L3 5800

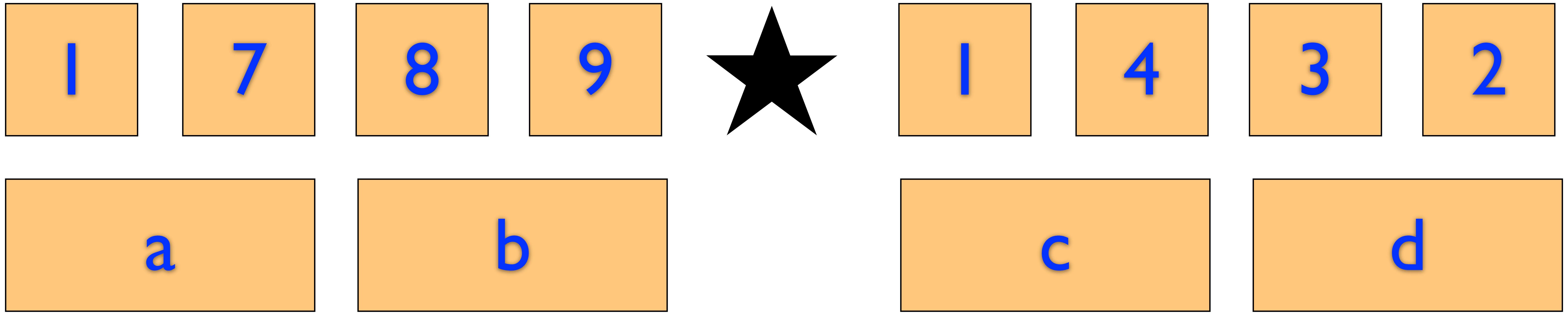
jan 25 2022

shelat

Warmup

$$\left(\frac{13}{11}\right)^{54} = 2^x$$

Karatsuba algorithm



Recursively compute

1. $ac, bd, (a+b)(c+d)$ $\quad 3T(n/2) + 2O(n)$
2. $ad + bc = (a+b)(c+d) - ac - bd$ $\quad 2O(n)$
3. $ac100^2 + (ad+bc)100 + bd$ $\quad 2O(n)$

Karatsuba algorithm

1 7 8 9 ★ 1 4 3 2

Karatsuba(ab, cd)

Base case: return b*d if inputs are 1-digit

ac = Karatsuba(a,c)

bd = Karatsuba(b,d)

t = Karatsuba((a+b),(c+d))

$$3T(n/2) + 2n$$

mid = t - ac - bd $\quad 4n$

RETURN ac*100^2 + mid*100 + bd $\quad 3n$

$$T(n) = 3T(n/2) + O(n)$$

calculations:

$$T(n) = 9n + 3 \cdot \frac{9n}{2} + 3^2 \cdot \frac{9n}{2^2} + \cdots + 3^{\lceil \log n \rceil} \cdot \frac{9n}{2^{\lceil \log n \rceil}}$$

$$= 9n \left[1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \cdots \left(\frac{3}{2}\right)^{\lceil \log n \rceil} \right]$$

$$= 9n \left[\frac{\left(\frac{3}{2}\right)^{\lceil \log n \rceil + 1} - 1}{\left(\frac{3}{2}\right) - 1} \right] = 9n \cdot 2 \cdot \frac{3}{2} \cdot \left(\frac{3}{2}\right)^{\lceil \log n \rceil} - 18n$$

$$= 27n \cdot \frac{2^{(\log_2 3)\lceil \log n \rceil}}{2^{\lceil \log n \rceil}} - 18n = 27 \cdot n^{\log_2 3} - 18n$$

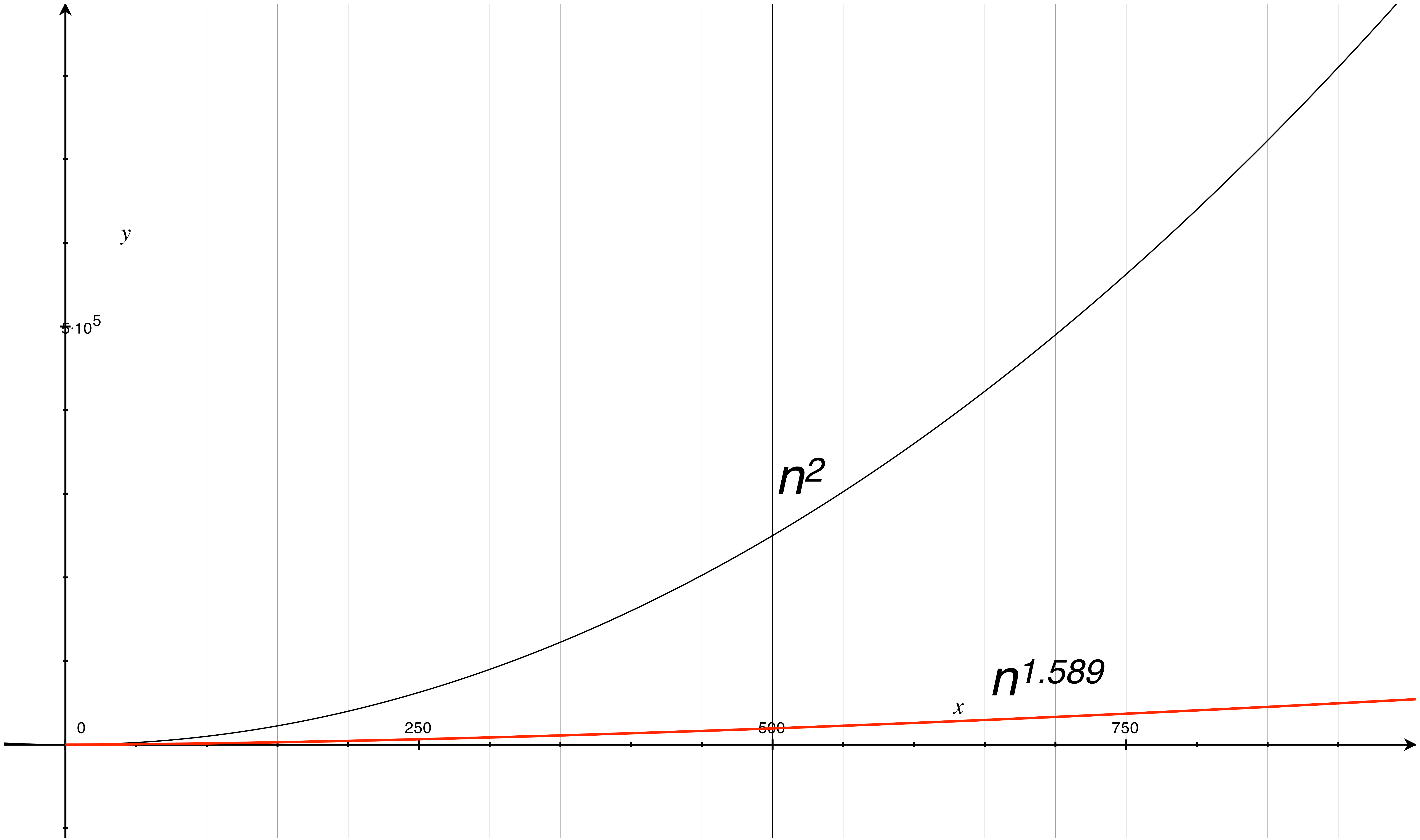
$$= O\left(n^{\log_2 3}\right)$$

$$T(n) = 3T(n/2) + 9n$$

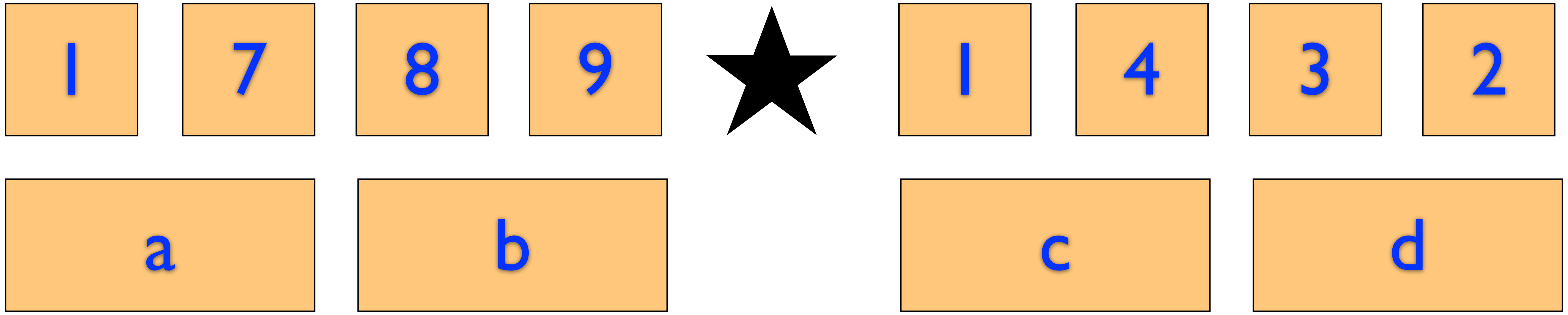
$$O(n^{\log_2(3)})$$

$$T(n) = 3T(n/2) + 9n$$

$$O(n^{\log_2(3)}) \quad O(n^{1.589})$$



y
5·10^5
n^2
n^1.589
x
0
250
500
750



$$T(n) = 3T(n/2) + 9n$$

$$T(n) = 4T(n/2) + 3n$$

simpler proof technique?

