRS SHORTEST PATH



First approach: run BF from each node.
What is the running time?

$$O(EV^2)$$

NEW APPROACH TO ALL-PAIRS

$G = (V, E)$

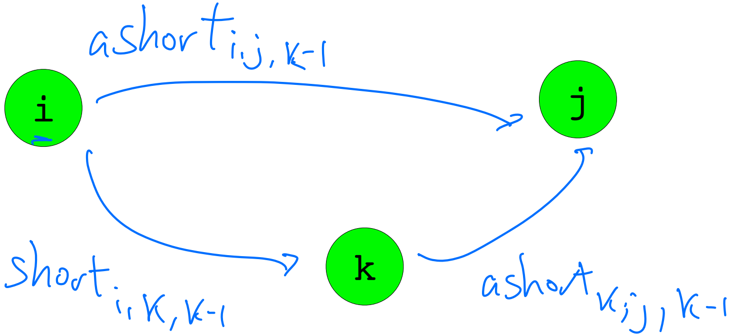
$\mathrm{ASHORT}_{i,j,k} =$ length of the shortest path from i to j that only uses vertices $\{1 \ldots k\}$ as intermediates.

NEW APPROACH TO ALL-PAIRS

$\text{ASHORT}_{i,j,k} =$ Length of the shortest path from i to j that only traverses nodes $1, \ldots, k$.

$$\mathrm{ASHORT}_{i,j,k} = \min \begin{cases} w(i,j) \\ \mathrm{ashort}_{i,j,k-1} \\ \mathrm{ashort}_{i,k,k-1} + \mathrm{ashort}_{k,j,k-1} \end{cases}$$

Q: how does node k help in getting from i to j?



$$\mathrm{ASHORT}_{i,j,k} =$$

$i,j,0 \curvearrowleft i,j,1 \curvearrowleft i,j,2$

$$\text{ASHORT}_{i,j,k} = \begin{cases} w_{i,j} & k = 0 \\ \min \begin{cases} \text{ASHORT}_{i,j,k-1} \\ \text{ASHORT}_{i,k,k-1} + \text{ASHORT}_{k,j,k-1} \end{cases} & k \geq 1 \end{cases}$$

$\text{ashort}_{i,j,|V|}$

our goal is to compute this value for all pairs (i,j).

FLOYD-WARSHALL(G,W)

INITIALIZE ASHORT$_{i,j,0}$

$\Theta(V^3)$

versus

$\Theta(EV^2)$

for k = 1 to V

for i = 1 to V

for j = 1 to V

$$ashort_{i,j,k} = \min \begin{cases} ashort_{i,j,k-1} \\ ashort_{i,k,k-1} + ashort_{k,j,k-1} \end{cases}$$

(greedy) Dijkstra $\Theta(E \cdot \log V)$

unit weight edges

BFS

$\Theta(E+V)$

negative edge weights

BF

$\Theta(E \cdot V)$

All pairs

$\Theta(V^3)$

(DP)