❶ classic

induction redux

goal: prove that some property P(k) is true for all k

$\forall k, P(k)$ holds

❶ classic one long proof...

goal: prove that some property P(k) is true for all k

$\forall k, P(k) \text{ holds}$

❶

Induction

classic
base case: $P(1)$

classic
inductive step:

$$\left.\begin{array}{c} P(1) \\ \cdots \\ P(k) \end{array}\right\} \text{ implies } P(k+1) \text{ true}$$

❷ induction redux

asymptotic style

base case: $P(n^*)$

inductive step: $\left.\begin{matrix} P(n^*) \\ \dots \\ P(k) \end{matrix}\right\}$ implies $P(k+1)$ true

simpler proof (guess +chk)

$$T(n) = 3T(n/2) + 9n$$

simpler proof

$$T(n) = 3T(n/2) + cn$$

simpler proof

$$T(n) = 3T(n/2) + cn$$

hypothesis: `T(n) < 400cn1.59`

The hypothesis is true for n=1. Suppose it is true for n<n0.

Now consider $T(n_0 + 1) = 3T((n_0 + 1)/2) + c(n_0 + 1)$

$$< 3 \cdot 400c[(n_0 + 1)/2)^{1.59} + c(n_0 + 1)$$

By the hypothesis because $(n_0+1)/2$ is less than n_0.

$$< \frac{3 \cdot 400c}{2^{1.59}}(n_0 + 1)^{1.59} + c(n_0 + 1)$$

$$< 399c(n_0 + 1)^{1.59} + c(n_0 + 1)$$

$$< 400c(n_0 + 1)^{1.59}$$

Becase $c(n_0+1) < c(n_0+1)^{1.59}$

Notice this conclusion EXACTLY matches the hypothesis.
This is essential for an induction proof.
What we have shown is that if $T(n) < 400cn^{1.59}$, then $T(n+1) < 400c(n+1)^{1.59}$

mergesort

goal:

technique:

6	4	9	12	2	5	8	7

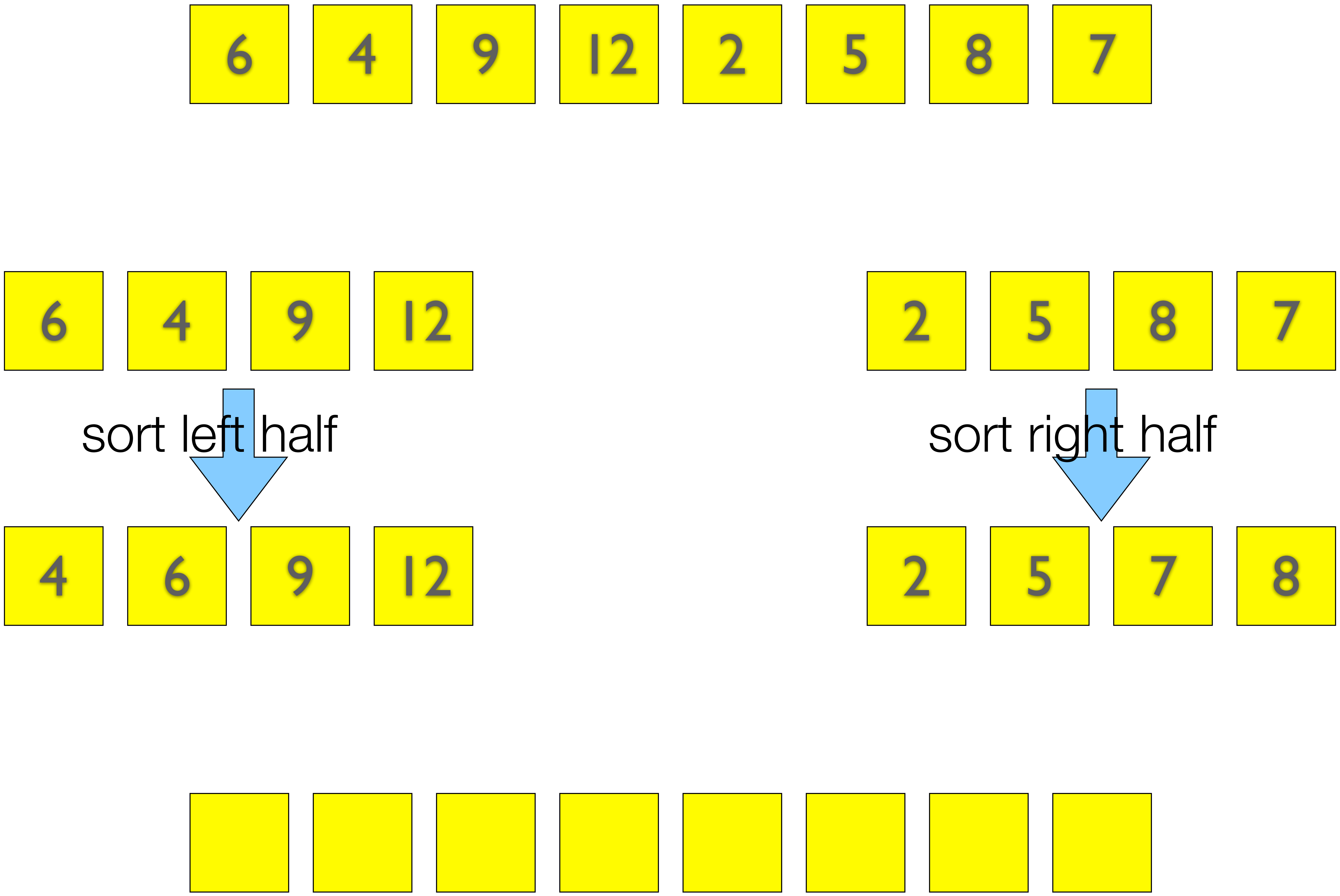
mergesort

6	4	9	12	2	5	8	7

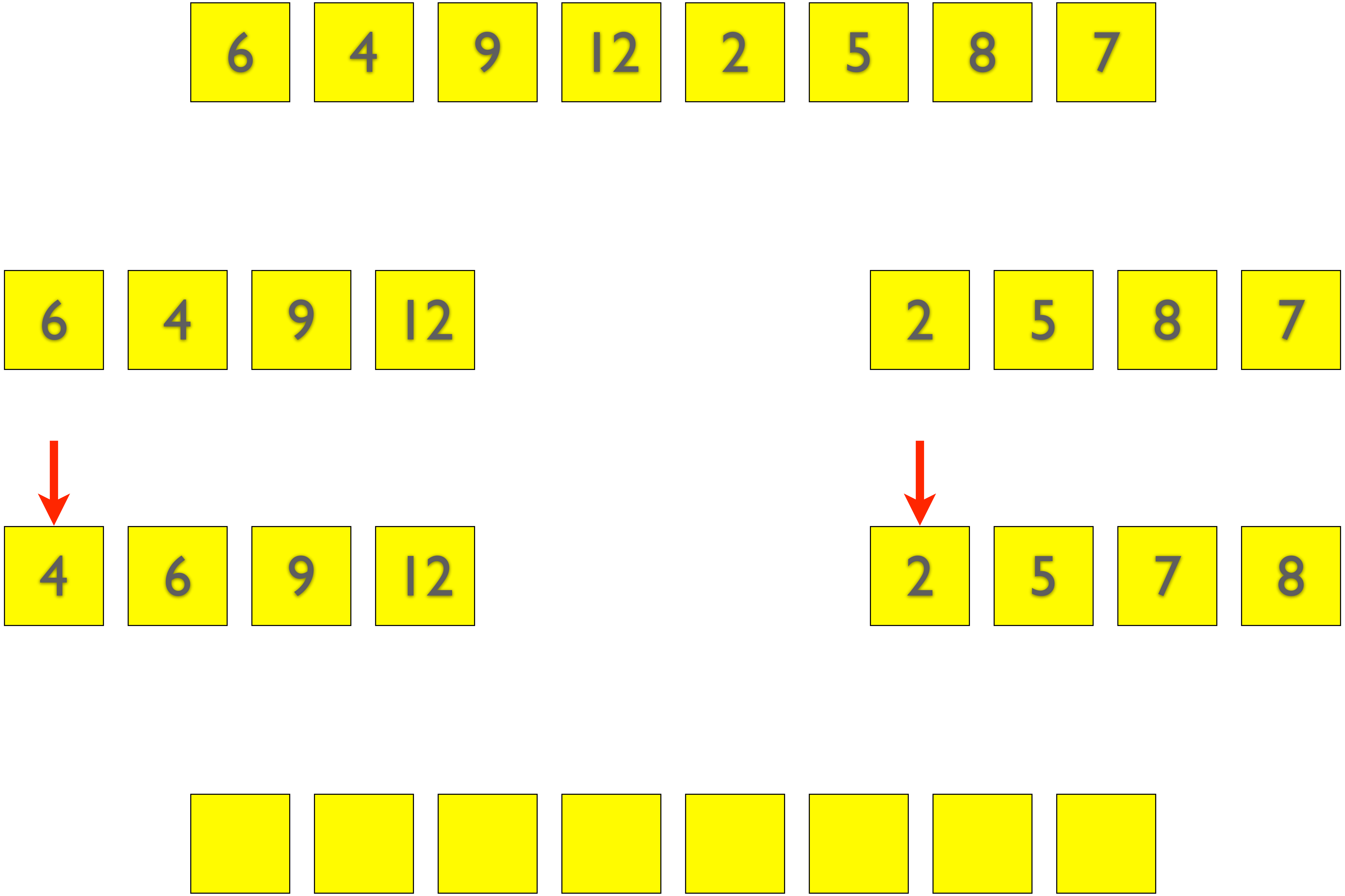
6	4	9	12

2	5	8	7

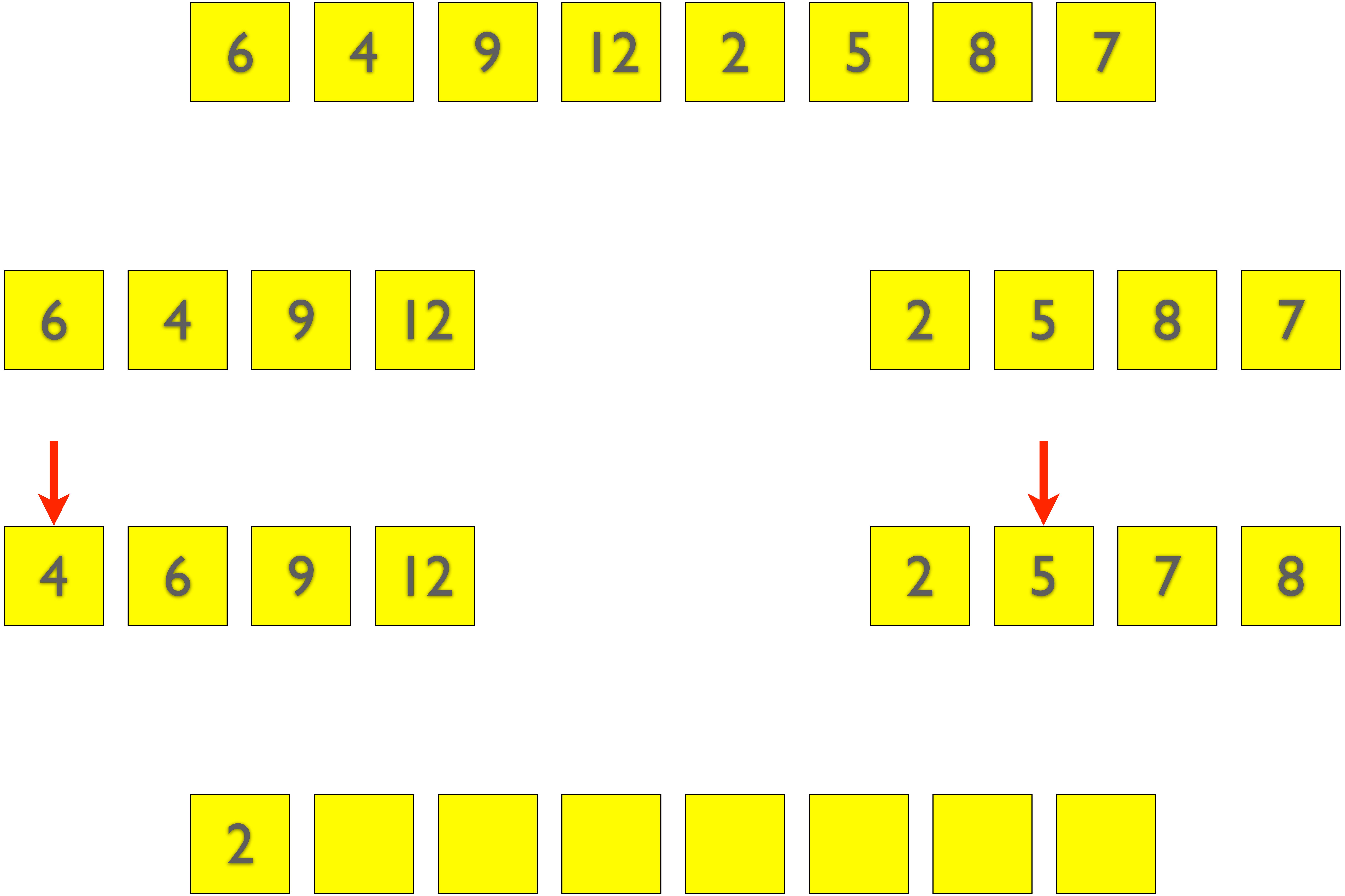
mergesort


6 4 9 12 2 5 8 7
6 4 9 12
2 5 8 7
sort left half
sort right half
4 6 9 12
2 5 7 8

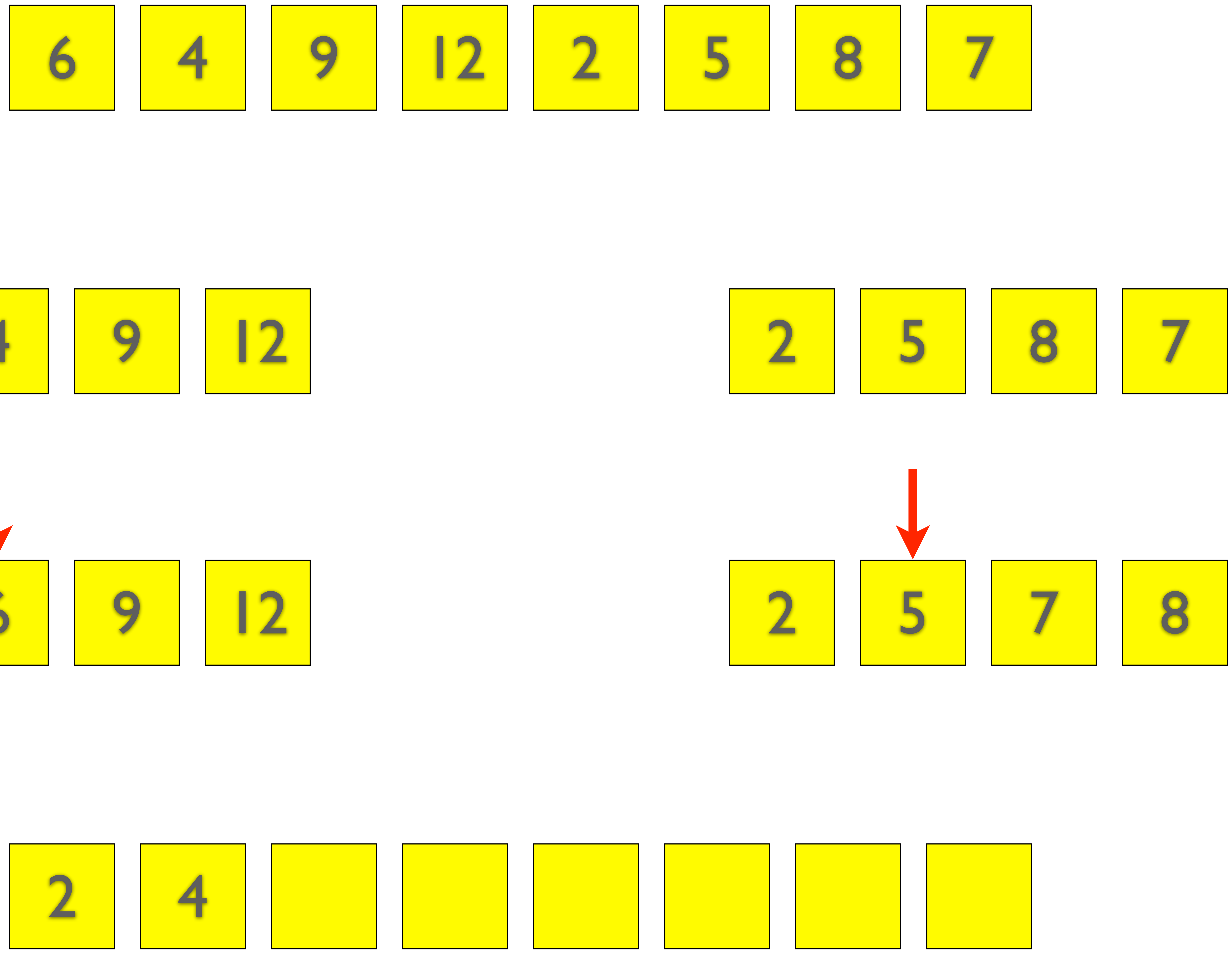
mergesort



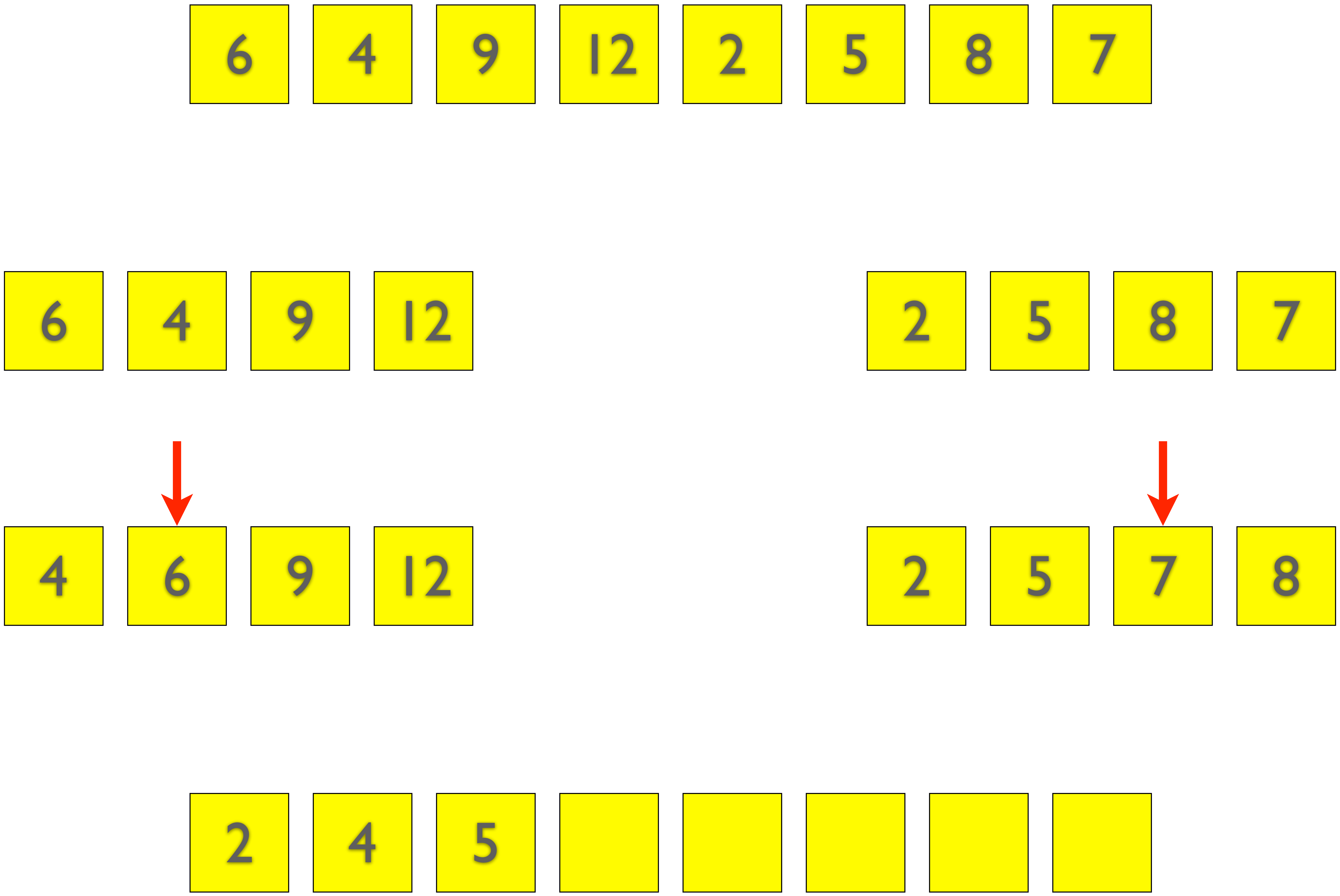
mergesort



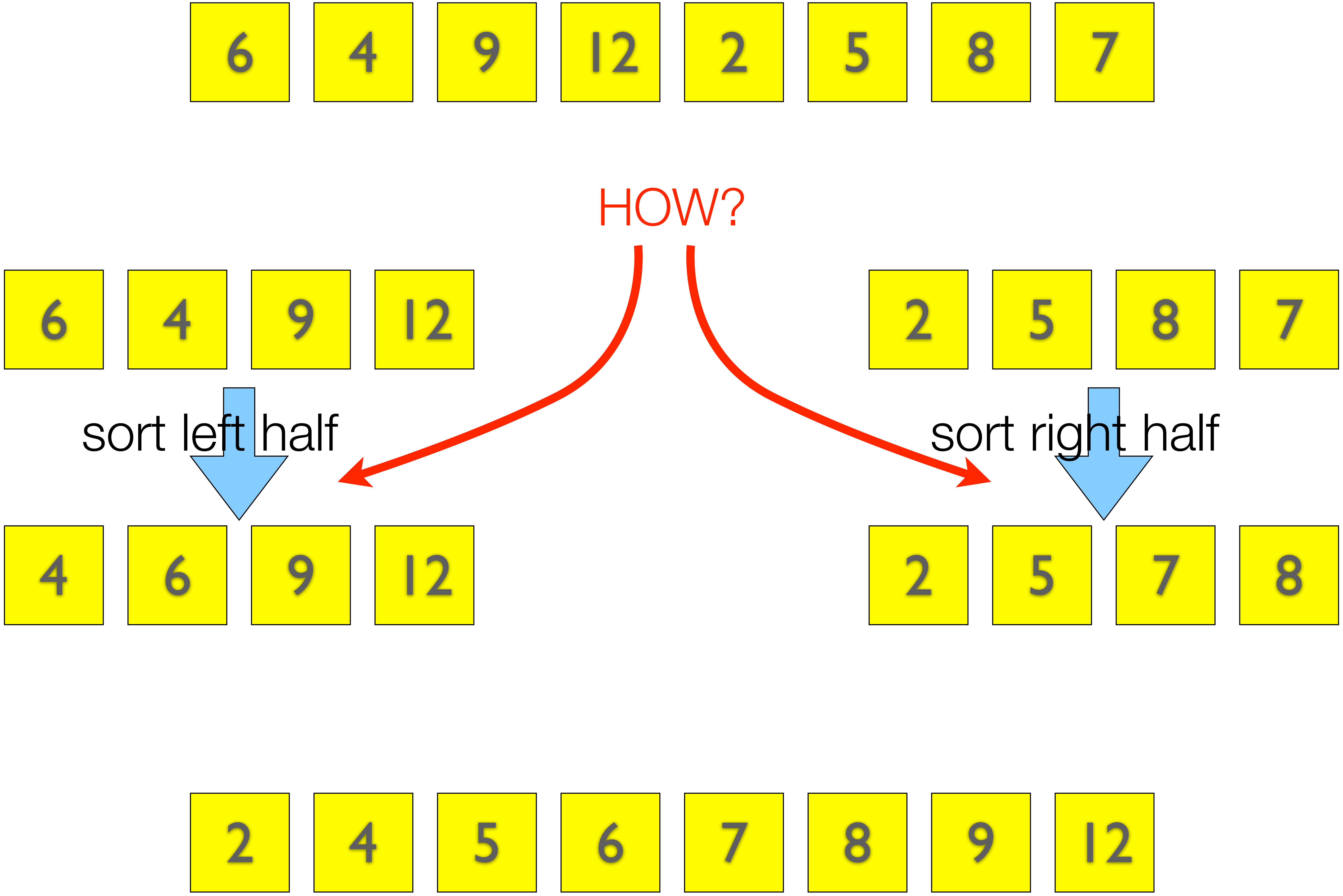
mergesort



mergesort



mergesort



6 4 9 12 2 5 8 7
HOW?
6 4 9 12
2 5 8 7
sort left half
sort right half
4 6 9 12
2 5 7 8
2 4 5 6 7 8 9 12

mergesort(`A,start,end`)

❶

❷

❸

❹

❺

mergesort(A,start,end)

1. `if start < end`
2. $q \leftarrow \lfloor (\text{start} + \text{end})/2 \rfloor$
3. `mergesort(A,start,q)`
   `mergesort(A,q+1,end)`
4. `merge(A,start,q,end)`
5. `else base case, return.`

mergesort(A,start,end)

1 `if start < end`

2 $q \leftarrow \lfloor (\text{start} + \text{end})/2 \rfloor$

3
```
    mergesort(A,start,q)
    mergesort(A,q+1,end)
```

4
```
    merge(A,start,q,end)
```

5 `else ...`

MERGE($A[1..n], m$):

$i \leftarrow 1$; $j \leftarrow m+1$
for $k \leftarrow 1$ to n
  if $j > n$
    $B[k] \leftarrow A[i]$; $i \leftarrow i+1$
  else if $i > m$
    $B[k] \leftarrow A[j]$; $j \leftarrow j+1$
  else if $A[i] < A[j]$
    $B[k] \leftarrow A[i]$; $i \leftarrow i+1$
  else
    $B[k] \leftarrow A[j]$; $j \leftarrow j+1$

for $k \leftarrow 1$ to n
  $A[k] \leftarrow B[k]$

jeff erickson

mergesort(`A,start,end`)

running time?

```
1  if start < end
```

2 $q \leftarrow \lfloor (\text{start} + \text{end})/2 \rfloor$

```
3      mergesort(A,start,q)
       mergesort(A,q+1,end)

4      merge(A,start,q,end)

5  else ...
```

$$T(n) = 2T(n/2) + n$$

prove:

hypothesis:

base case:

inductive step:

$$T(n) = 2T(n/2) + n$$

prove: $T(n) = O(n \log n)$

property: $T(n) < cn \log n$ for c>1

base case:

inductive step:

$$T(n) = 2T(n/2) + n$$

goal is to show $T(n) = \Theta(n \log n)$

show: $T(n) \le n \log n$

Proof: Base case holds for $n \le 5$. Assume that the hypothesis holds for all $k \le n$. Consider

$$T(n+1) = 2T\left(\frac{n+1}{2}\right) + (n+1)$$

$\frac{n+1}{2} < n, \Rightarrow T\left(\frac{n+1}{2}\right) \le \left(\frac{n+1}{2}\right) \log\left(\frac{n+1}{2}\right)$ by ind hypothesis

$$\le 2\left(\frac{n+1}{2}\right) \log\left(\frac{n+1}{2}\right) + n+1$$

$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$

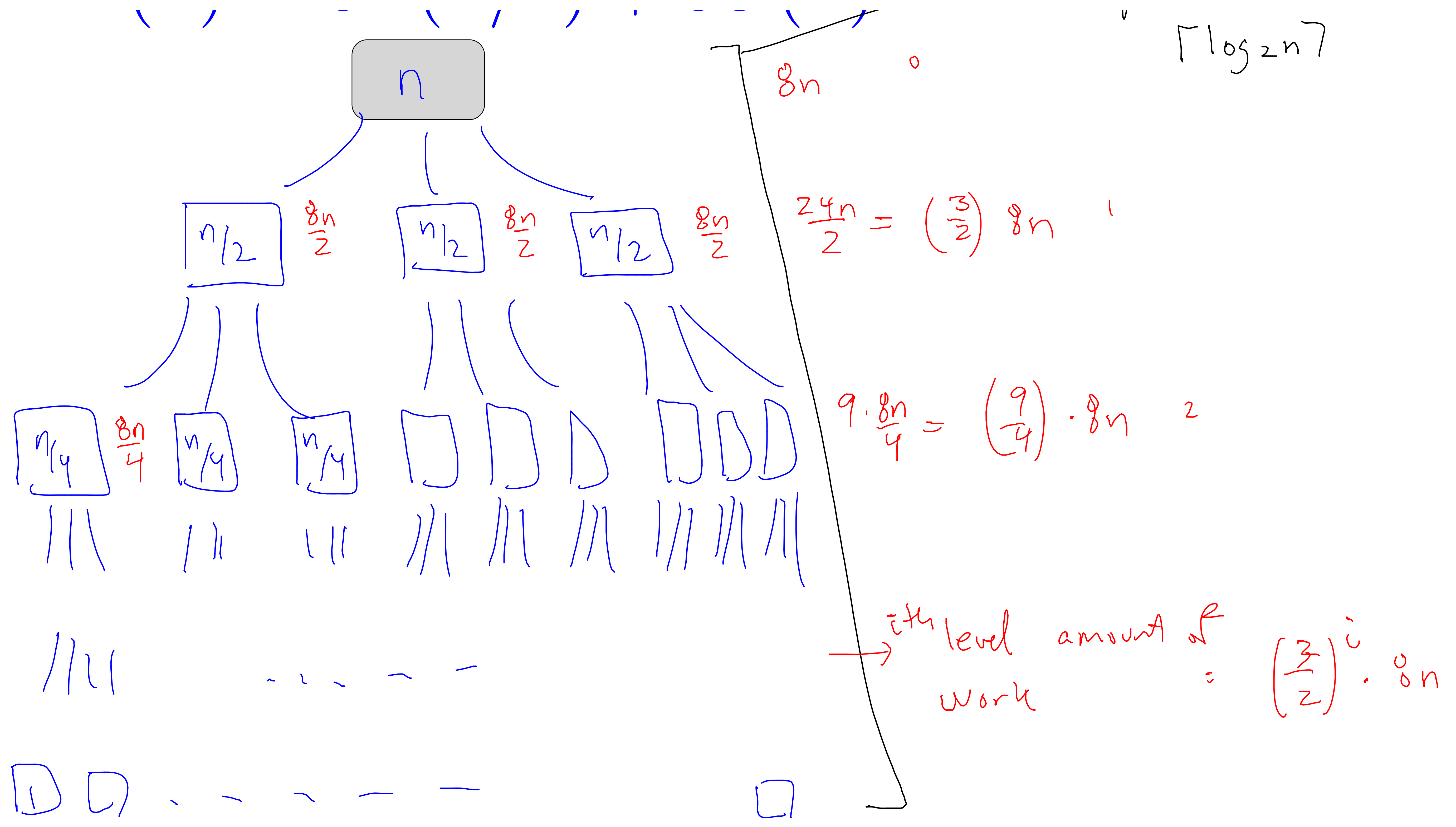
$$= (n+1)\left[\log(n+1) - 1\right] + n+1$$

$$= (n+1)\log(n+1) - (n+1) + n+1$$

$$= (n+1)\log(n+1)$$

$$T(n) = 3T(n/2) + 9n$$

$$O(n^{1.589}) \qquad O(n^{\log_2(3)})$$



$$T(n) = 3T(n/2) + 9n$$ (guess +chk)

Lets prove that $T(n) \le n^{\log_2 3} - 20n$

Assuming T(1) = 1

$$T(n) = 3T(n/2) + 9n$$ (guess +chk)

Lets prove that $T(n) \leq n^{\log_2 3} - 20n$

Assuming T(1) = 1

By inspection, indeed, $T(n) \leq n^{\log_2 3} - 20n$ when n<1024.

$$T(n) = 3T(n/2) + 9n$$ (guess +chk)

Lets prove that $T(n) \leq n^{\log_2 3} - 20n$ Assuming T(1) = 1

By inspection, indeed, $T(n) \leq n^{\log_2 3} - 20n$ when n<1024.

A1: Lets assume that $T(n) \leq n^{\log_2 3} - 20n$ when $n < n_0$

$$T(n) = 3T(n/2) + 9n$$ (guess +chk)

Lets prove that $T(n) \leq n^{\log_2 3} - 20n$

Assuming T(1) = 1

By inspection, indeed, $T(n) \leq n^{\log_2 3} - 20n$ when n<1024.

A1: Lets assume that $T(n) \leq n^{\log_2 3} - 20n$ when $n < n_0$

Consider the case of $T(n_0 + 1)$

$$T(n_0 + 1) = 3T((n_0 + 1)/2) + 9(n_0 + 1)$$ By definition

$$T(n) = 3T(n/2) + 9n$$ (guess +chk)

Lets prove that $T(n) \leq n^{\log_2 3} - 20n$

Assuming T(1) = 1

By inspection, indeed, $T(n) \leq n^{\log_2 3} - 20n$ when n<1024.

A1: Lets assume that $T(n) \leq n^{\log_2 3} - 20n$ when $n < n_0$

Consider the case of $T(n_0 + 1)$

$$T(n_0 + 1) = 3T((n_0 + 1)/2) + 9(n_0 + 1)$$ By definition

But since $(n_0 + 1)/2 < n_0$ and A1, it follows that

$$T(n_0 + 1) < 3\left[\left(\frac{n_0 + 1}{2}\right)^{\log 3} - 20\left(\frac{n_0 + 1}{2}\right)\right] + 9(n_0 + 1)$$

$$T(n_0 + 1) < 3\left[\left(\frac{n_0 + 1}{2}\right)^{\log 3} - 20\left(\frac{n_0 + 1}{2}\right)\right] + 9(n_0 + 1)$$

$$T(n_0 + 1) < 3\left[\left(\frac{n_0 + 1}{2}\right)^{\log 3} - 20\left(\frac{n_0 + 1}{2}\right)\right] + 9(n_0 + 1)$$

$$< (n_0 + 1)^{\log 3} - 30(n_0 + 1) + 9(n_0 + 1)$$

$$T(n_0+1) < 3\left[\left(\frac{n_0+1}{2}\right)^{\log 3} - 20\left(\frac{n_0+1}{2}\right)\right] + 9(n_0+1)$$

$$< (n_0+1)^{\log 3} - 30(n_0+1) + 9(n_0+1)$$

$$< (n_0+1)^{\log 3} - 20(n_0+1)$$

$$T(n_0+1) < 3\left[\left(\frac{n_0+1}{2}\right)^{\log 3} - 20\left(\frac{n_0+1}{2}\right)\right] + 9(n_0+1)$$

$$< (n_0+1)^{\log 3} - 30(n_0+1) + 9(n_0+1)$$

$$< (n_0+1)^{\log 3} - 20(n_0+1)$$

This expression matches our Assumption A1.

A1: Lets assume that $T(n) \leq n^{\log_2 3} - 20n$ when $n < n_0$

Thus, we can conclude the proof via induction.

This establishes that $T(n) = O(n^{\log_2 3})$

Induction summary

1 $T(n) \leq n^{\log_2 3} - 20n$ IS TRUE for one case.

2 $T(n) \leq n^{\log_2 3} - 20n$ Suppose TRUE for $n < n_0$

3 Showed that 1,2 imply that

$$T(n_0 + 1) \leq (n_0 + 1)^{\log_2 3} - 20(n_0 + 1)$$

4 (Induction)

What happens if
we skip the -20n?

$$T(n) = 3T(n/2) + 9n$$ (guess +chk)

Lets prove that $T(n) \leq n^{\log_2 3} - 20n$

By inspection, indeed, $T(n) \leq n^{\log_2 3} - 20n$ when n<1024.

A1: Lets assume that $T(n) \leq n^{\log_2 3} - 20n$ when $n < n_0$

Consider the case of $T(n_0 + 1)$

$$T(n_0 + 1) = 3T((n_0 + 1)/2) + 9(n_0 + 1)$$ By definition

$T(n) = 3T(n/2) + 9n$ (guess +chk)

Lets prove that $T(n) \leq n^{\log_2 3} - 20n$

By inspection, indeed, $T(n) \leq n^{\log_2 3} - 20n$ when n<1024.

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$$T(n_0 + 1) = 3T((n_0 + 1)/2) + 9(n_0 + 1)$$ By definition

But since $(n_0 + 1)/2 < n_0$ and A1, it follows that

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$$T(n_0 + 1) < 3\left[\left(\frac{n_0+1}{2}\right)^{\log 3} - 20\left(\frac{n_0+1}{2}\right)\right] + 9(n_0 + 1)$$

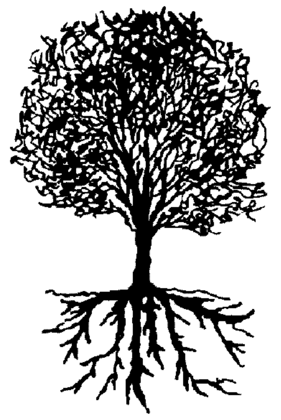
$$T(n_0+1) < 3\left[\left(\frac{n_0+1}{2}\right)^{\log 3} - 20\left(\frac{n_0+1}{2}\right)\right] + 9(n_0+1)$$

$$< (n_0+1)^{\log 3} - 30(n_0+1) + 9(n_0+1)$$

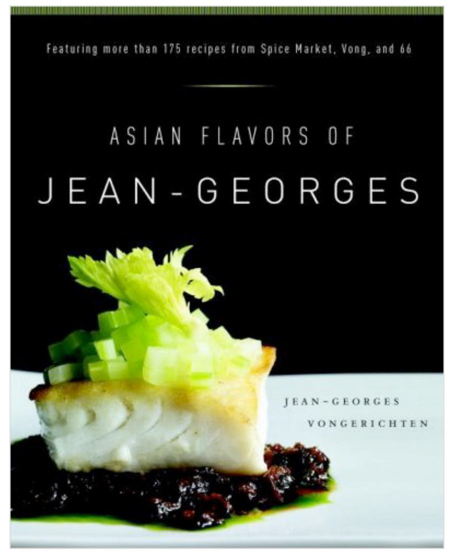
This expression DOES NOT matches our Assumption A1.

So the induction STOPS! $T(n) \leq n^{\log_2 3}$

$$T(n) = 8T(n/2) + \Theta(n^2)$$ (guess +chk)

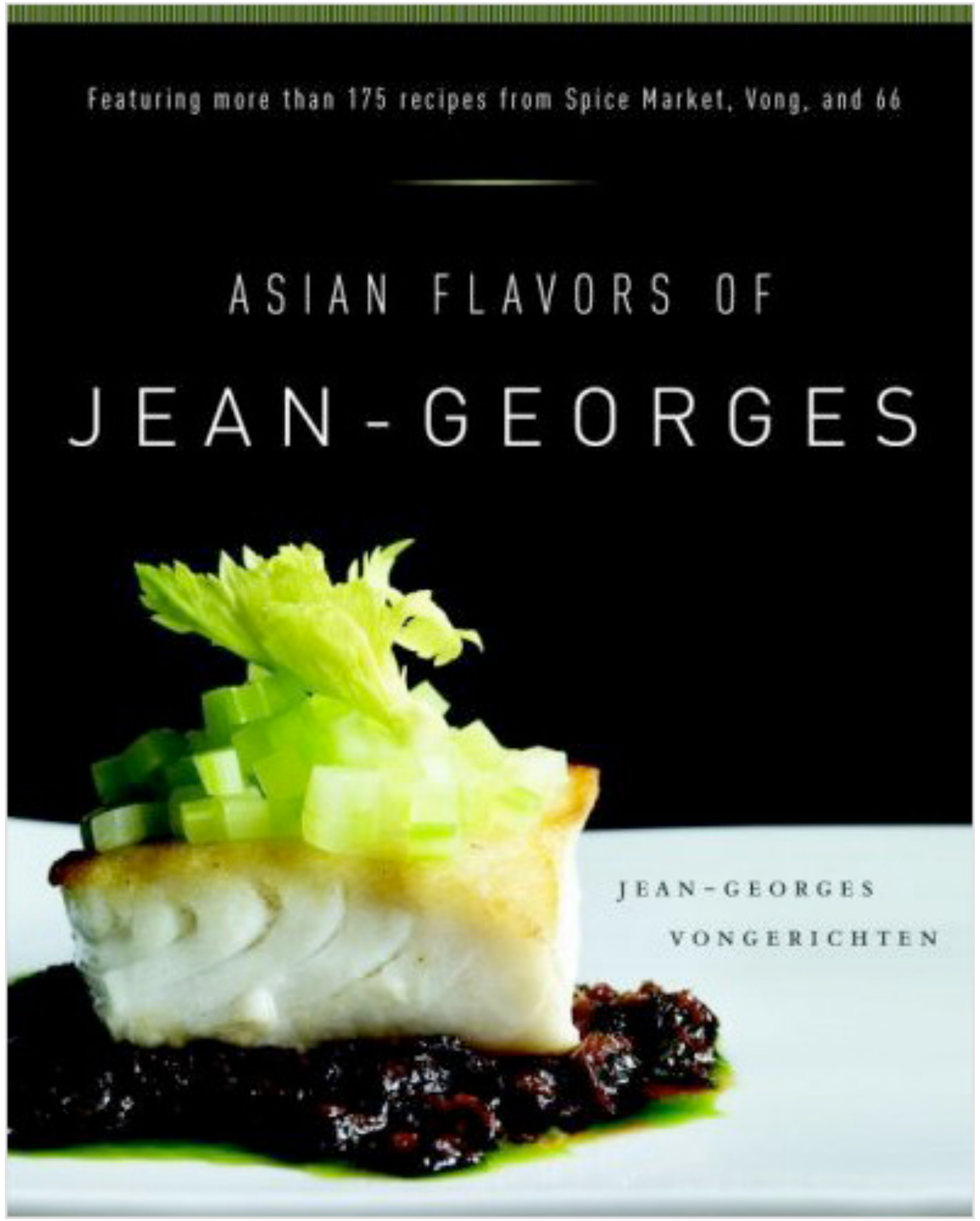


?-✓



http://www.drblank.com/law301.jpg

cookbook



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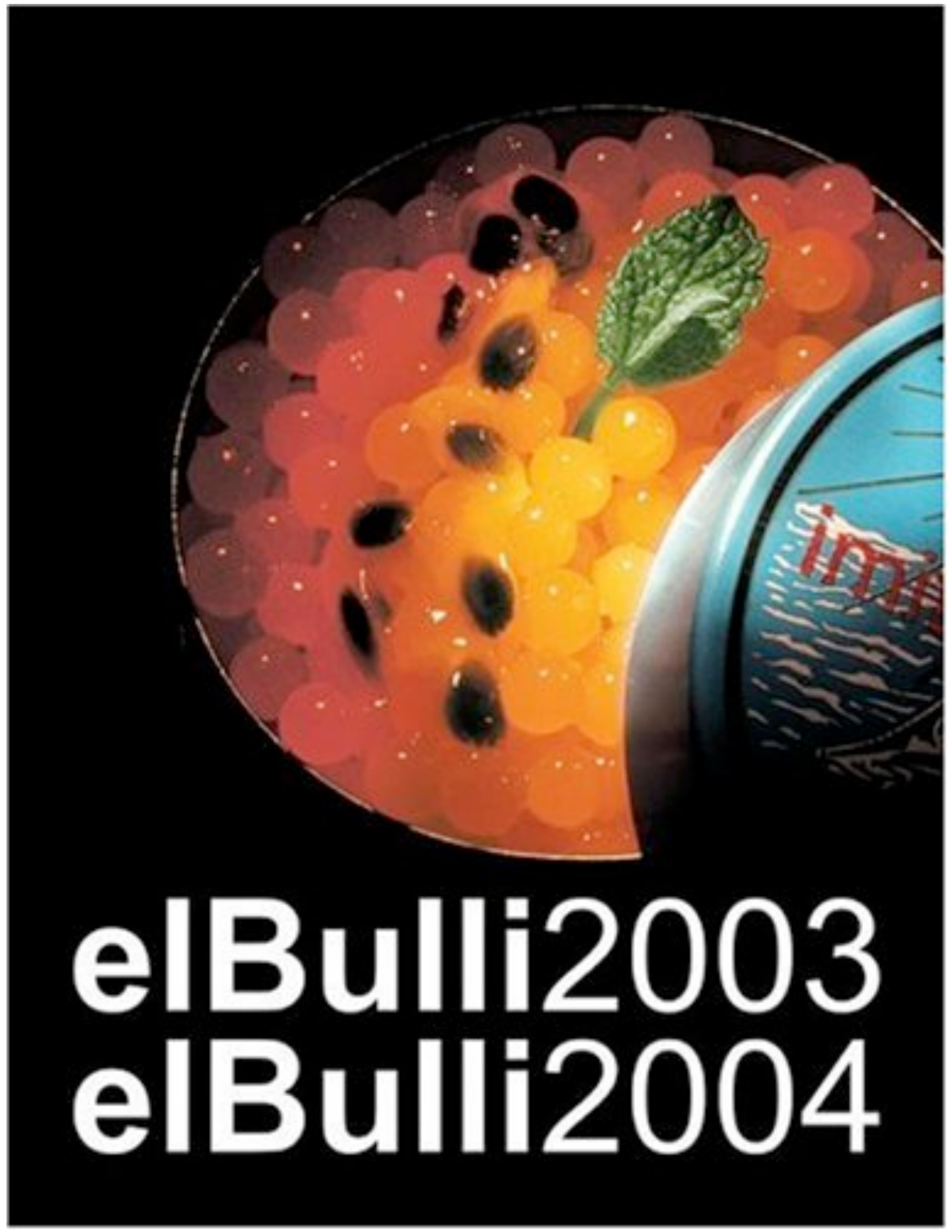
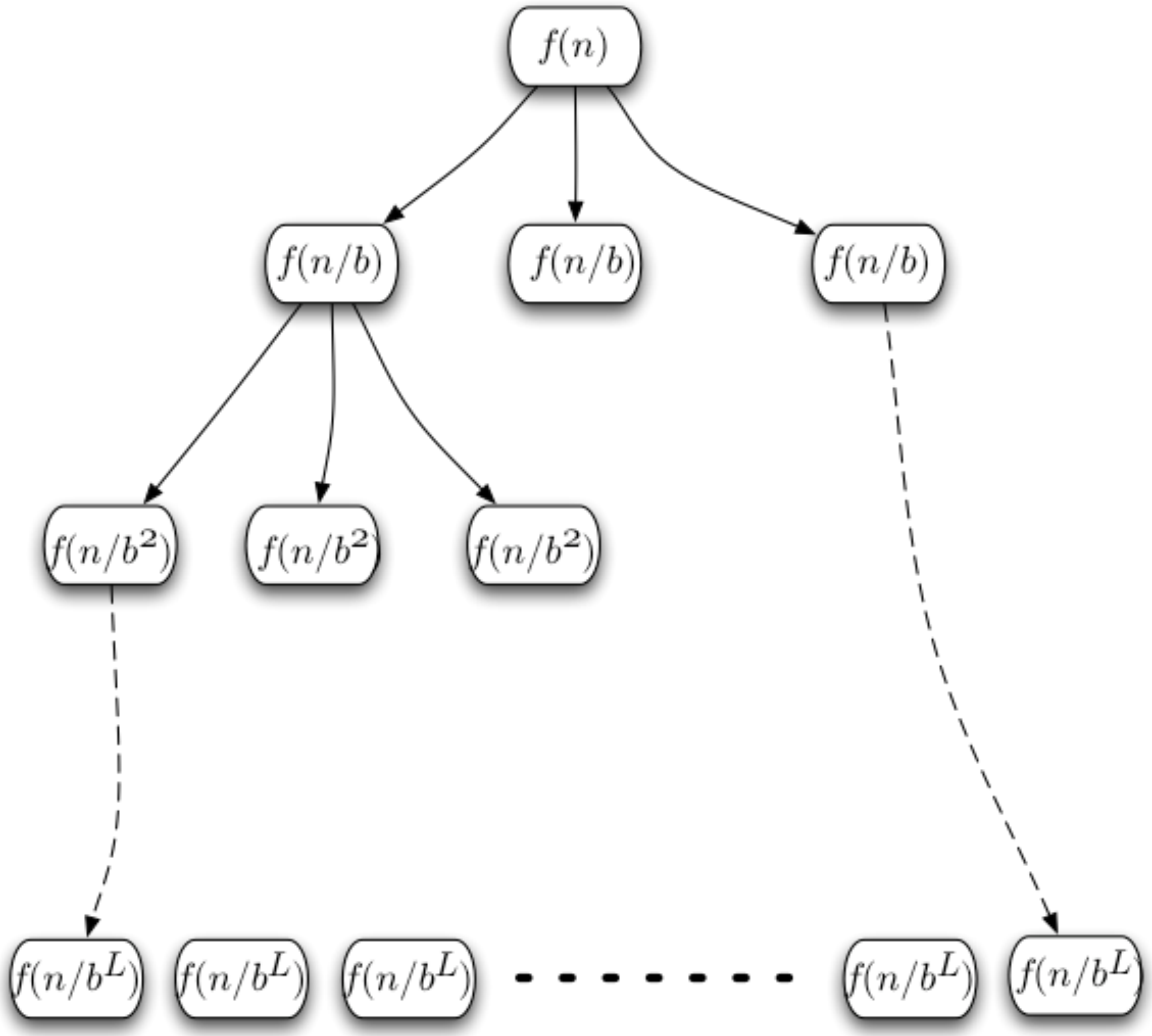
JEAN-GEORGES
JEAN-GEORGES
VONGERICHTEN



elBulli2003
elBulli2004

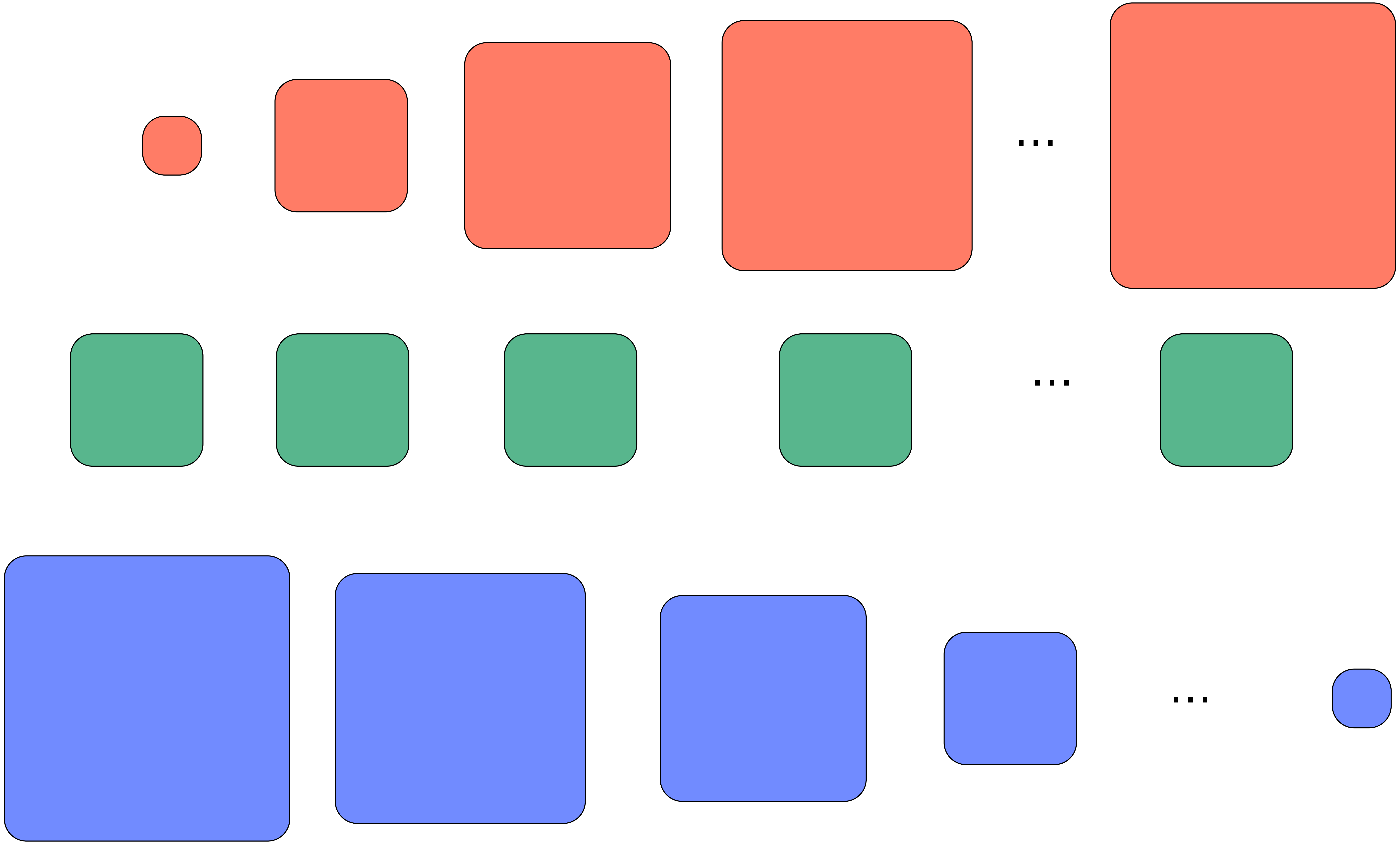
$$T(n) = aT(n/b) + f(n)$$

$$T(n) = aT(n/b) + f(n)$$



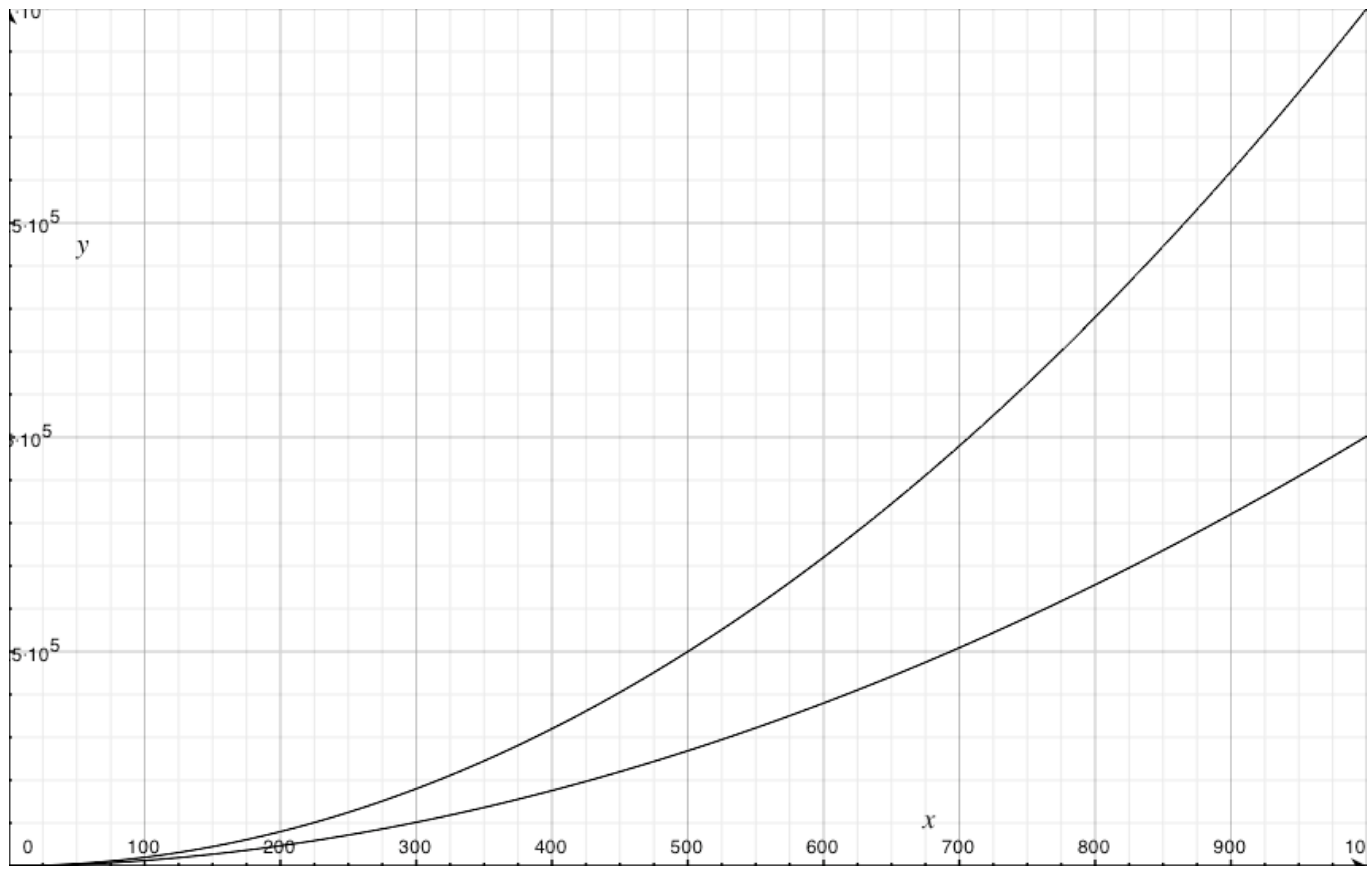
$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$



$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1: $f(n) = O(n^{\log_b a - \epsilon})$



y
x
0
100
200
300
400
500
600
700
800
900

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1: $f(n) = O(n^{\log_b a - \epsilon})$

example: $T(n) = 4T(n/2) + n$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 1: $f(n) = O(n^{\log_b a - \epsilon})$

$$1 + b + b^2 + \cdots + b^L =$$

$$1 + b + b^2 + \cdots + b^L =$$

a^L

$b^{\epsilon L}$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

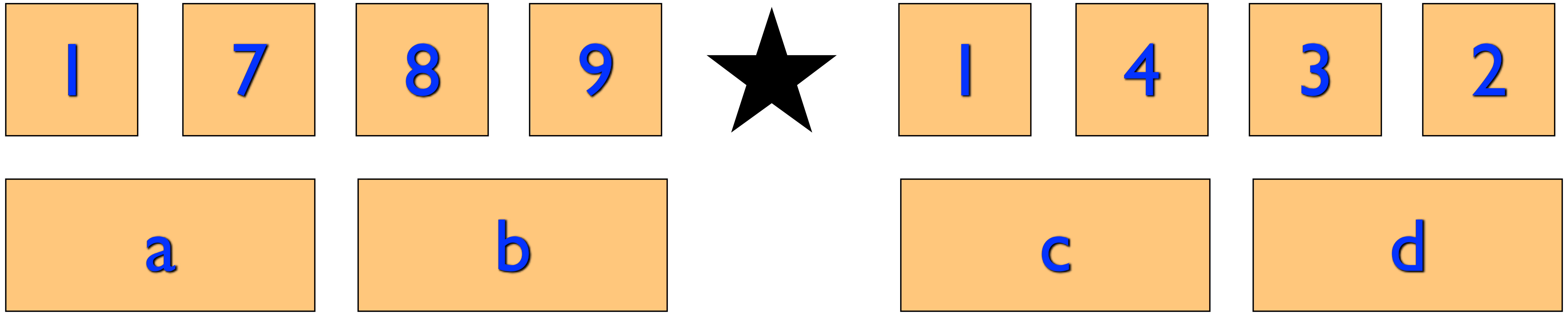
case 1 (cont):

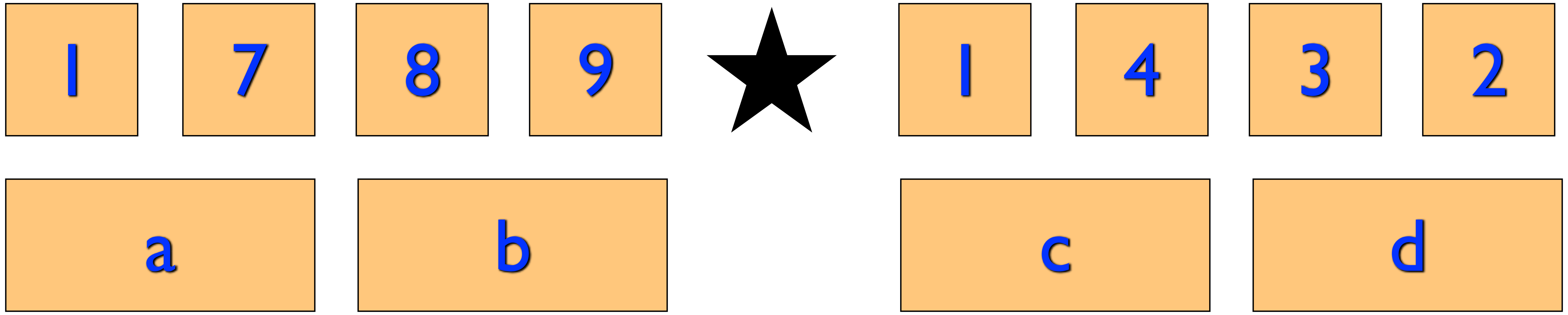
$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

case 2: $f \in \Theta(n^{\lg_b a})$

case 3: $f \in \Omega(n^{\lg_b a} + \epsilon)$ and...

example 2: $T(n) = 8T(n/2) + \Theta(n^2)$


1
7
8
9
1
4
3
2
a
b
c
d



$$T(n) = 4T(n/2) + 3O(n)$$

example 2:

$$T(n) = T\left(\frac{14}{17}n\right) + 24$$

$$T(n) = 2T(n/2) + n^3$$

$$T(n) = 16T(n/4) + n^2$$


8
13
FIFA

$$T(n) = 2T\left(\sqrt{n}\right) + \lg n$$

leo of pisa (1170-1250) aka fib

rule of rabbits

1 month to mature

once mature, have 2 children each month
(ad nauseam)

R_n :

n-2

n-1

n

Objective: Solve $R(n) = R(n-1) + R(n-2)$

$$A(x) = R_0 + R_1 x + R_2 x^2 + R_3 x^3 + \cdots$$

$$A(x) = \frac{x}{1 - x - x^2}$$

method of partial fractions

$$(1 - x - x^2) =$$

$$A(x) = \frac{x}{(1-\phi x)(1-\hat{\phi} x)}$$

$$A(x) = \frac{A_1}{(1-\phi x)} + \frac{A_2}{(1-\hat{\phi} x)}$$

$$A(x) = \frac{1}{\sqrt{5}} \left[\frac{1}{(1 - \phi x)} - \frac{1}{(1 - \hat{\phi} x)} \right]$$

$$\frac{1}{1 - ax}$$

$$1 - ax \overline{\big) \ 1 \qquad\qquad\qquad\qquad\qquad\qquad}$$

$$A(x) = \frac{1}{\sqrt{5}} \left[\frac{1}{(1 - \phi x)} - \frac{1}{(1 - \hat{\phi} x)} \right]$$

$$R_i = \left(\frac{1}{\sqrt{5}}\right)\left(\phi^i - \hat{\phi}^i\right)$$

Review of generating functions method: