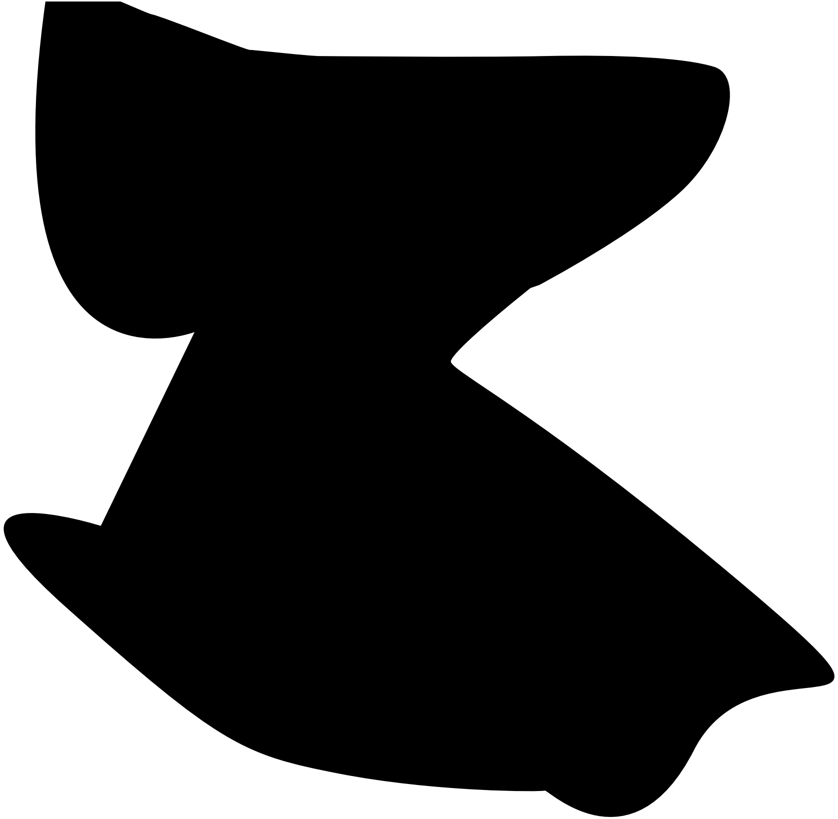
L5 5800

feb 1/3 2022

shelat

divide

& conquer



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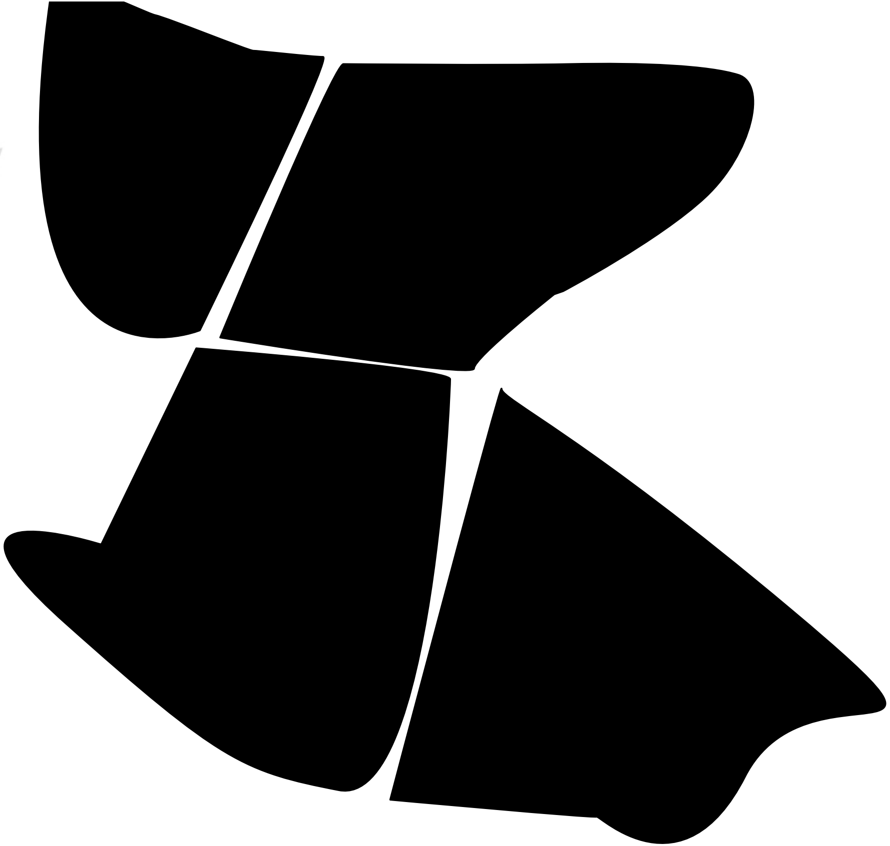

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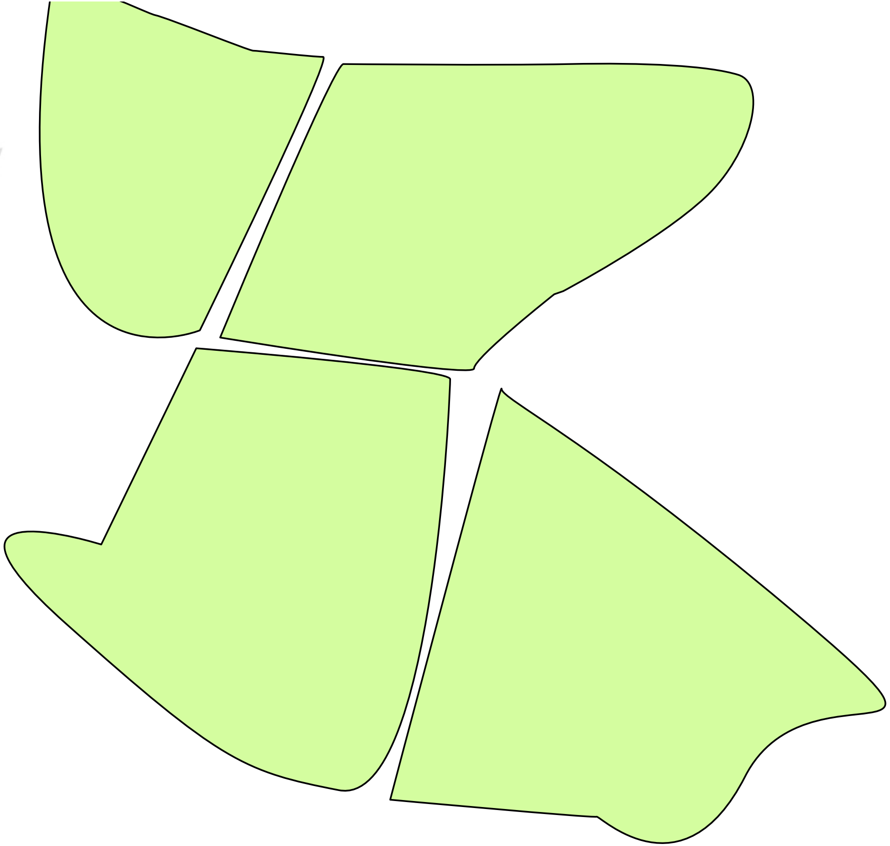
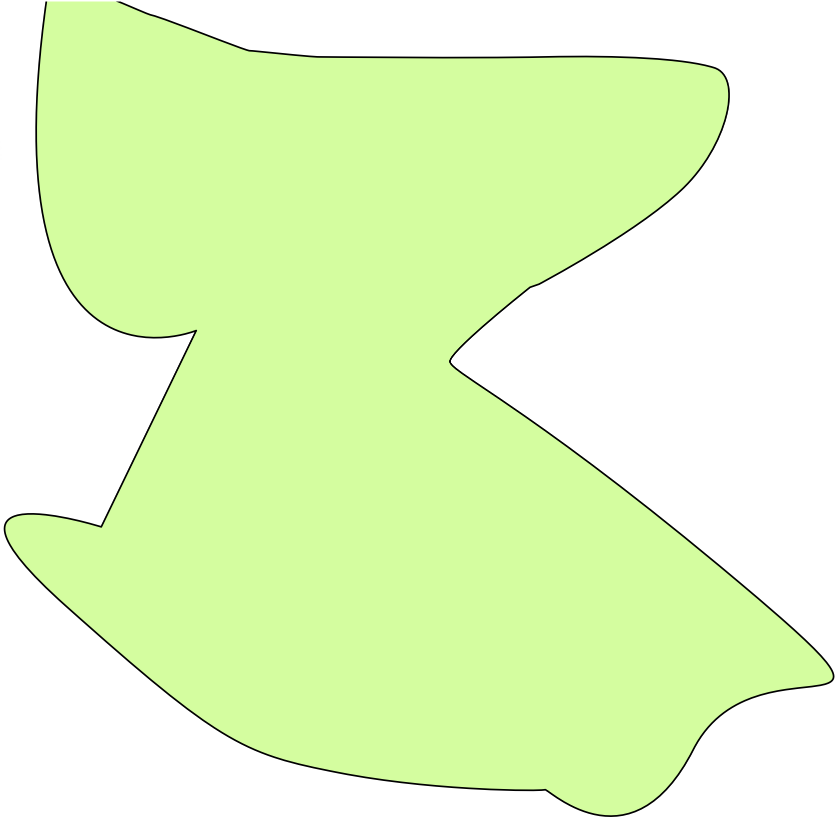


http://www.kitchenknifedrawer.com/files/1696205/uploaded/K6615D.jpg

2nd insight:

how to recombine.



http://www.kitchenknifedrawer.com/files/1696205/uploaded/K6615D.jpg

Examples we will discuss

- Mergesort
- Arbitrage
- Closest pair of points
- Median
- Matrix mult
- Fast Fourier Transform

Merge

merge-sort (A, p, r)
    if $p < r$
        $q \leftarrow \lfloor (p+r)/2 \rfloor$
        merge-sort (A, p, q)
        merge-sort $(A, q+1, r)$
        merge(A, p, q, r)

MERGE$(A[1..n], m)$:
    $i \leftarrow 1$; $j \leftarrow m+1$
    for $k \leftarrow 1$ to n
        if $j > n$
            $B[k] \leftarrow A[i]$; $i \leftarrow i+1$
        else if $i > m$
            $B[k] \leftarrow A[j]$; $j \leftarrow j+1$
        else if $A[i] < A[j]$
            $B[k] \leftarrow A[i]$; $i \leftarrow i+1$
        else
            $B[k] \leftarrow A[j]$; $j \leftarrow j+1$
    for $k \leftarrow 1$ to n
        $A[k] \leftarrow B[k]$

jeff erickson

5	2	4	7	1	3	2	6

merge-sort (A, p, r)
if $p < r$
$q \leftarrow \lfloor (p+r)/2 \rfloor$
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MERGE$(A[1\,..\,n], m)$:
$i \leftarrow 1;\ j \leftarrow m+1$
for $k \leftarrow 1$ to n
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$B[k] \leftarrow A[i];\ i \leftarrow i+1$
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else
$B[k] \leftarrow A[j];\ j \leftarrow j+1$
for $k \leftarrow 1$ to n
$A[k] \leftarrow B[k]$

jeff erickson

5	2	4	7	1	3	2	6

5	2	4	7

1	3	2	6

merge-sort (A, p, r)
    if $p < r$
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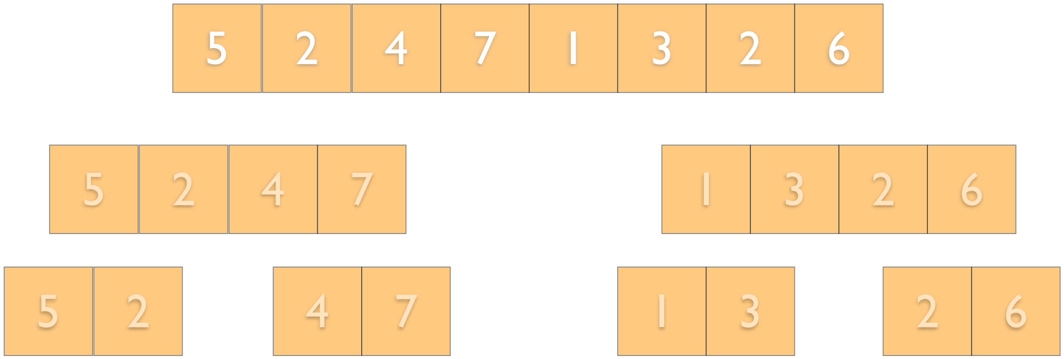
        merge(A, p, q, r)

$\underline{\textsc{Merge}(A[1\,..\,n], m):}$
  $i \leftarrow 1;\ \ j \leftarrow m + 1$
  for $k \leftarrow 1$ to n
    if $j > n$
      $B[k] \leftarrow A[i];\ \ i \leftarrow i + 1$
    else if $i > m$
      $B[k] \leftarrow A[j];\ \ j \leftarrow j + 1$
    else if $A[i] < A[j]$
      $B[k] \leftarrow A[i];\ \ i \leftarrow i + 1$
    else
      $B[k] \leftarrow A[j];\ \ j \leftarrow j + 1$
  for $k \leftarrow 1$ to n
    $A[k] \leftarrow B[k]$

jeff erickson



merge-sort (A, p, r)

if $p < r$

$q \leftarrow \lfloor (p+r)/2 \rfloor$

merge-sort (A, p, q)

merge-sort $(A, q+1, r)$

merge(A, p, q, r)

MERGE$(A[1..n], m)$:

$i \leftarrow 1$; $j \leftarrow m+1$

for $k \leftarrow 1$ to n

  if $j > n$

    $B[k] \leftarrow A[i]$; $i \leftarrow i+1$

  else if $i > m$

    $B[k] \leftarrow A[j]$; $j \leftarrow j+1$

  else if $A[i] < A[j]$

    $B[k] \leftarrow A[i]$; $i \leftarrow i+1$

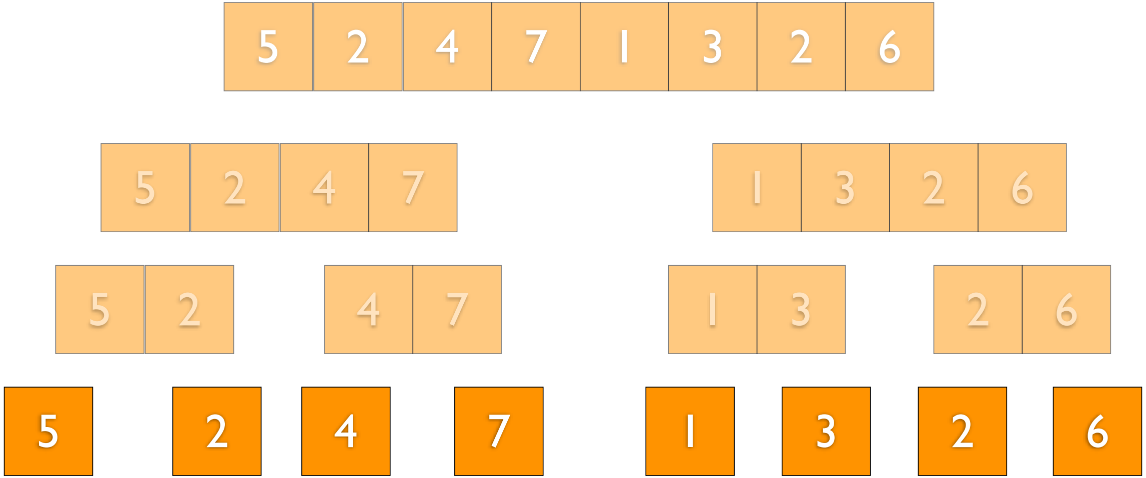
  else

    $B[k] \leftarrow A[j]$; $j \leftarrow j+1$

for $k \leftarrow 1$ to n

  $A[k] \leftarrow B[k]$

jeff erickson



merge-sort (A, p, r)
  if $p < r$
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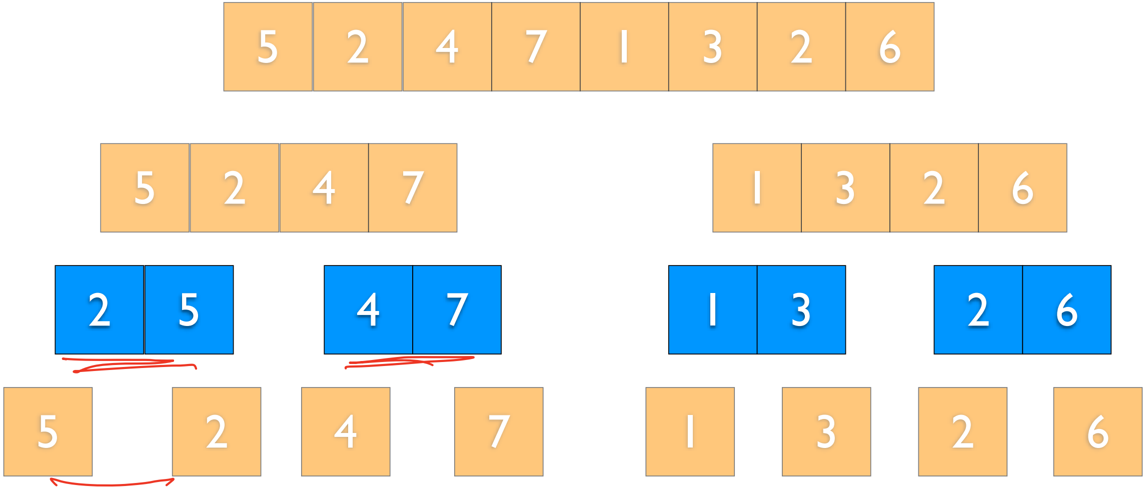
    merge-sort (A, p, q)
    merge-sort $(A, q+1, r)$
    merge(A, p, q, r)



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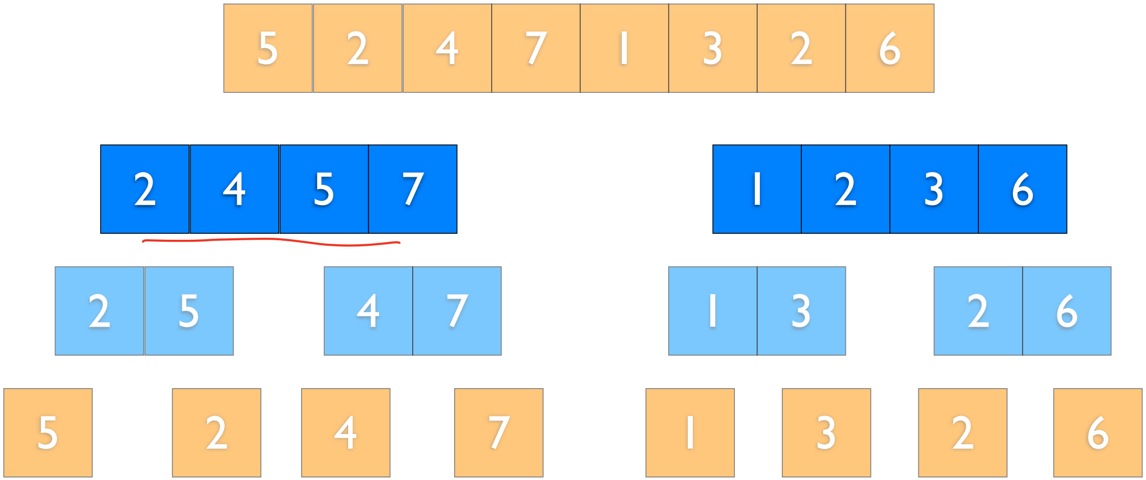

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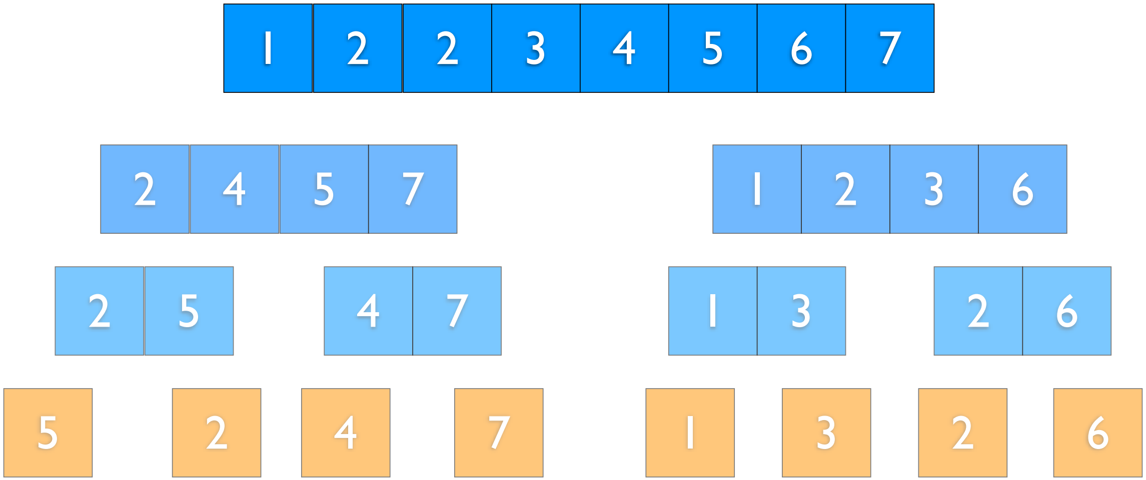

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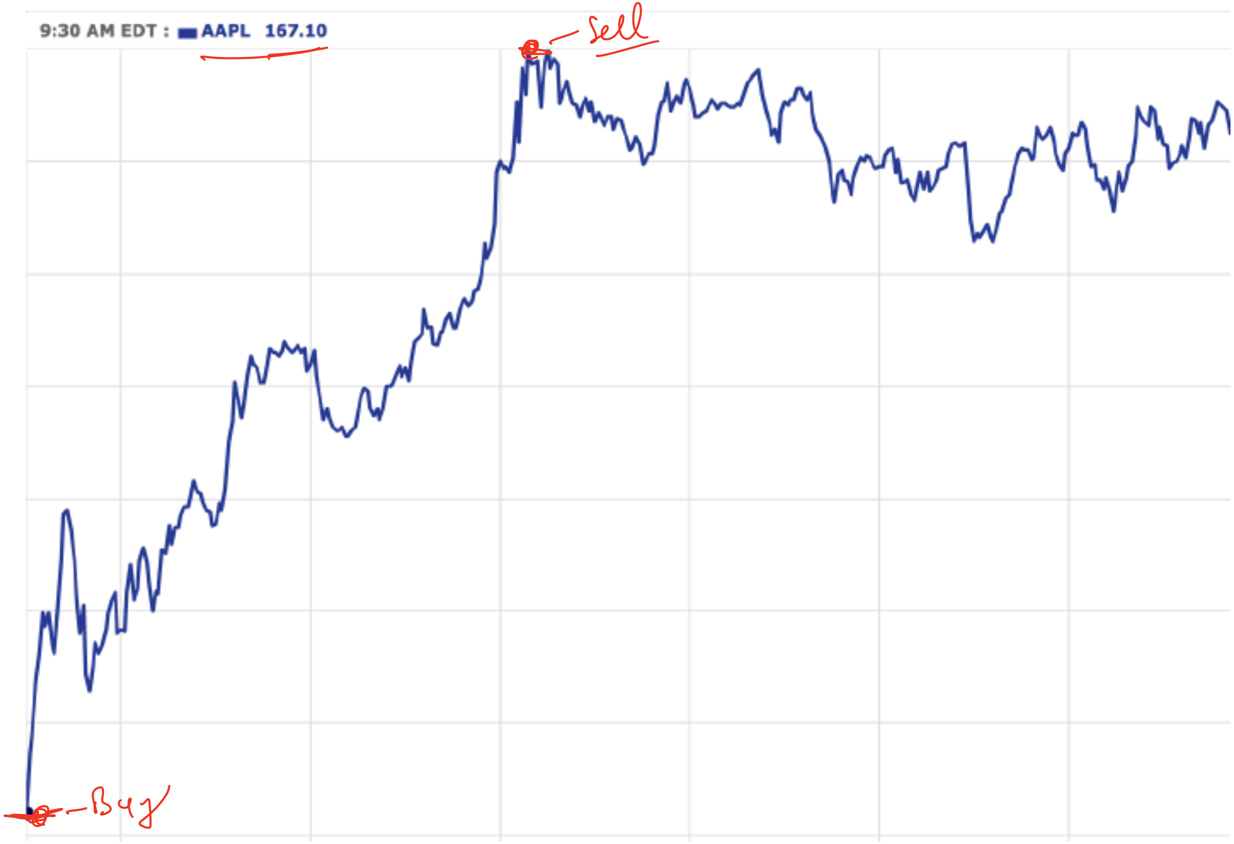
    if $p < r$
        $q \leftarrow \lfloor (p+r)/2 \rfloor$
        merge-sort (A, p, q)
        merge-sort $(A, q+1, r)$
        merge(A, p, q, r)

$\Theta(n)$

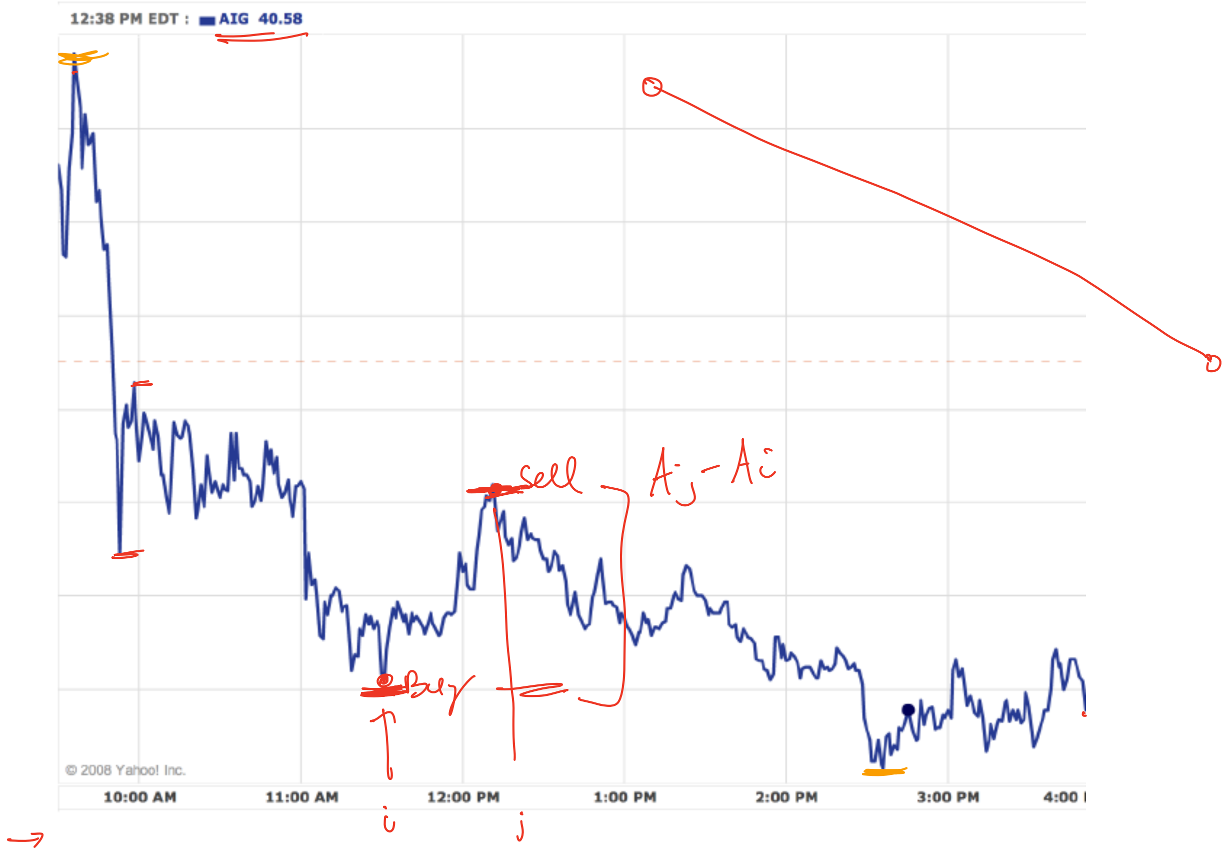
$$T(n) = 2T(n/2) + \Theta(n)$$

$$= \Theta(n \log n)$$

arbitrage

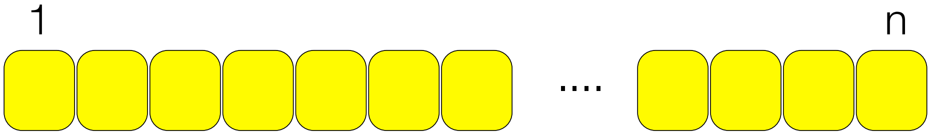

9:30 AM EDT : AAPL 167.10
Sell
Buy

Open 27.46
Close 27.07
Low 26.65
High 27.69
Vol 33.79K
% Chg -75.68%
BPT 27.07
YAHOO! FINANCE



12:38 PM EDT : AIG 40.58
© 2008 Yahoo! Inc.
10:00 AM
11:00 AM
12:00 PM
1:00 PM
2:00 PM
3:00 PM
4:00
sell
Buy
$A_j - A_i$
i
j

input: array of n numbers



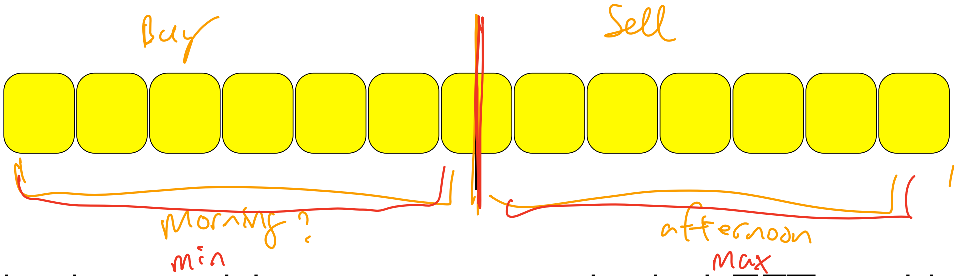
goal: is to find the best buy-sell opportunity.

i.e. the pair (i, j) such that $i < j$

and $A_j - A_i$ is the largest among

all pairs $(i, j) \in [1, n]^2$

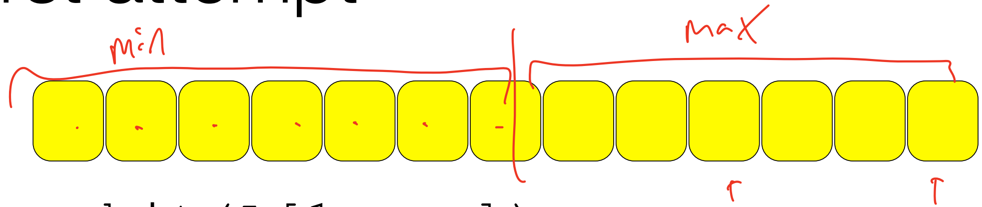
Main idea



Find the best arbitrage opportunity in LEFT and in RIGHT.

Then look for opportunities when you buy on the left and sell on the right.

first attempt




```
arbit(A[1...n])
```

if $|A| = 2$, then base case

else

$lg \leftarrow$ arbit(A[1...n/2])

$rg \leftarrow$ arbit(A[n/2+1,...n])

min, max $\leftarrow$ MIN(left(A)), MAX(right(A))

return max$\{lg, rg, \text{max} - \text{min}\}$

first attempt

```
arbit(A[1...n])
    base case if |A|<=2
    lg = arbit(left(A))
    rg = arbit(right(A))
    minl = min(left(A))
    maxr = max(right(A))
    return max{maxr-minl,lg,rg}
```

min = ∞
for i=1, n/2
if A[i] < min,
min = A[i]

return $(A_2 - A_1)$.

$T(n/2)$

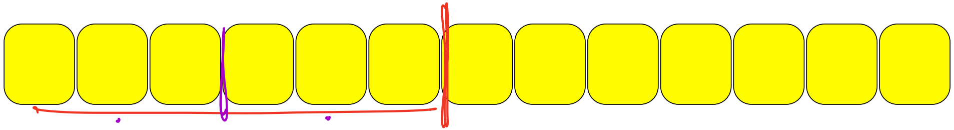
$T(n/2)$

$\Theta(n)$

case 2

$T(n) = 2T(\frac{n}{2}) + \Theta(n) \Longrightarrow \Theta(n \log n)$

first attempt: time $\underline{\Theta(n \log n)}$



```
arbit(A[1...n])
    base case if |A|<=2                  1
    lg = arbit(left(A))                  T(n/2)
    rg = arbit(right(A))                 T(n/2)
    minl = min(left(A))
    maxr = max(right(A))                 which takes Θ(n)
    return max{maxr-minl,lg,rg}          1
```

better approach

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

can this be reduced ?

better approach

Can we find a solution that has $T(n) = 2T(n/2) + O(1)$?

if we could, then the solution would be

case 1

$$\Theta(n^{\log_2 2}) = \Theta(n)$$

better approach

Can we find a solution that has T(n) = 2T(n/2) + O(1) ?

```
minl = min(left(A))
maxr = max(right(A))

return max{maxr-minl,lg,rg}
```

→ Have arbit function ALSO return min, max of the array

second attempt

```
arbit+(A[1...n])
   base case if |A|<=2
```

$lg, lmin, lmax \leftarrow arbit+(left(A))$ $\quad T(\frac{n}{2})$

$rg, rmin, rmax \leftarrow arbit+(right(A))$ $\quad T(\frac{n}{2})$

$\rightarrow$ $mid = rmax - lmin$ $\quad 1$

return $\max\{lg, rg, mid\}$, $\quad 1$

$\min\{lmin, rmin\}$, $\quad 1$

$\max\{lmax, rmax\}$ $\quad 1$

second attempt

```
arbit+(A[1...n])
    base case if |A|<=2, ...
    (lg,minl,maxl) = arbit(left(A))
    (rg,minr,maxr) = arbit(right(A))

      return max{maxr-minl,lg,rg},
             min{minl, minr},
             max{maxl, maxr}
```

if $|A|=1$, $\{0, A_1, A_1\}$

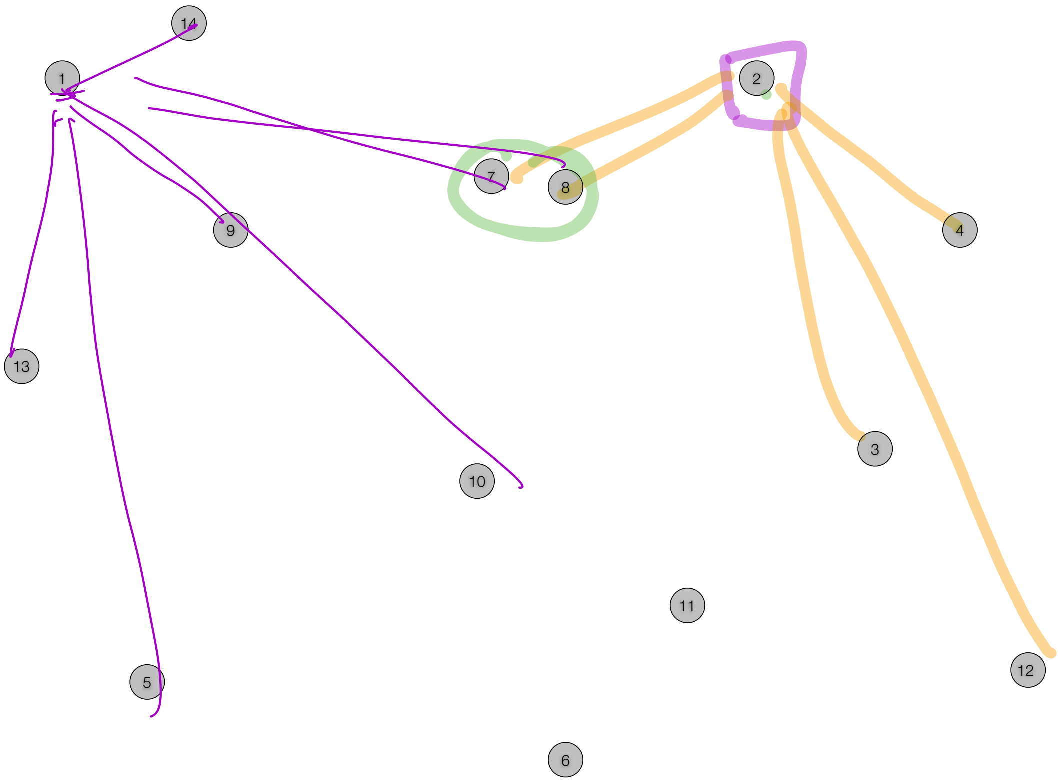
return A_2-A_1, $\min(A_1,A_2)$, $\max(A_1,A_2)$

$T(n) = 2T(\frac{n}{2}) + \Theta(1) \Rightarrow T(n) = \Theta(n)$ by case I.

closest pair

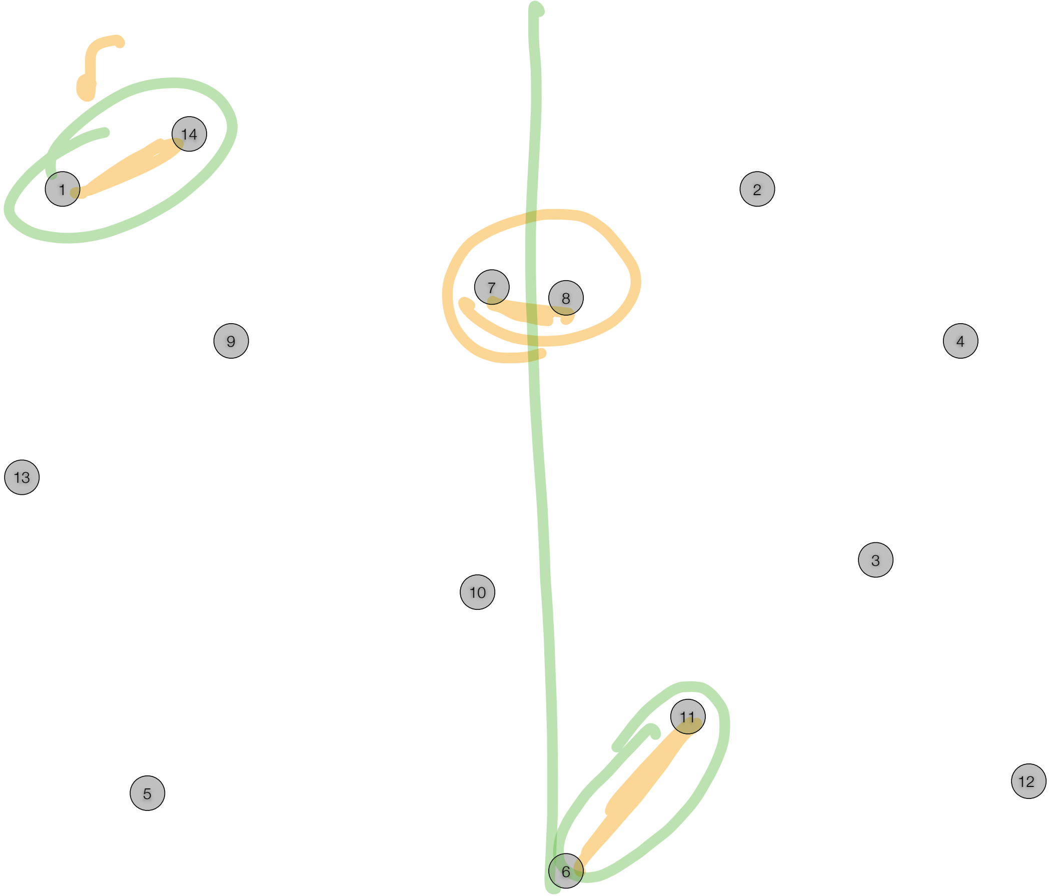
of points

Simple brute force approach takes $\Theta(n^2)$

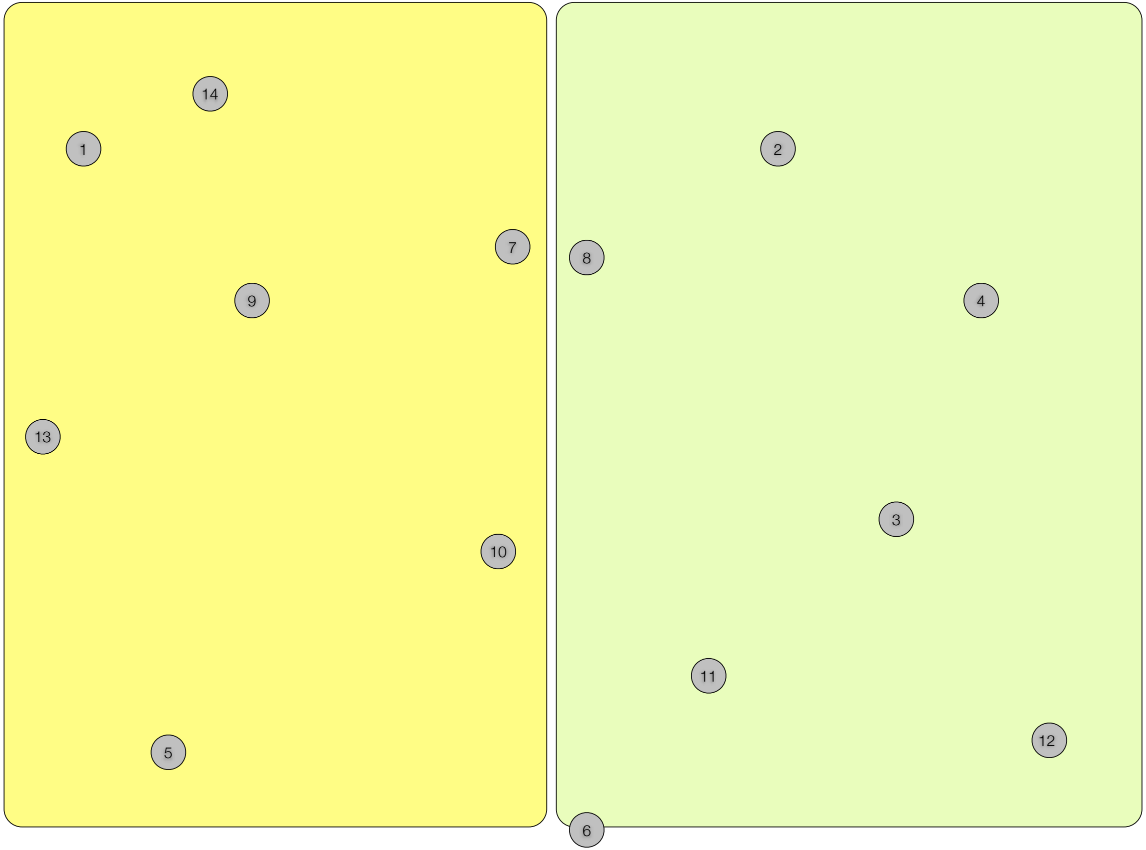


solve the large problem by

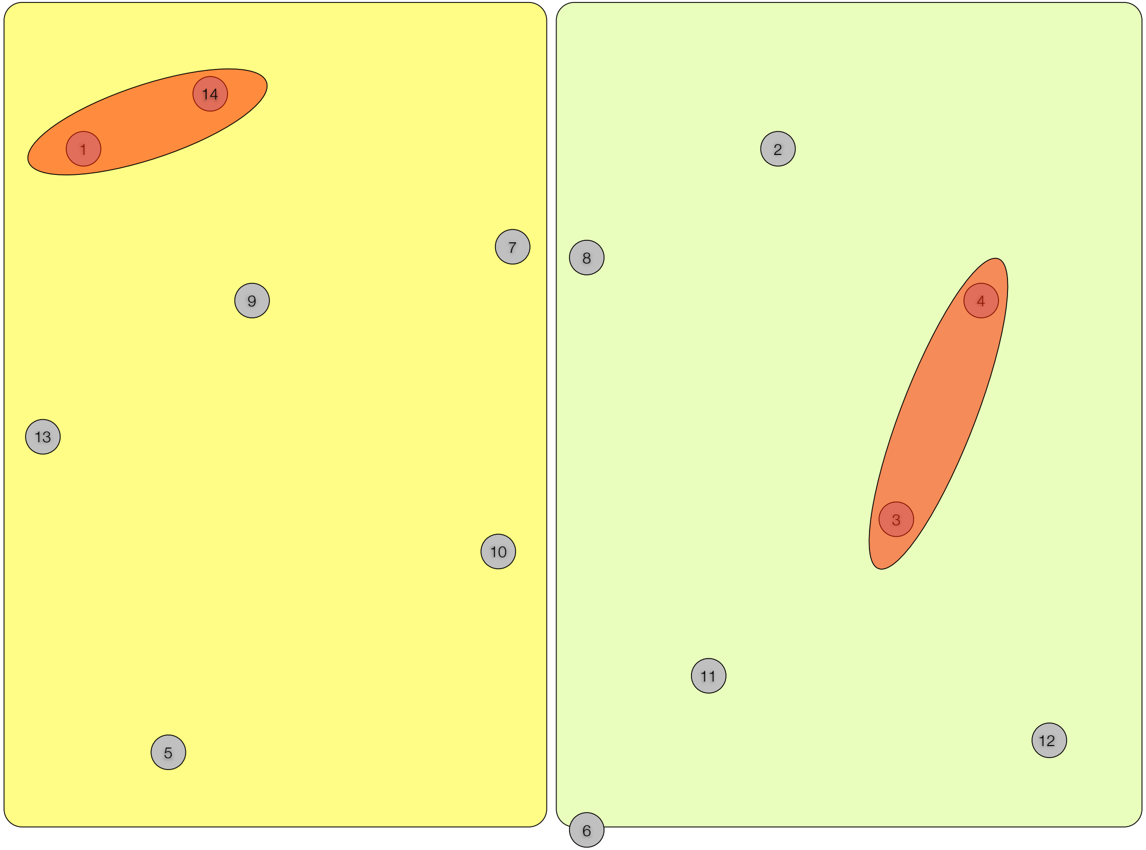
solving smaller problems
and combining solutions


14
1
2
7
8
9
4
13
3
10
11
12
5
6

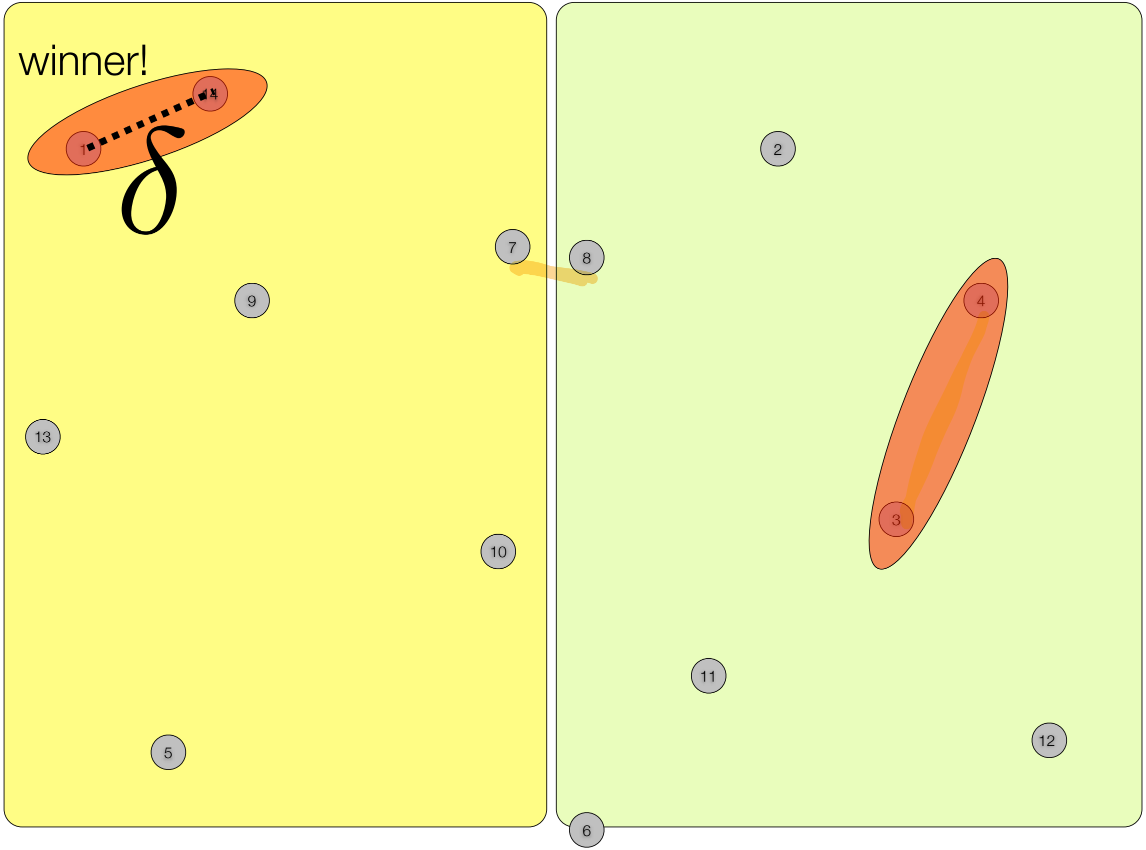
Divide & Conquer



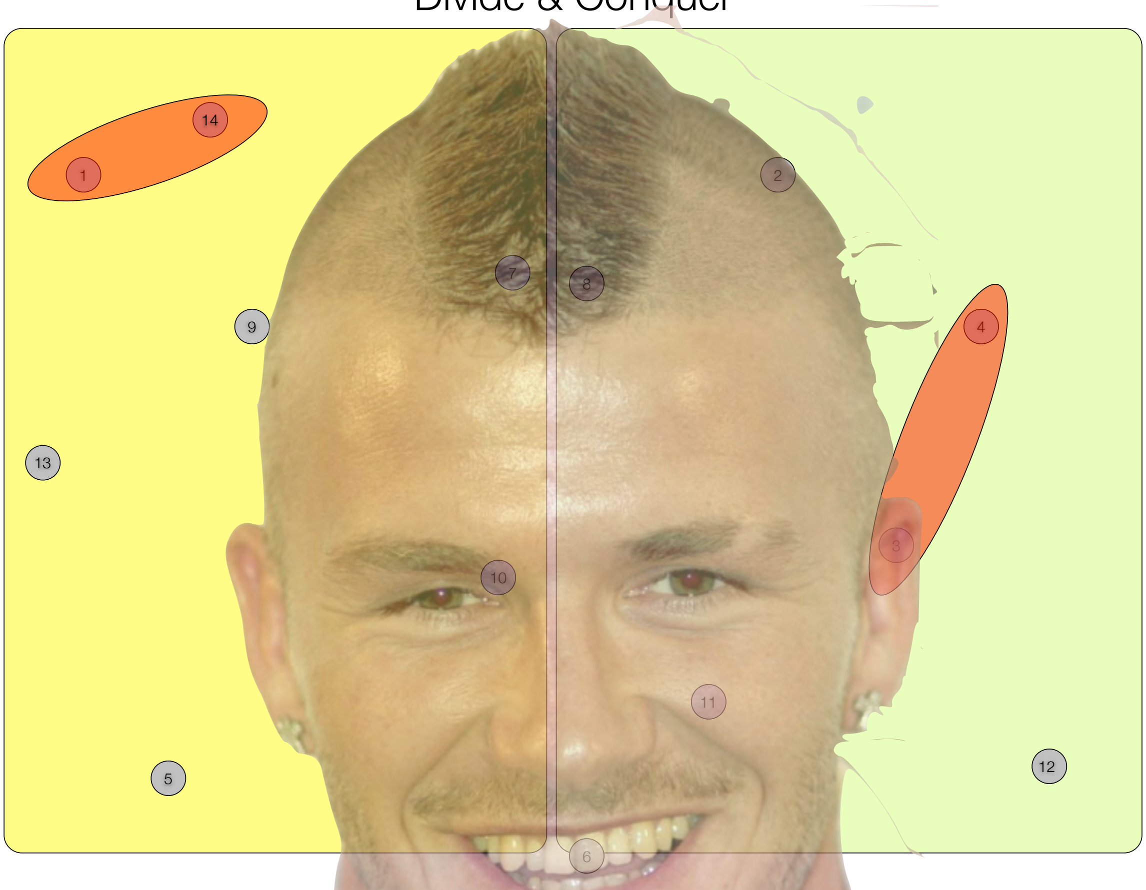
Divide & Conquer



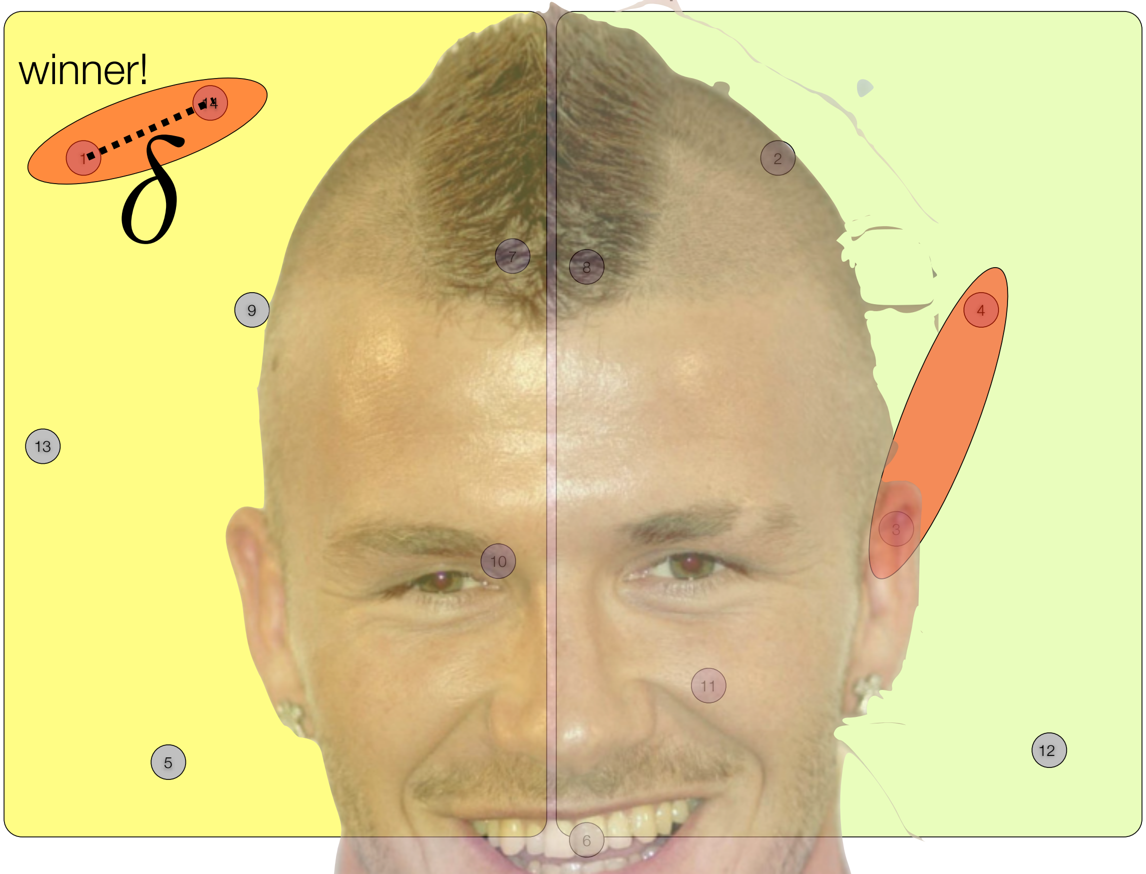
Divide & Conquer



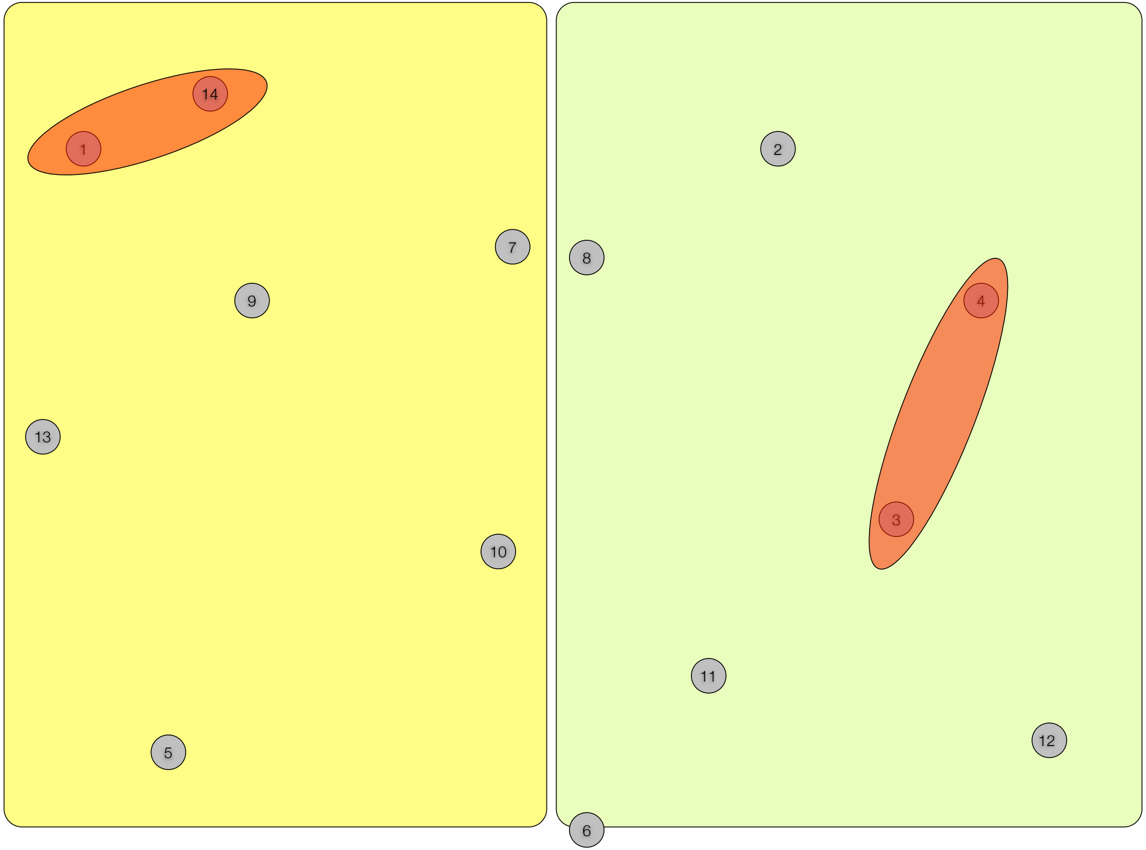
Divide & Conquer



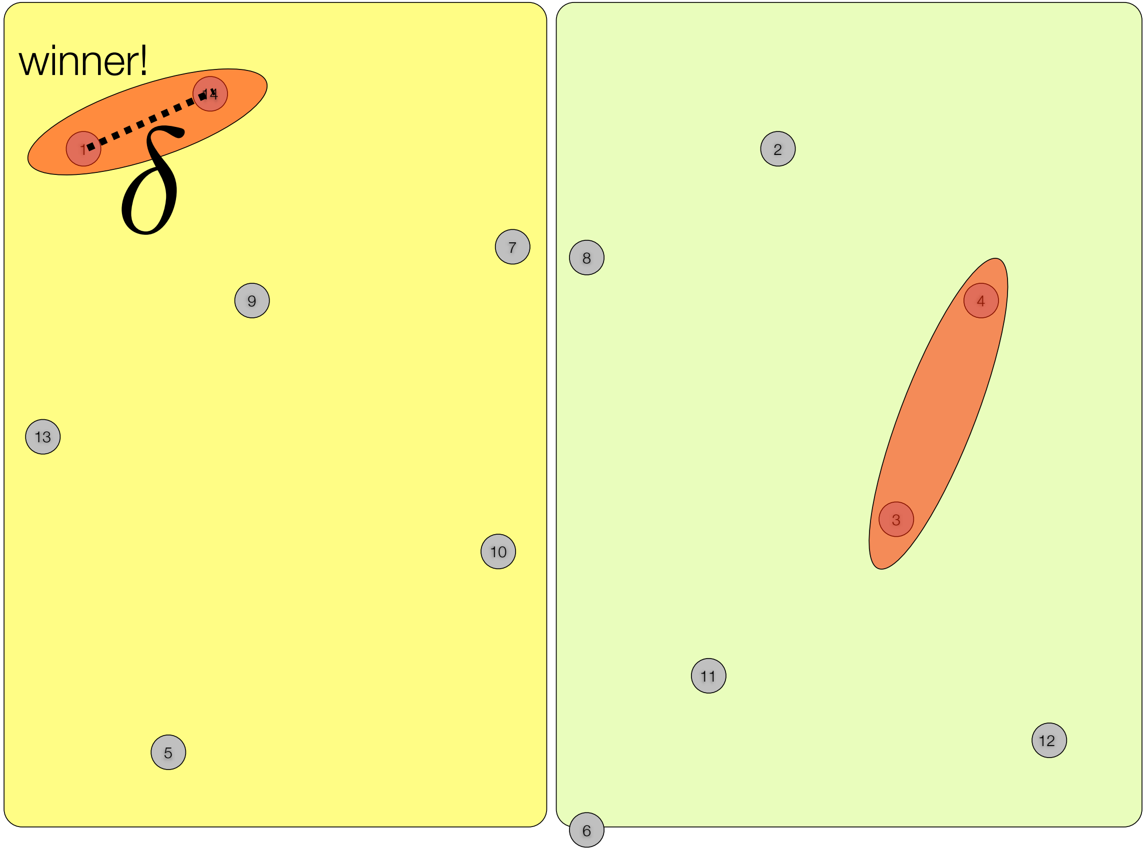
Divide & Conquer



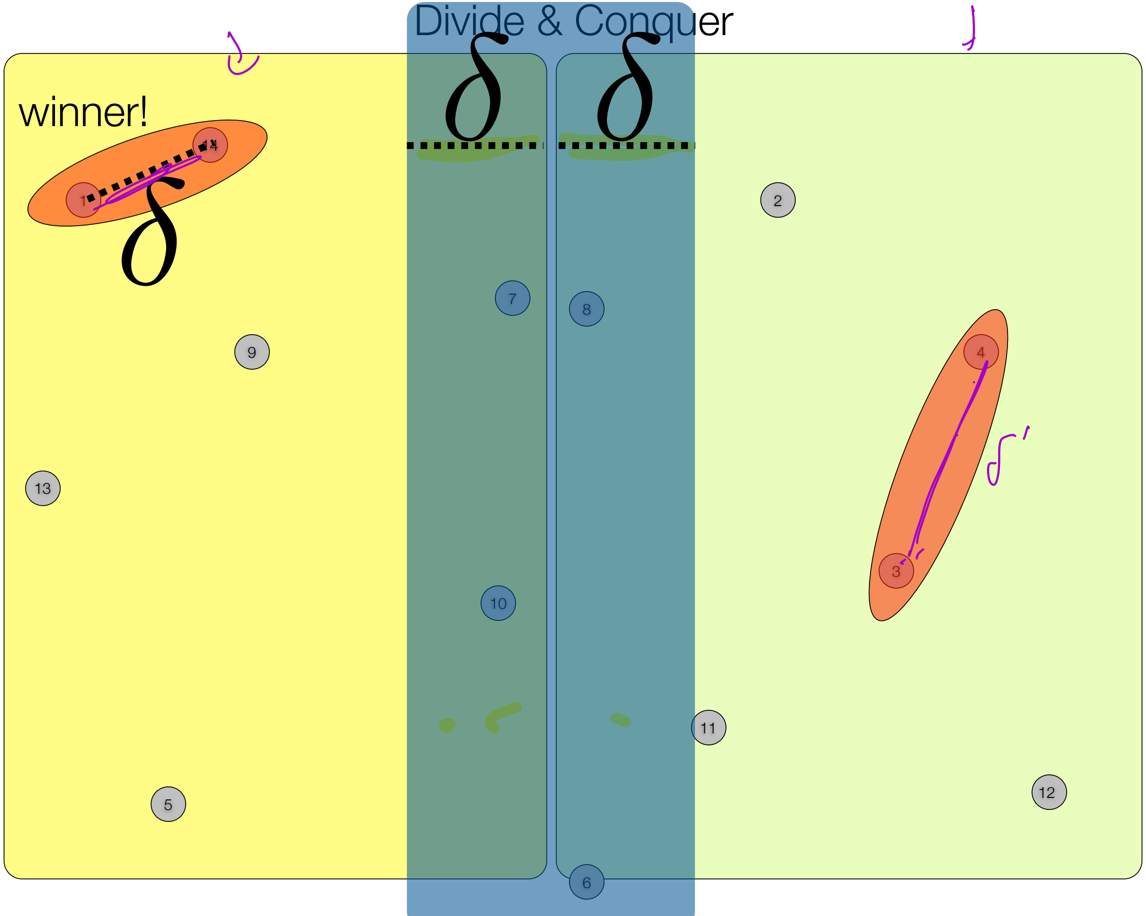
Divide & Conquer



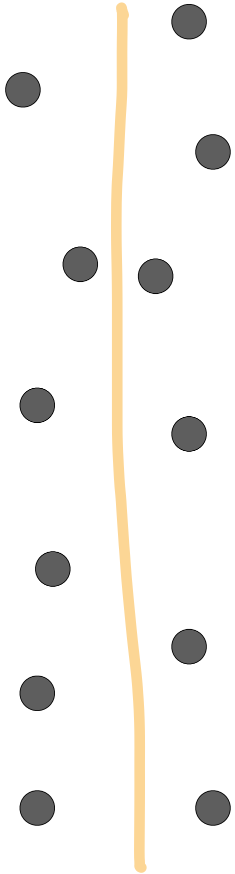
Divide & Conquer



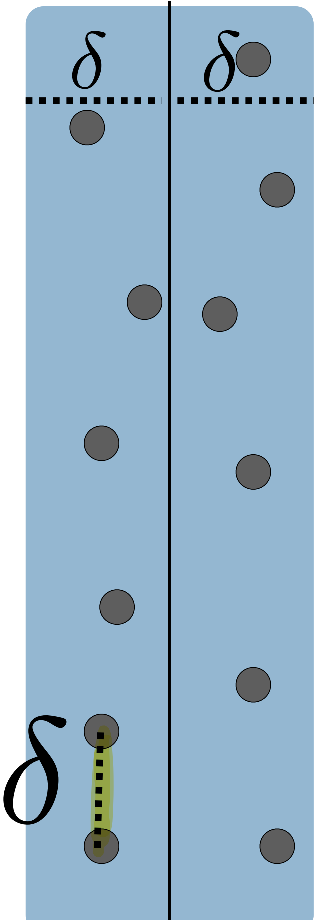
Divide & Conquer



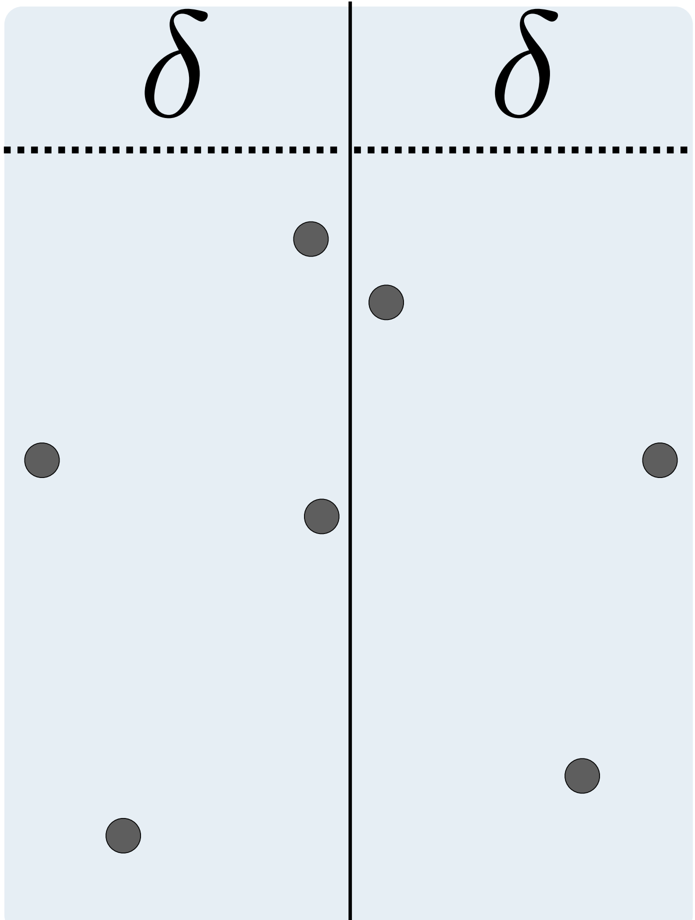
What if the input points are distributed like this?



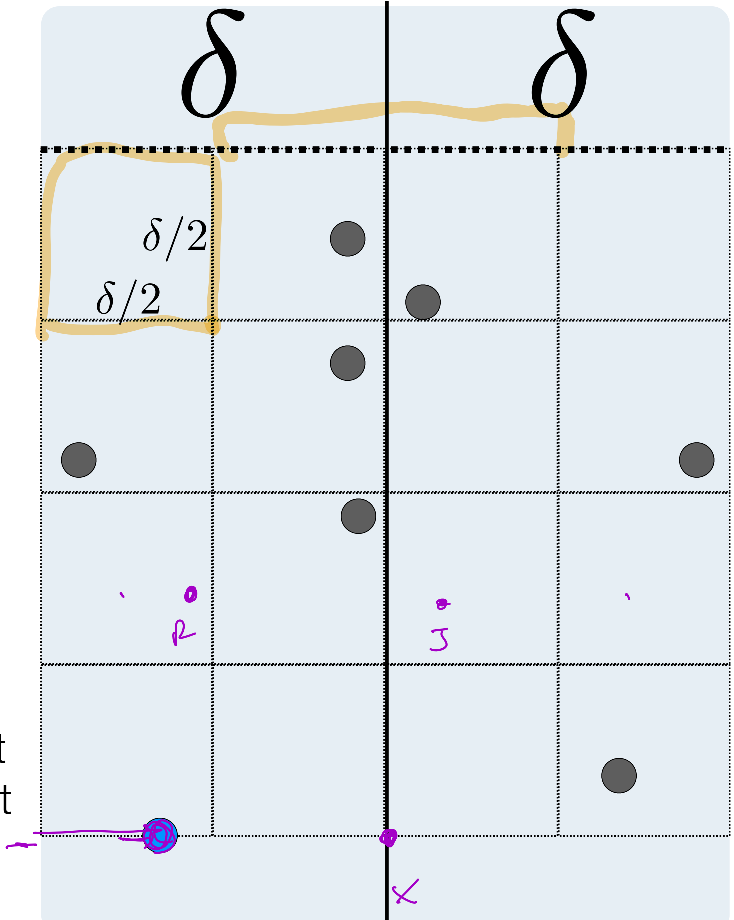
Then all of the points are within δ of the middle. If we need to check all of the points, we are back to $O(n^2)$

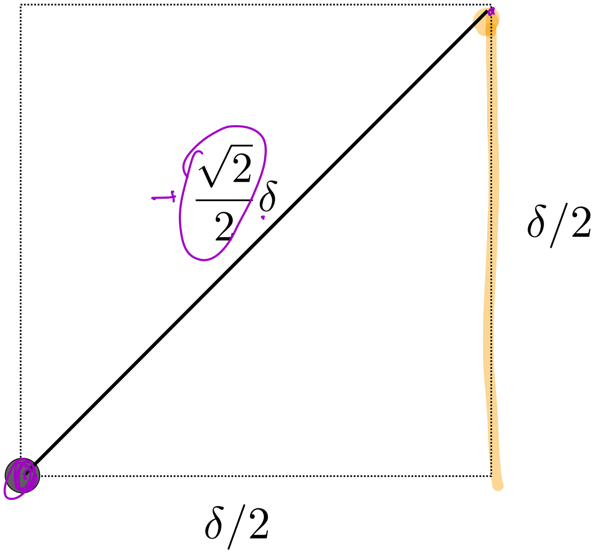


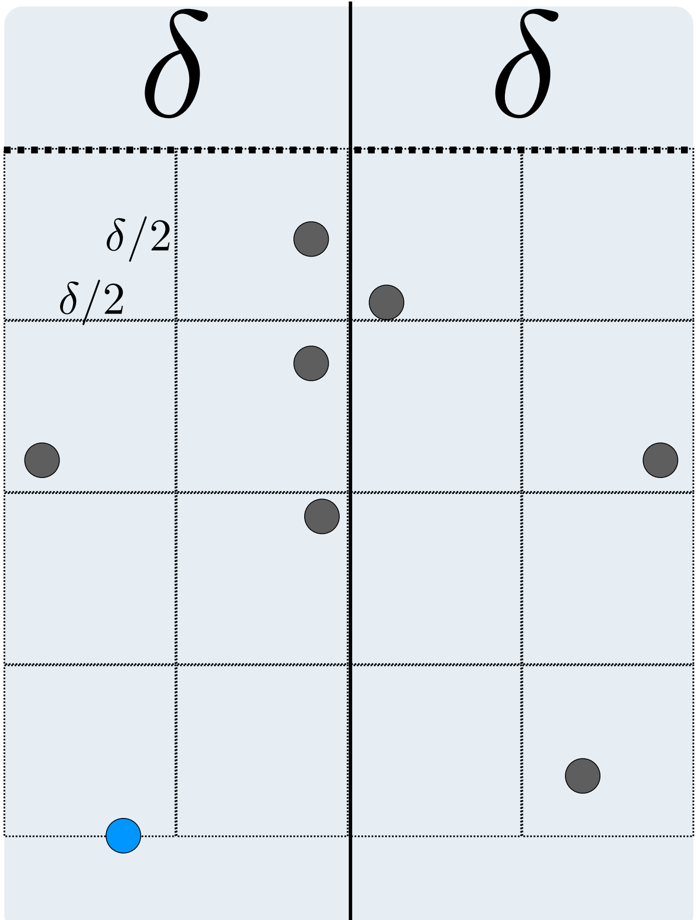
But we have extra information! The only candidates for closest pair are within δ of each other. How can we use this info?



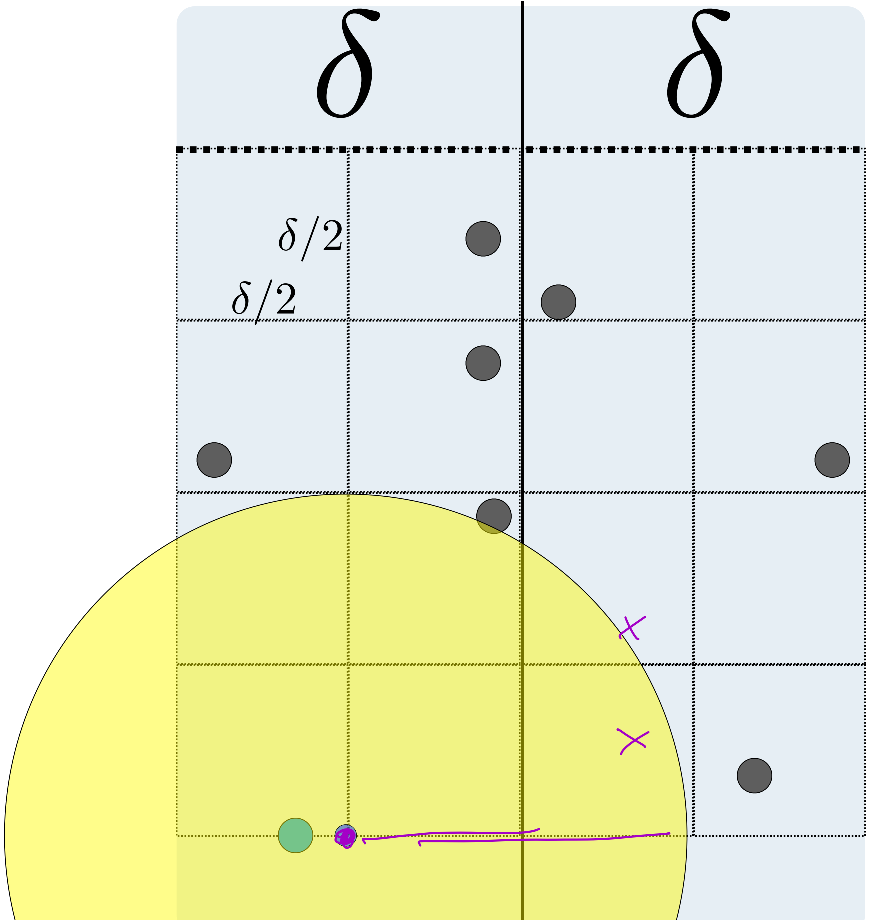
Imagine there is a grid of cubbies starting at the lowest Y point



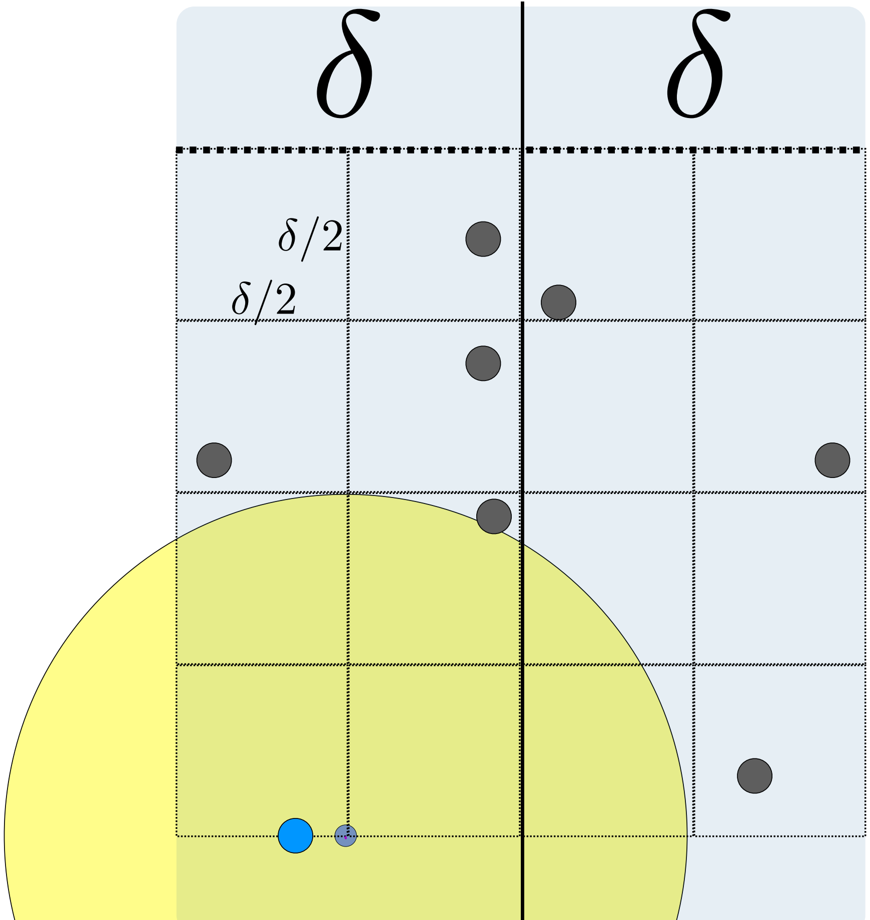

$\frac{\sqrt{2}}{2}\delta$
$\delta/2$
$\delta/2$



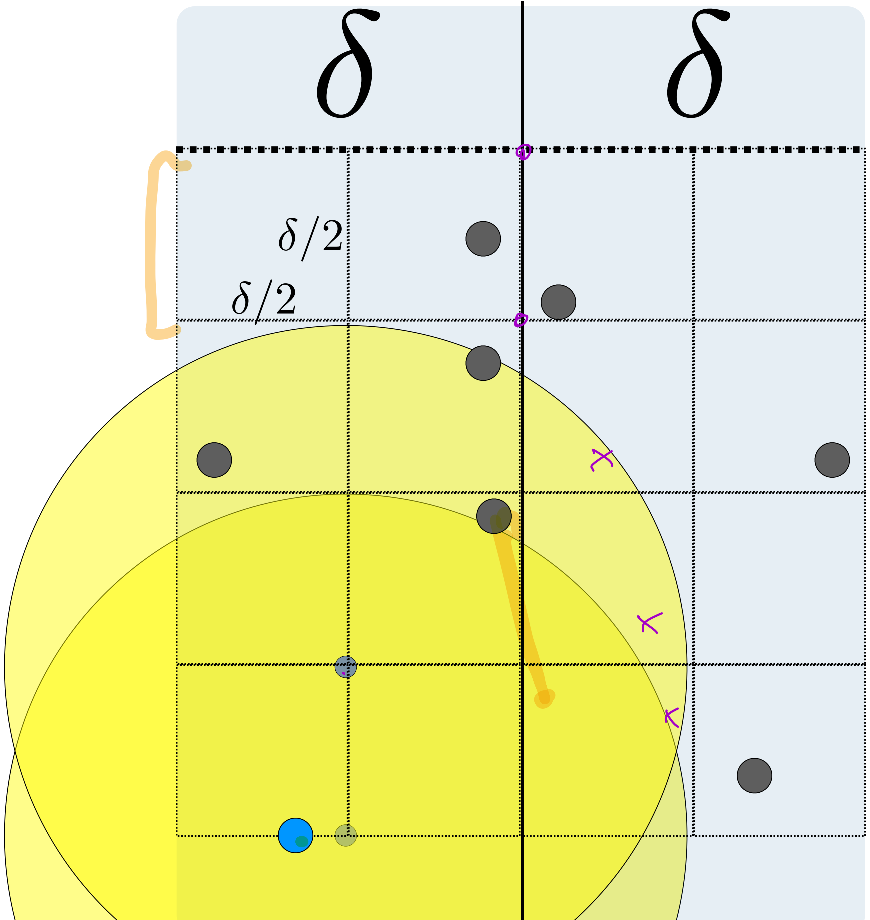
FACT: At most 1 point in each cubby



FACT: <=1 point per cubby

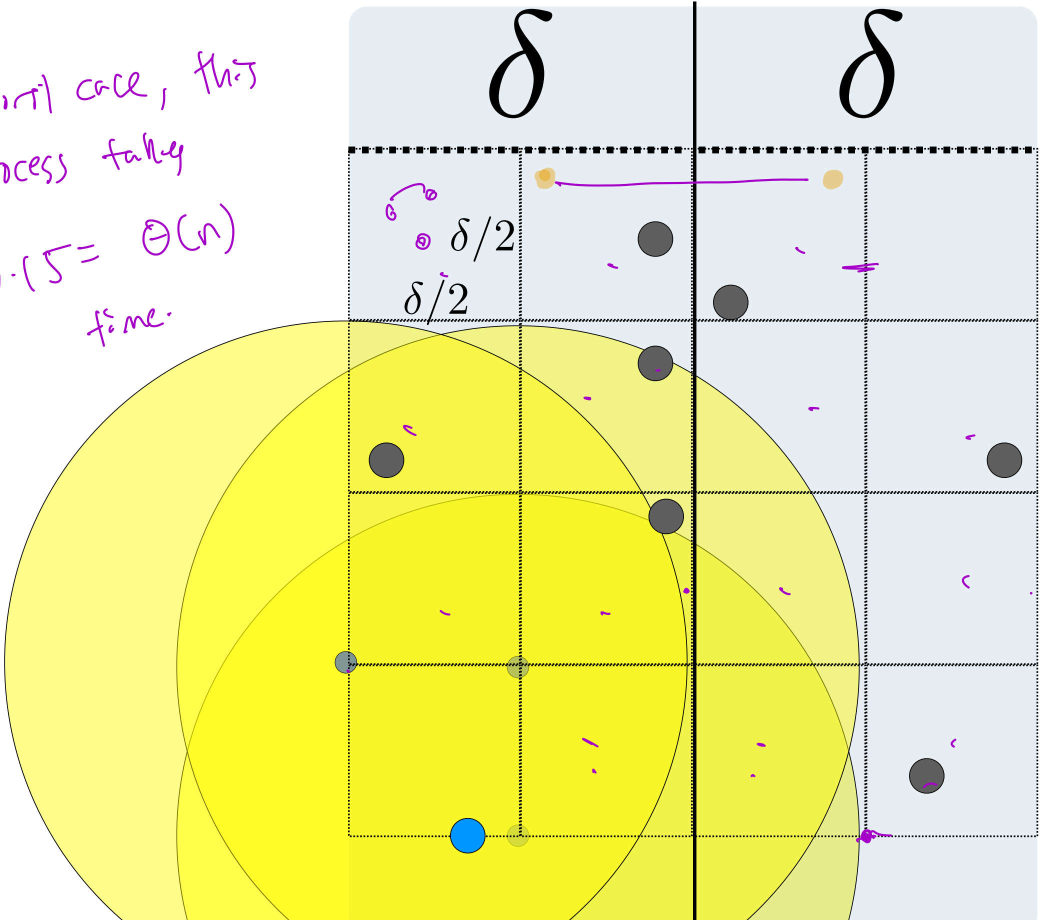


FACT: <=1 point per cubby



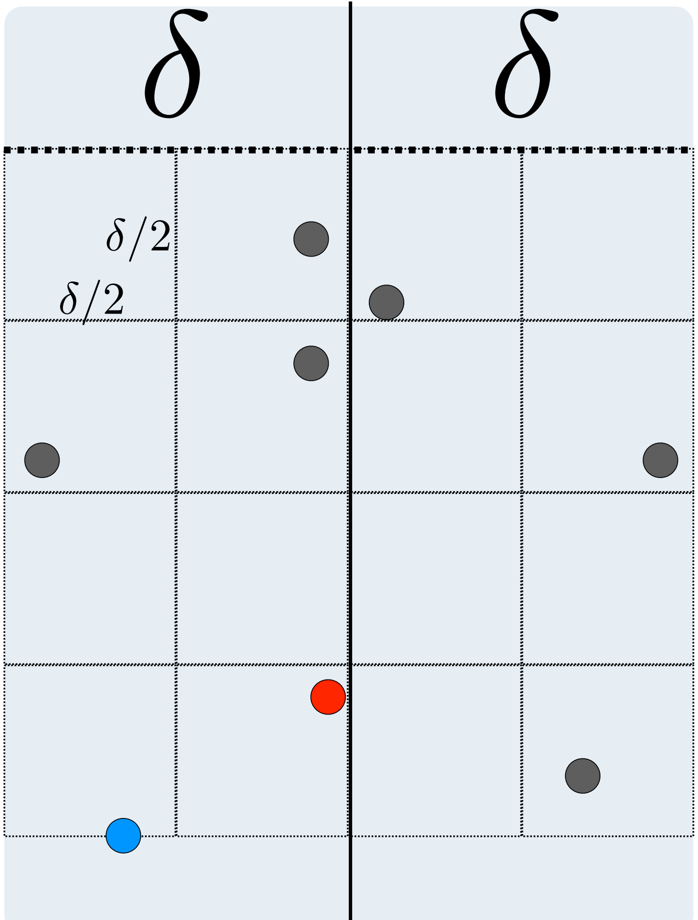
FACT: <=1 point per cubby

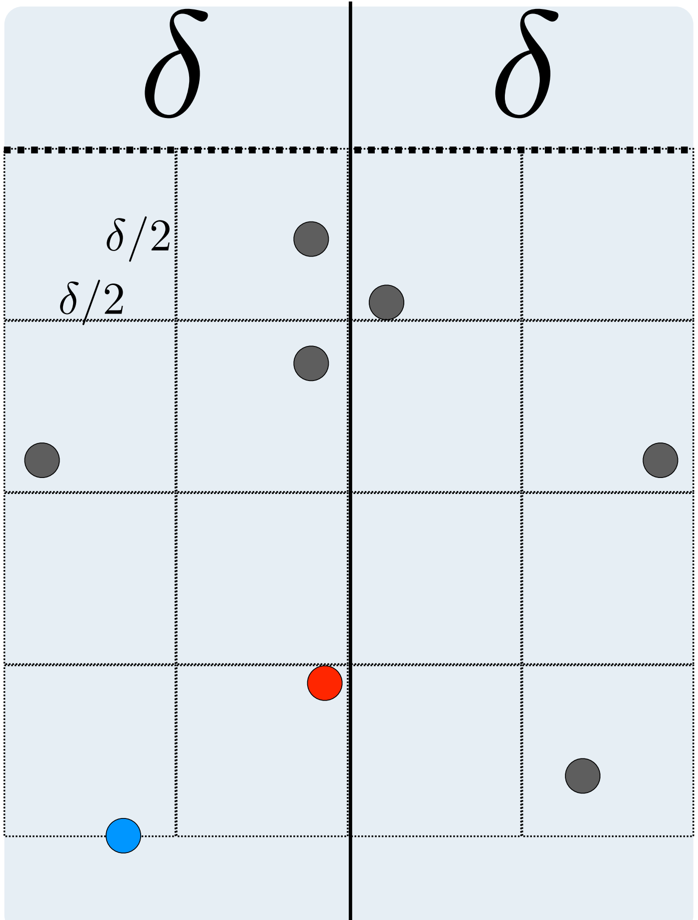
worst case, this process takes $n \cdot 15 = \Theta(n)$ time.



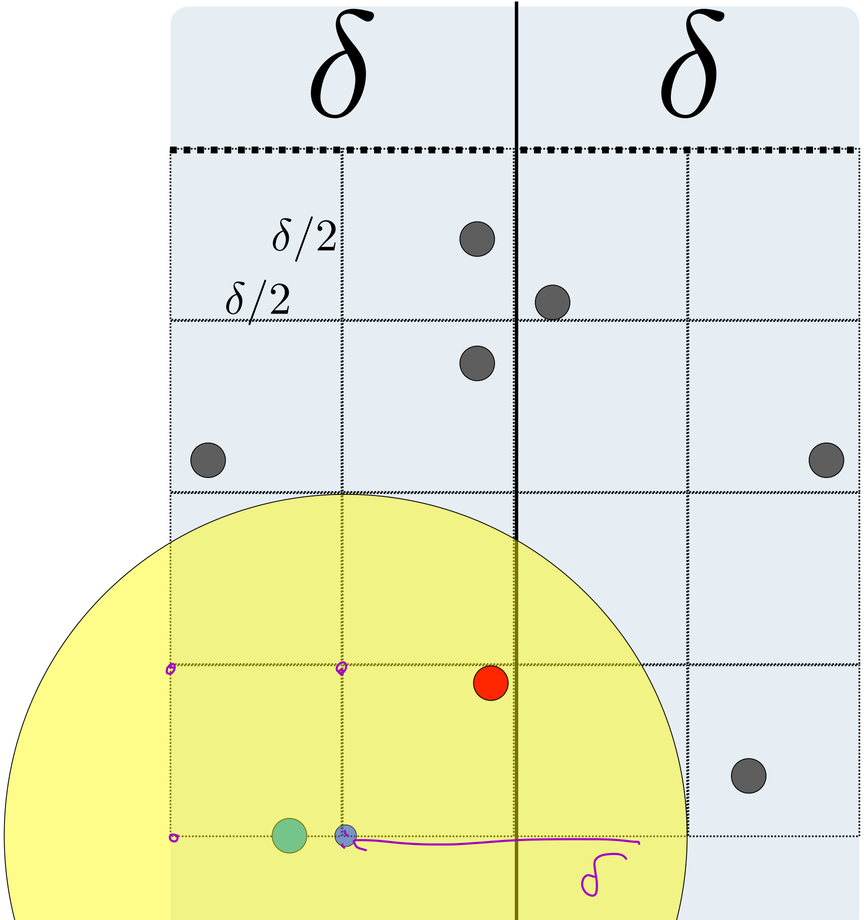
FACT: <=1 point per cubby

15 cubbies in which Romeo can find Juliet.

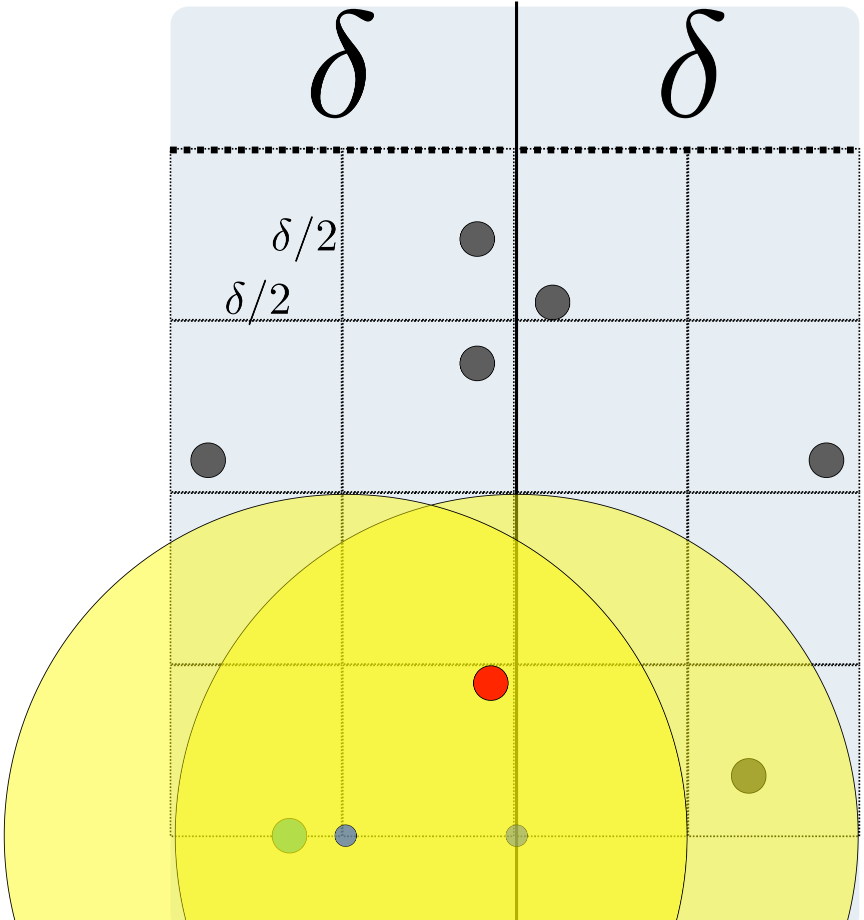

δ
δ
δ/2
δ/2



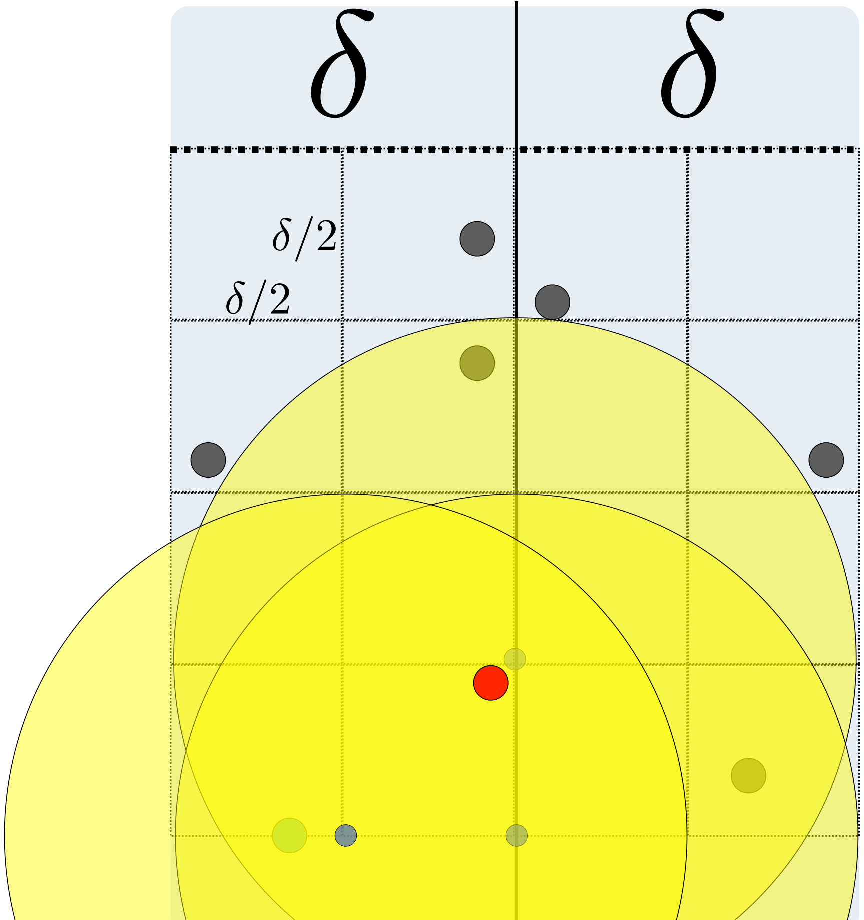
δ
δ
$\delta/2$
$\delta/2$

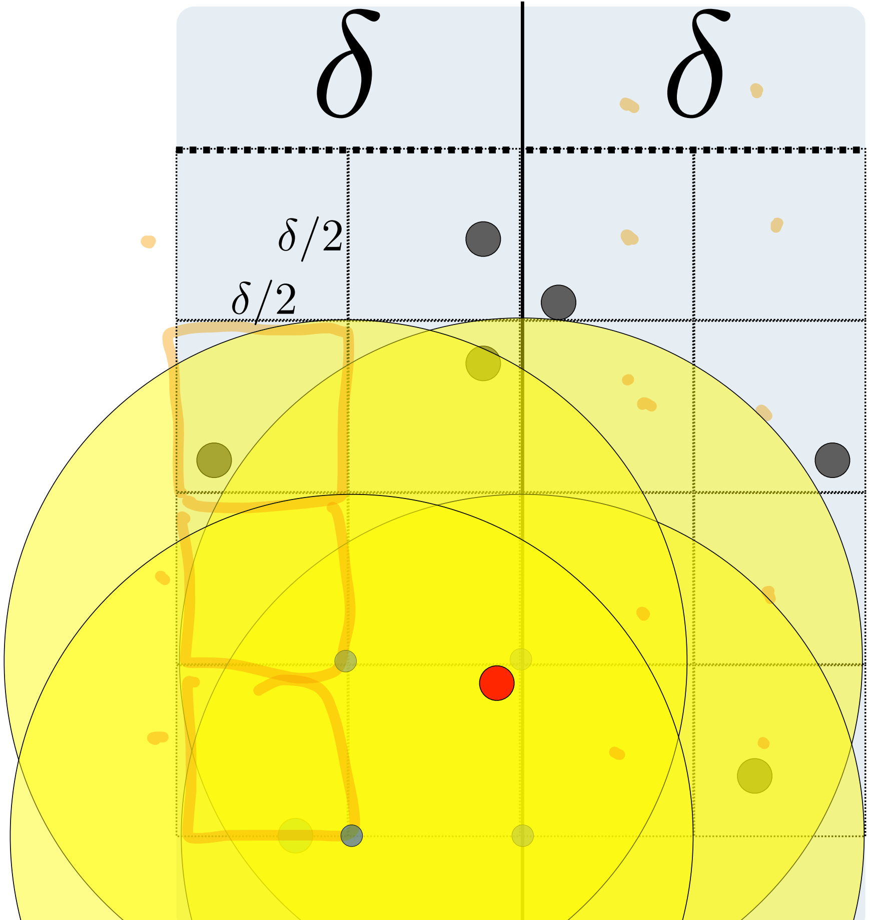


δ
δ
δ/2
δ/2
δ



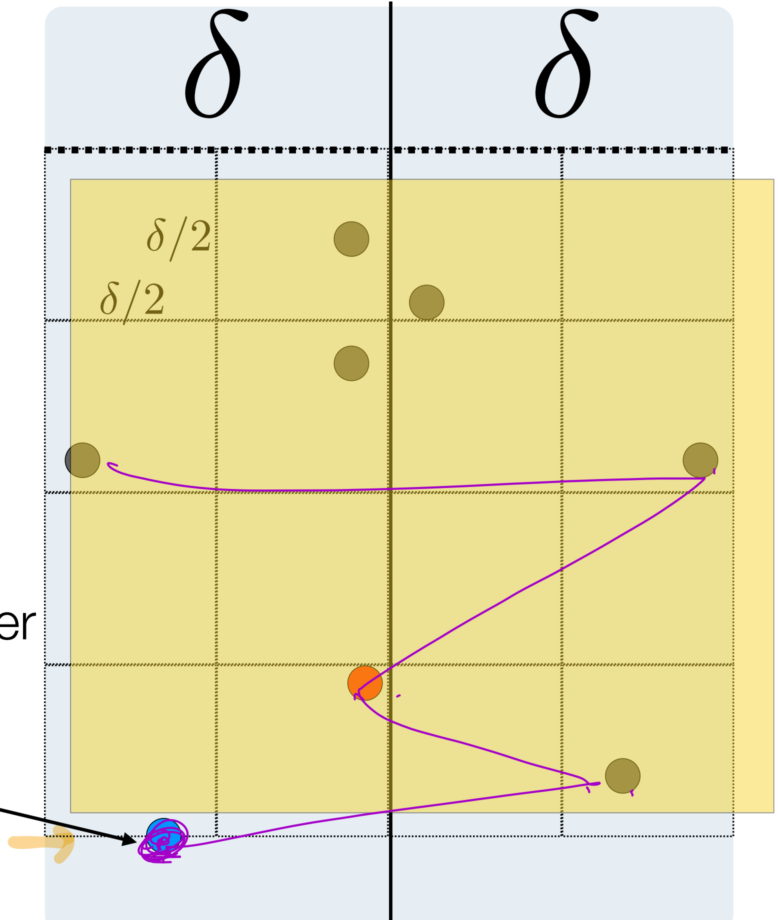
δ
δ
δ/2
δ/2


δ
δ
δ/2
δ/2



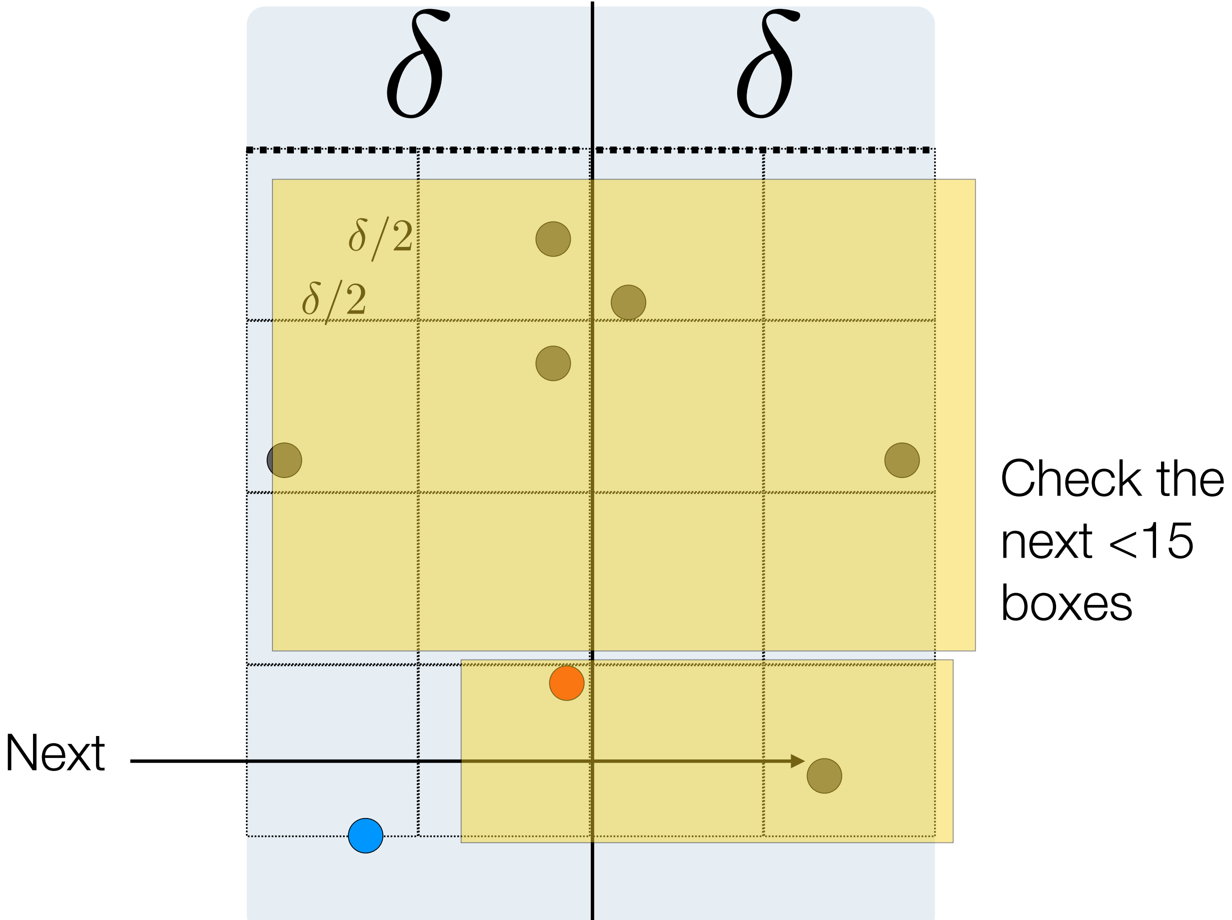
δ
δ
δ/2
δ/2

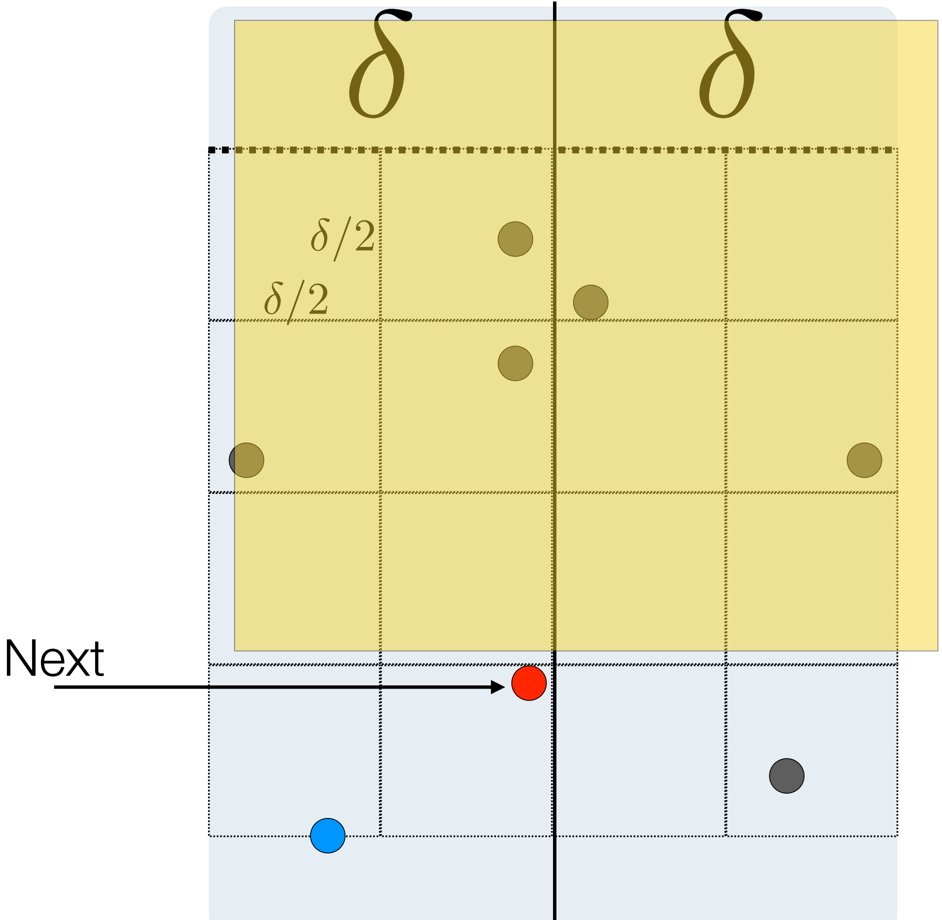
Visit its
by y-order



Check the
next 15
boxes

Start


δ
δ
$\delta/2$
$\delta/2$
Check the next <15 boxes
Next



Check the next <15 boxes

Closest(P)

)

Closest(P)

Base Case: If <8 points, brute force.

1. Let q be the "middle-element" of points
2. Divide P into Left, Right according to q
3. delta,r,j = MIN(Closest(Left) , Closest(Right))
4. Mohawk = { Scan P, add pts that are delta from q.x }
5. For each point x in Mohawk (in y-order):

   Compute distance to its next 15 neighbors

   Update delta,r,j if any pair (x,y) is < delta

6. Return (delta,r,j)

Closest(P)

Base Case: If <8 points, brute force.

1. Let q be the “middle-element” of points

2. Divide P into Left, Right according to q

3. delta,r,j = MIN(Closest(Left) , Closest(Right))

4. Mohawk = { Scan P, add pts that are delta from q.x }

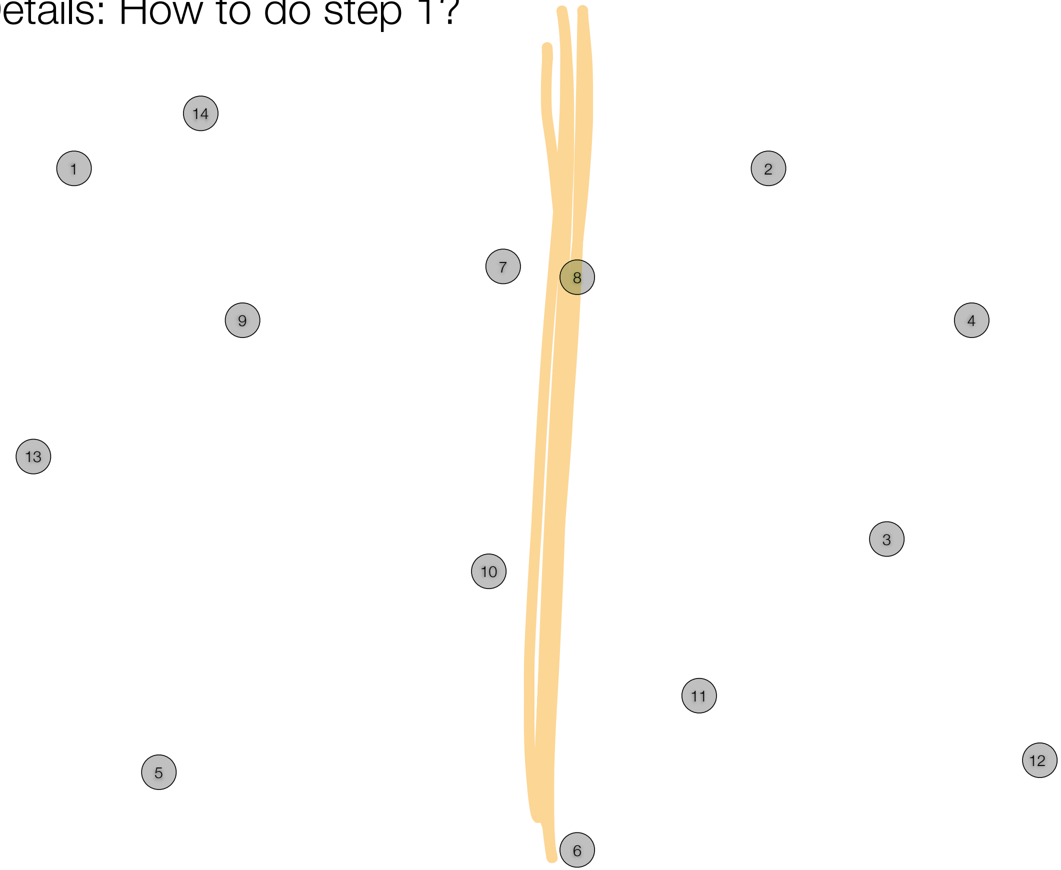
5. For each point x in Mohawk (in y-order):

   Compute distance to its next 15 neighbors
   Update delta,r,j if any pair (x,y) is < delta

6. Return (delta,r,j)

Can be reduced to 7!

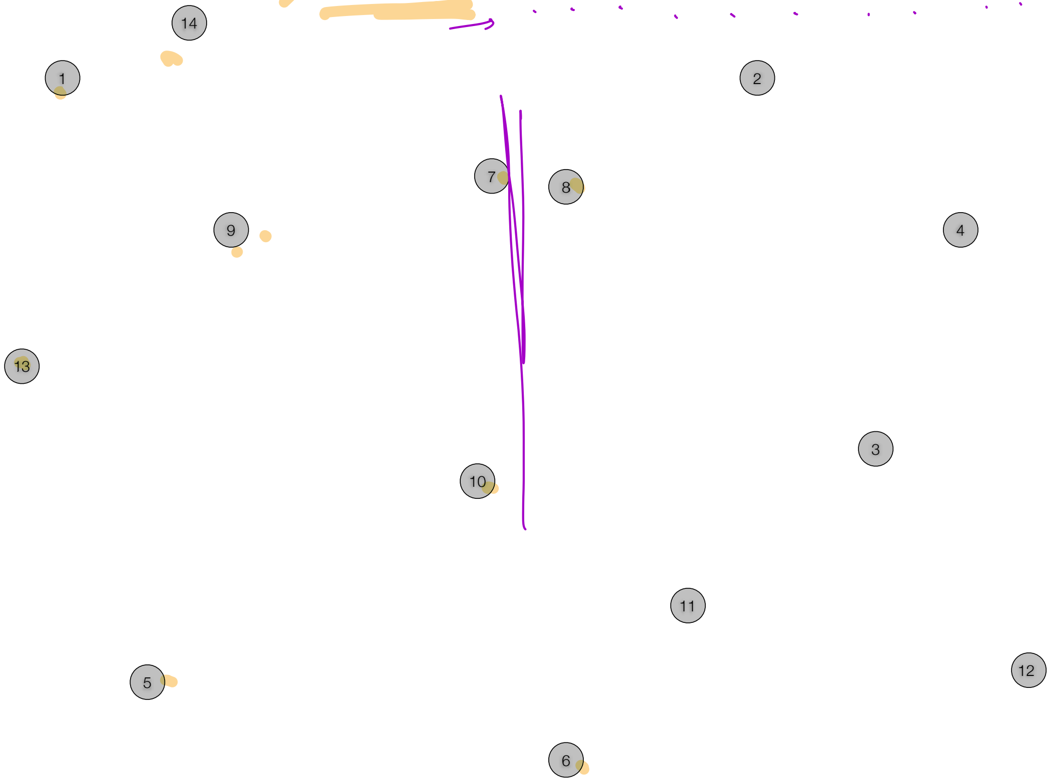
Details: How to do step 1?



6

sorted in X: 13 1 5 14 9 10 7 9 8 11 2 3 4 12

sorted in Y: 6 5 12 11 10 3 13 4 9 8 7 2 1 14



sorted in X: 13 1 5 14 9 10 7 9 8 11 2 3 4 12

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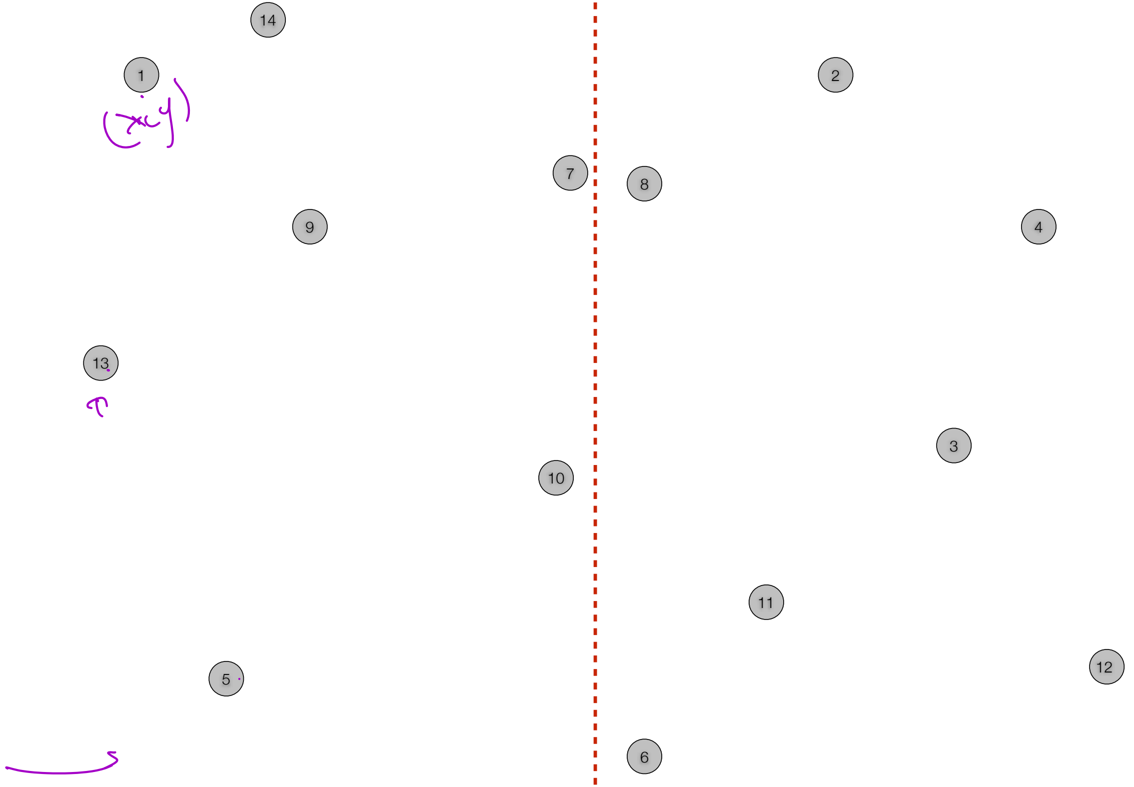

ClosestPair(P)

Compute Sorted-in-X list SX $\Theta(n \cdot \log n)$

Compute Sorted-in-Y list SY $\Theta(n \cdot \log n)$

Closest(P,SX,SY) $T(n) = \Theta(n \cdot \log n)$

$\text{Total} = \Theta(n \cdot \log n)$

Closest(P,SX,SY) sorted list

points

Let q be the middle-element of SX 1

Divide P into Left, Right according to q 1

delta,r,j = MIN(Closest(Left, LX, LY) Closest(Right, RX, RY))

$2T(\frac{n}{2})$

Mohawk = { Scan SY, add pts that are delta from q.x } $\Theta(n)$

For each point x in Mohawk (in order):

Compute distance to its next 15 neighbors

Update delta,r,j if any pair (x,y) is < delta

$\Theta(n)$

Return (delta,r,j)

$$T(n) = 2T(\frac{n}{2}) + \Theta(n) \Rightarrow \Theta(n\log n)$$

Closest(P,SX,SY)

Let q be the middle-element of SX

Divide P into Left, Right according to q

delta,r,j = MIN(Closest(Left, LX, LY) Closest(Right, RX, RY))

Mohawk = { Scan SY, add pts that are delta from q.x }

For each point x in Mohawk (in order):

Compute distance to its next 15 neighbors

Update delta,r,j if any pair (x,y) is < delta

Return (delta,r,j)

Can be reduced to 7!

sorted in X: 13 1 5 14 9 10 7 9 8 11 2 3 4 12

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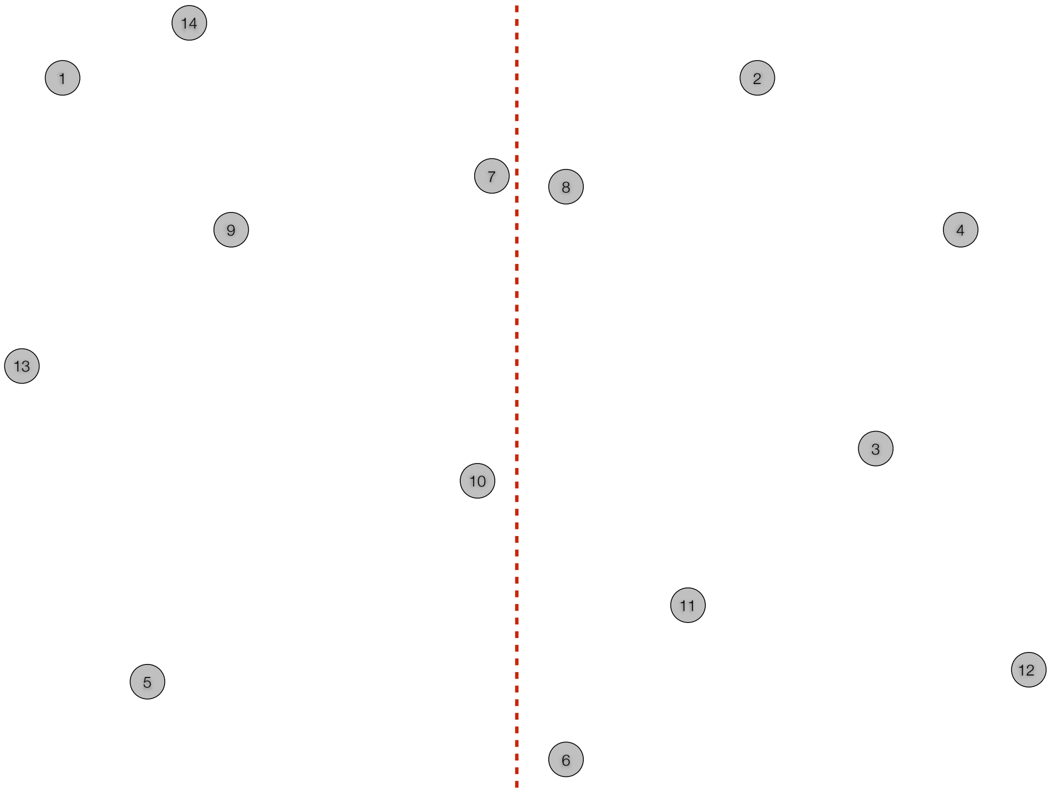

Closest(P,SX,SY)

Let q be the middle-element of SX

Divide P into Left, Right according to q. Scan to get LY, RY.

delta,r,j = MIN(Closest(Left, LX, LY) Closest(Right, RX, RY))

Mohawk = { Scan SY, add pts that are delta from q.x }

For each point x in Mohawk (in order):

Compute distance to its next 15 neighbors
Update delta,r,j if any pair (x,y) is < delta

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Can be reduced to 7!

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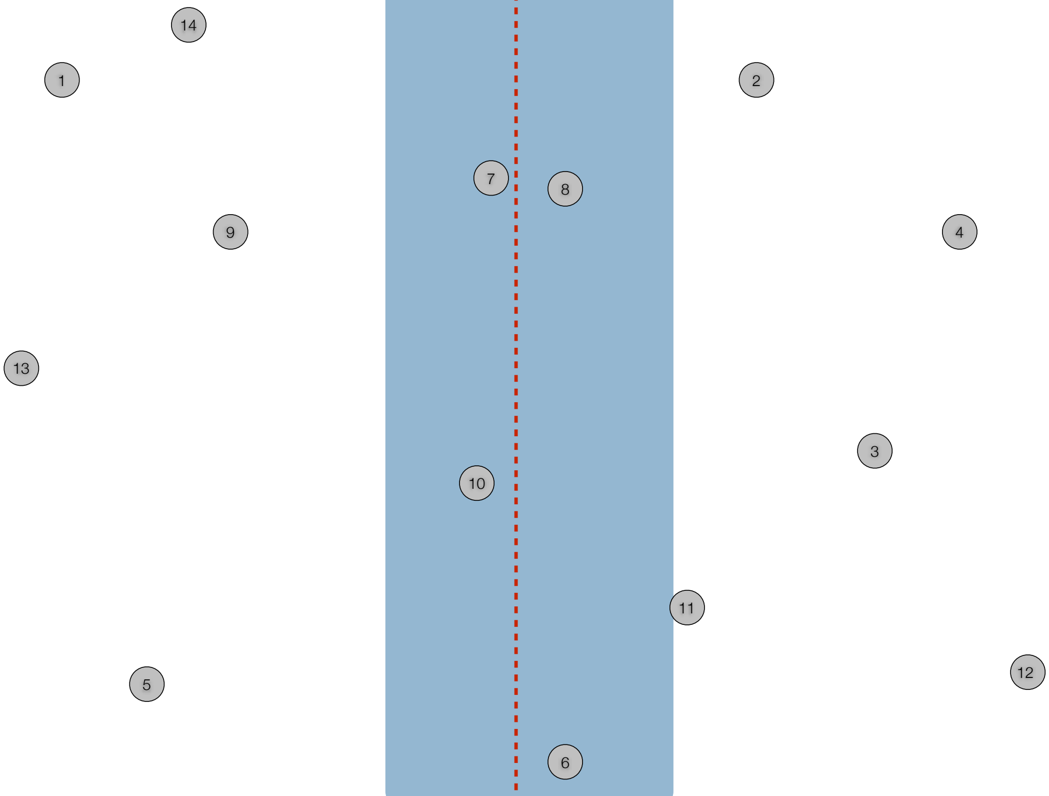

```
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        Compute distance to its next 15 neighbors
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    Return (delta,r,j)
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For each point x in Mohawk (in order):

Compute distance to its next 15 neighbors
Update delta,r,j if any pair (x,y) is < delta

Return (delta,r,j)

Can be reduced to 7!

Running time for Closest pair algorithm

$$T(n) =$$

$$T(n) = 2T(n/2) + \Theta(n) = \Theta(n \log n)$$

@author Robert Sedgewick
@author Kevin Wayne

http://algs4.cs.princeton.edu/99hull/ClosestPair.java.html

```
public ClosestPair(Point2D[] points) {
    int N = points.length;
    if (N <= 1) return;

    // sort by x-coordinate (breaking ties by y-coordinate)
    Point2D[] pointsByX = new Point2D[N];
    for (int i = 0; i < N; i++)
        pointsByX[i] = points[i];
    Arrays.sort(pointsByX, Point2D.X_ORDER);

    // check for coincident points
    for (int i = 0; i < N-1; i++) {
        if (pointsByX[i].equals(pointsByX[i+1])) {
            bestDistance = 0.0;
            best1 = pointsByX[i];
            best2 = pointsByX[i+1];
            return;
        }
    }

    // sort by y-coordinate (but not yet sorted)
    Point2D[] pointsByY = new Point2D[N];
    for (int i = 0; i < N; i++)
        pointsByY[i] = pointsByX[i];

    // auxiliary array
    Point2D[] aux = new Point2D[N];

    closest(pointsByX, pointsByY, aux, 0, N-1);
}
```

```
// find closest pair of points in pointsByX[lo..hi]
// precondition:  pointsByX[lo..hi] and pointsByY[lo..hi] are the same sequence of points, sorted by x,y-coord
private double closest(Point2D[] pointsByX, Point2D[] pointsByY, Point2D[] aux, int lo, int hi) {
    if (hi <= lo) return Double.POSITIVE_INFINITY;

    int mid = lo + (hi - lo) / 2;
    Point2D median = pointsByX[mid];

    // compute closest pair with both endpoints in left subarray or both in right subarray
    double delta1 = closest(pointsByX, pointsByY, aux, lo, mid);
    double delta2 = closest(pointsByX, pointsByY, aux, mid+1, hi);
    double delta = Math.min(delta1, delta2);

    // merge back so that pointsByY[lo..hi] are sorted by y-coordinate
    merge(pointsByY, aux, lo, mid, hi);

    // aux[0..M-1] = sequence of points closer than delta, sorted by y-coordinate
    int M = 0;
    for (int i = lo; i <= hi; i++) {
        if (Math.abs(pointsByY[i].x() - median.x()) < delta)
            aux[M++] = pointsByY[i];
    }

    // compare each point to its neighbors with y-coordinate closer than delta
    for (int i = 0; i < M; i++) {
        // a geometric packing argument shows that this loop iterates at most 7 times
        for (int j = i+1; (j < M) && (aux[j].y() - aux[i].y() < delta); j++) {
            double distance = aux[i].distanceTo(aux[j]);
            if (distance < delta) {
                delta = distance;
                if (distance < bestDistance) {
                    bestDistance = delta;
                    best1 = aux[i];
                    best2 = aux[j];
                    // StdOut.println("better distance = " + delta + " from " + best1 + " to " + best2);
                }
            }
        }
    }
    return delta;
}
```

Matrix

multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \bigstar \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \star \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & & & \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & & & \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{bmatrix}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & & & \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & & & \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{bmatrix}$$

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} \cdot b_{k,j}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & & & \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & & & \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

$$= \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

$$T(n) = 8T(n/2) + \Theta(n^2)$$

$$\Theta(n^3)$$

$$= \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

[Strassen]

$$P_1 = A(F - H)$$

$$P_2 = (A + B)H$$

$$P_3 = (C + D)E$$

$$P_4 = D(G - E)$$

$$P_5 = (A + D)(E + H)$$

$$P_6 = (B - D)(G + H)$$

$$P_7 = (A - C)(E + F)$$

$$= \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

$R = P_5 + P_4 - P_2 + P_6$

$S = P_1 + P_2$

$T = P_3 + P_4$

$U = P_5 + P_1 - P_3 - P_7$

[strassen]

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$P_2 = (A + B)H$

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$$M(n) = 7M(n/2) + 18n^2$$

$$= \Theta(n^{\log_2 7})$$

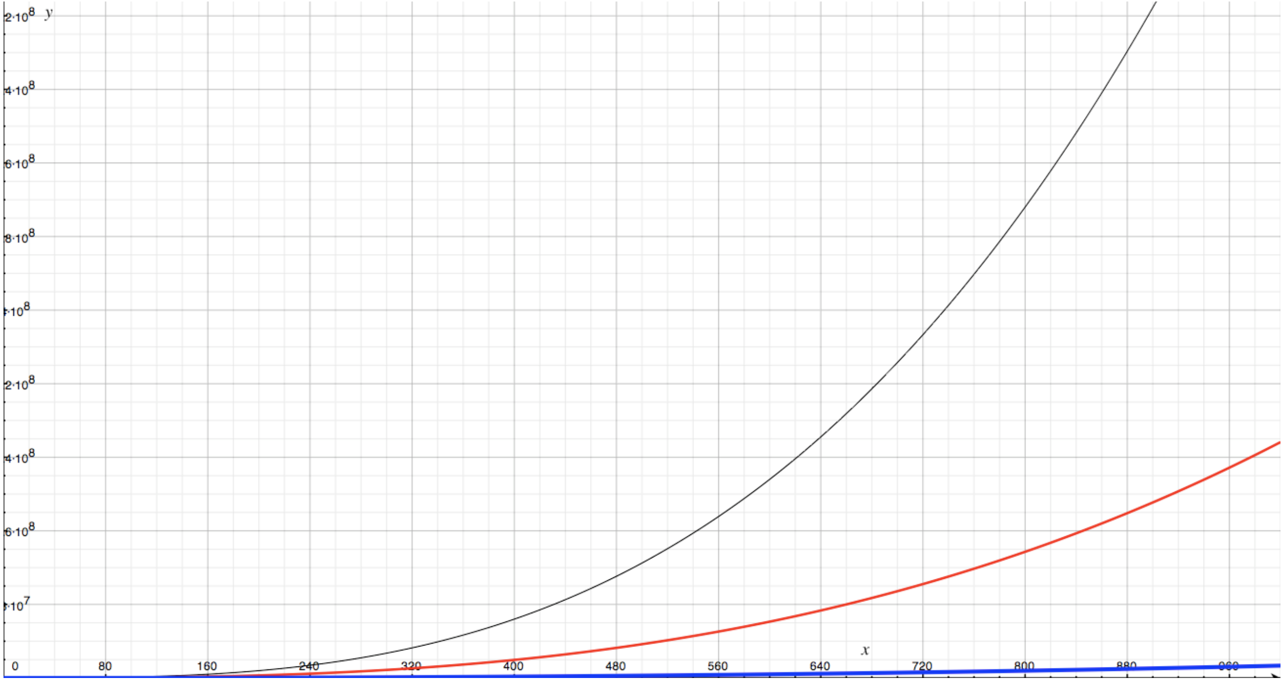
taking this idea further

3x3 matricies [Laderman’75]

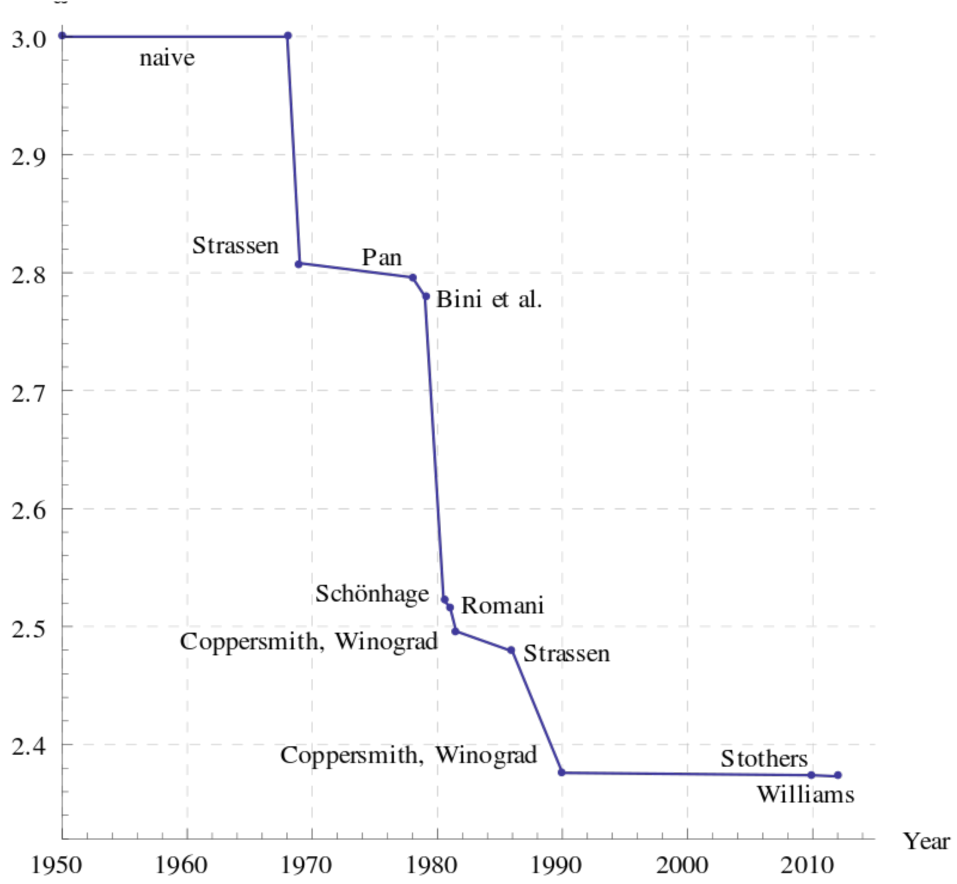
1978 victor pan method

70x70 matrix using 143640 mults

what is the recurrence:

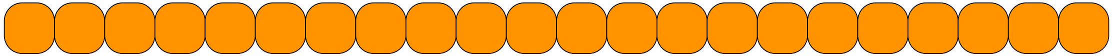


y
2·10^8
4·10^8
6·10^8
8·10^8
·10^8
2·10^8
4·10^8
6·10^8
·10^7
0
80
160
240
320
400
480
560
640
720
800
880
960
x

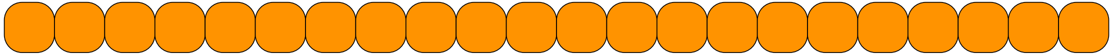


https://en.wikipedia.org/wiki/File:Bound_on_matrix_multiplication_omega_over_time.svg

MEDIAN



problem: given a list of **n** elements, find the element of rank **n**/2. (half are larger, half are smaller)

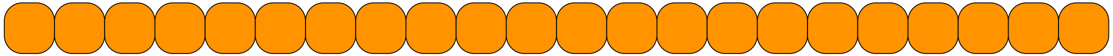


problem: given a list of **n** elements, find the element of rank n/2. (half are larger, half are smaller)

can generalize to **i**

first solution: sort and pluck.

$$O(n \log n)$$

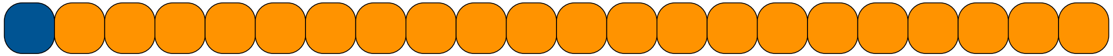


problem: given a list of **n** elements, find the element of rank **i**.

key insight:

we do not have to "fully" sort.

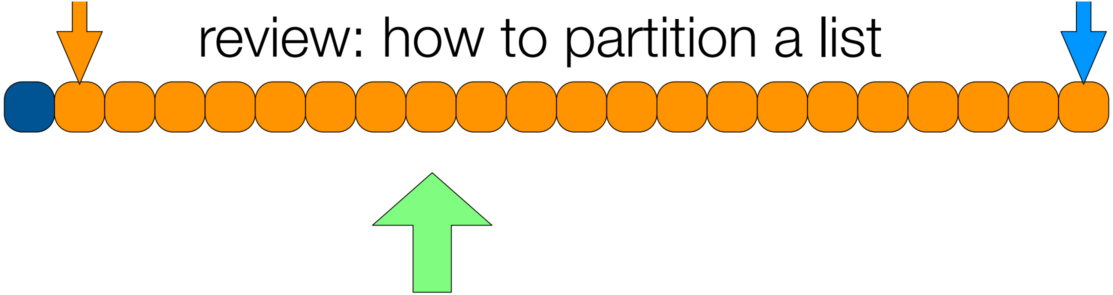
semi sort can suffice.



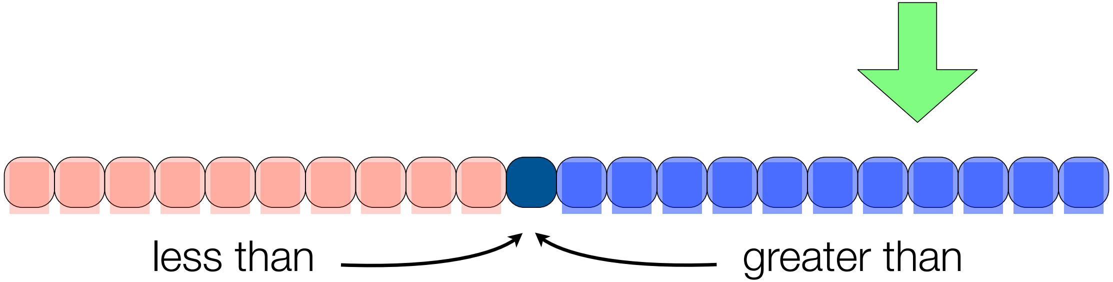
pick first element
partition list about this one
see where we stand

review: how to partition a list

review: how to partition a list



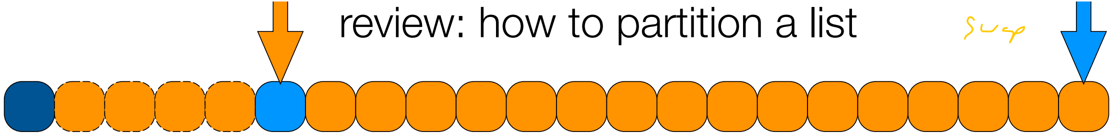
GOAL: start with THIS LIST and END with THAT LIST



review: how to partition a list

review: how to partition a list

swap



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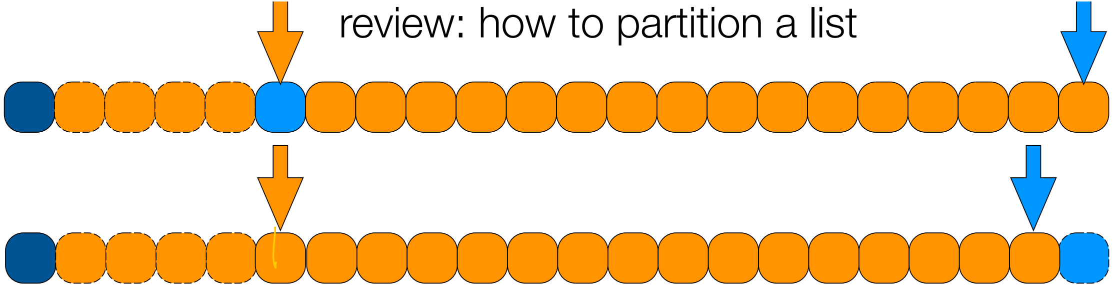

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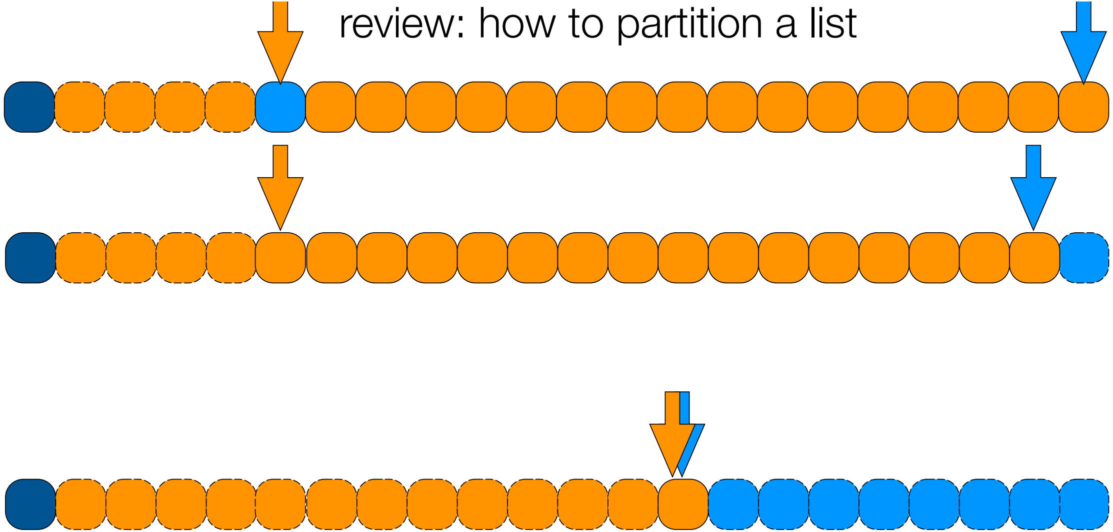

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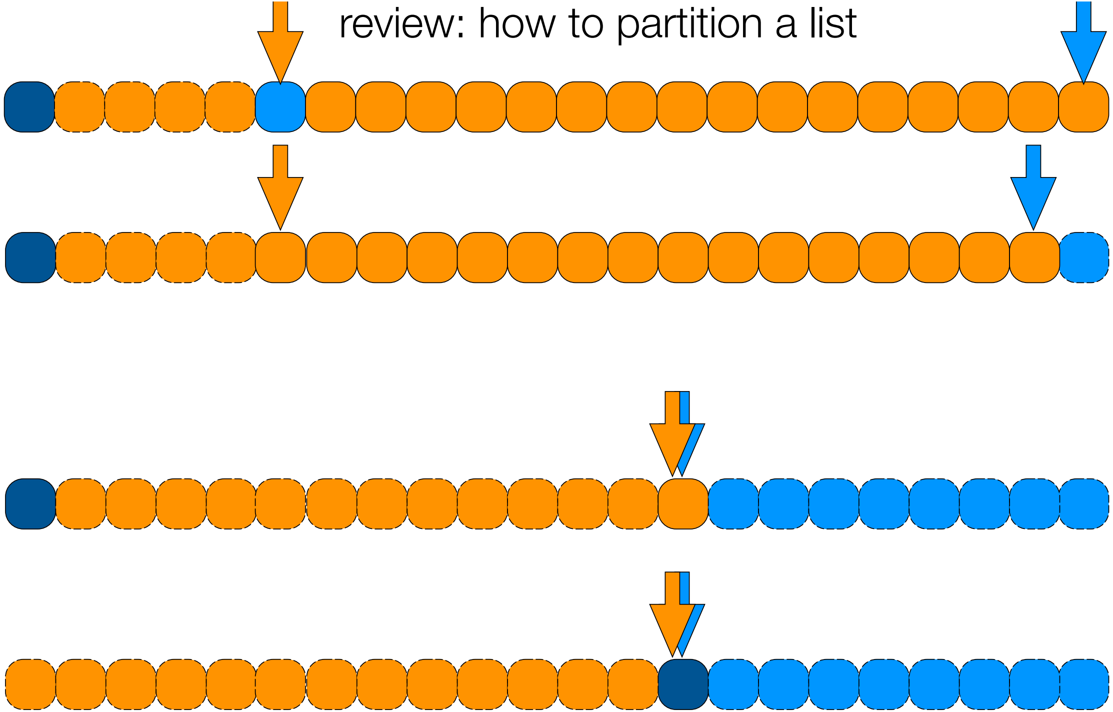

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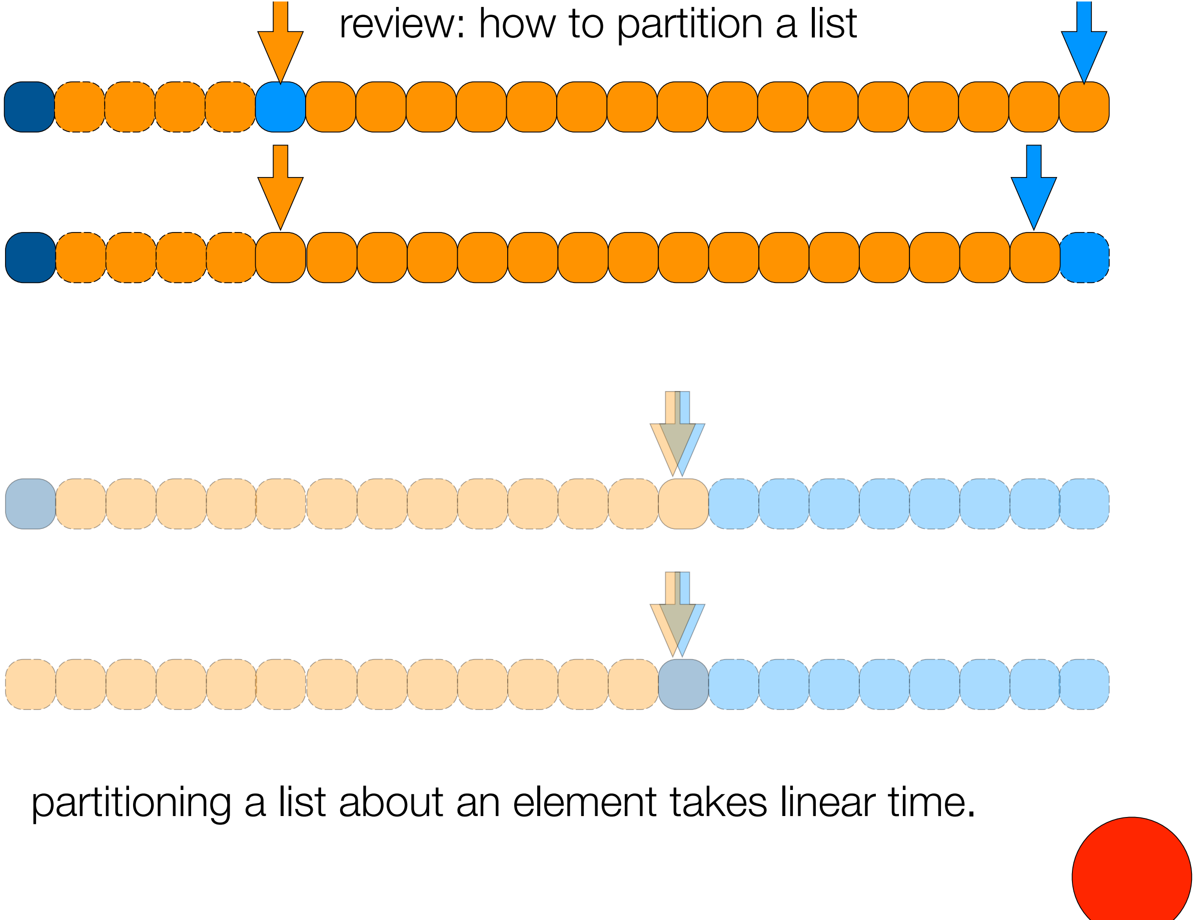
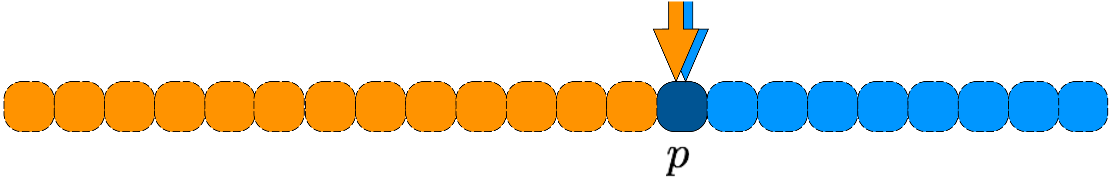
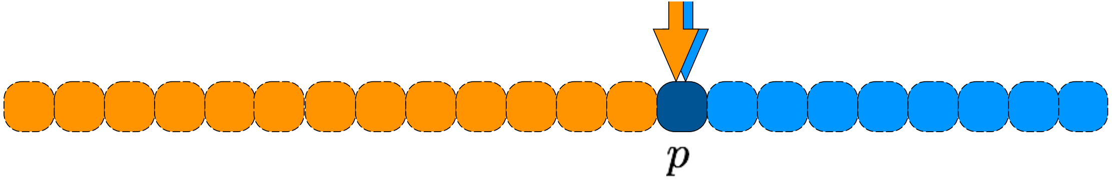


partitioning a list about an element takes linear time.



select $(i, A[1, \ldots, n])$



select $(i, A[1, \ldots, n])$

handle base case.

partition list about first element

if pivot p is position **i**, return pivot

else if pivot p is in position > **i** select $(i, A[1, \ldots, p-1])$

else select $((i-p-1), A[p+1, \ldots, n])$

select $(i, A[1, \ldots, n])$

Assume our partition always splits list into two eql parts

handle base case.

partition list about first element

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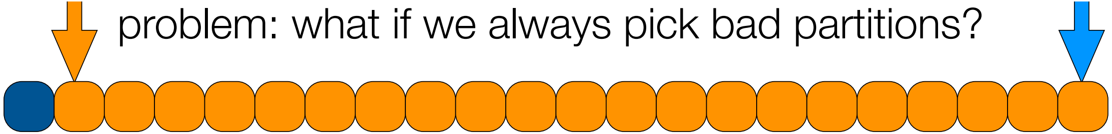
$$T(n) = T(n/2) + O(n)$$

$$\Theta(n)$$

problem: what if we always pick bad partitions?



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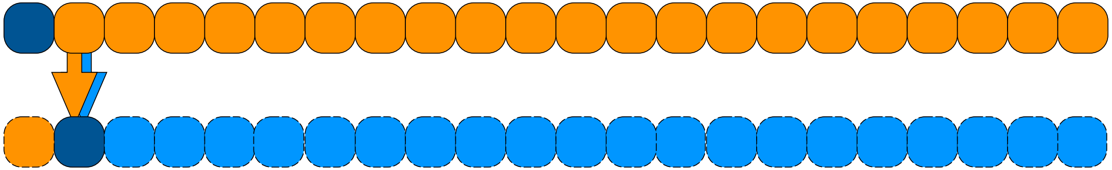

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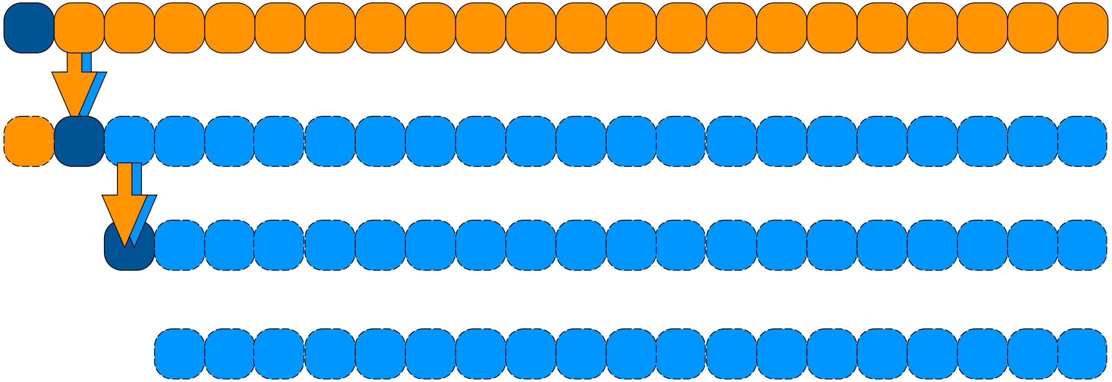

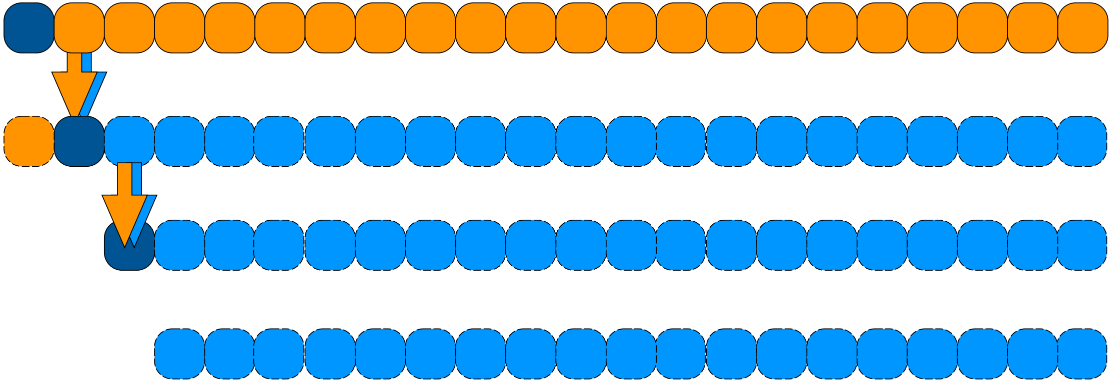

problem: what if we always pick bad partitions?

select $(i, A[1, \ldots, n])$

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$$T(n) = T(n-1) + O(n)$$

$$\Theta(n^2)$$

Needed:

a good partition element

partition $(A[1, \ldots, n])$

Needed:

a good partition element

partition $(A[1, \ldots, n])$

produce an element where
30% smaller, 30% larger

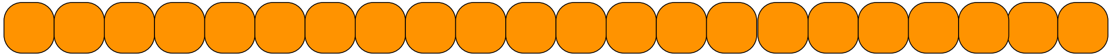
solution: bootstrap

image: gucci

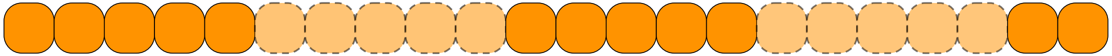
image: mark nason

image: d&g

partition $(A[1, \ldots, n])$



partition $(A[1,\ldots,n])$



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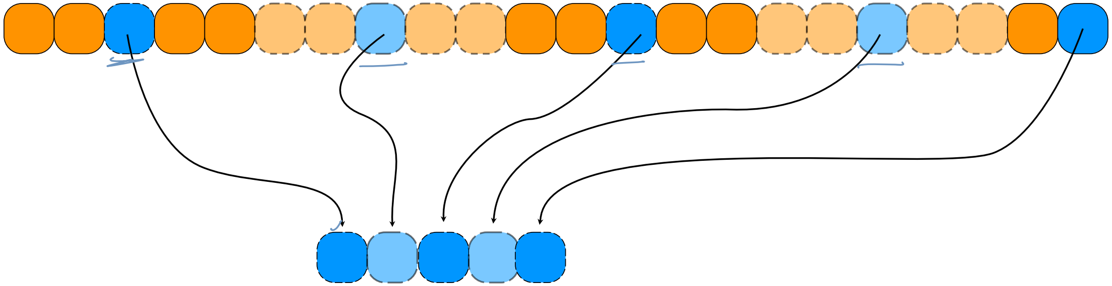
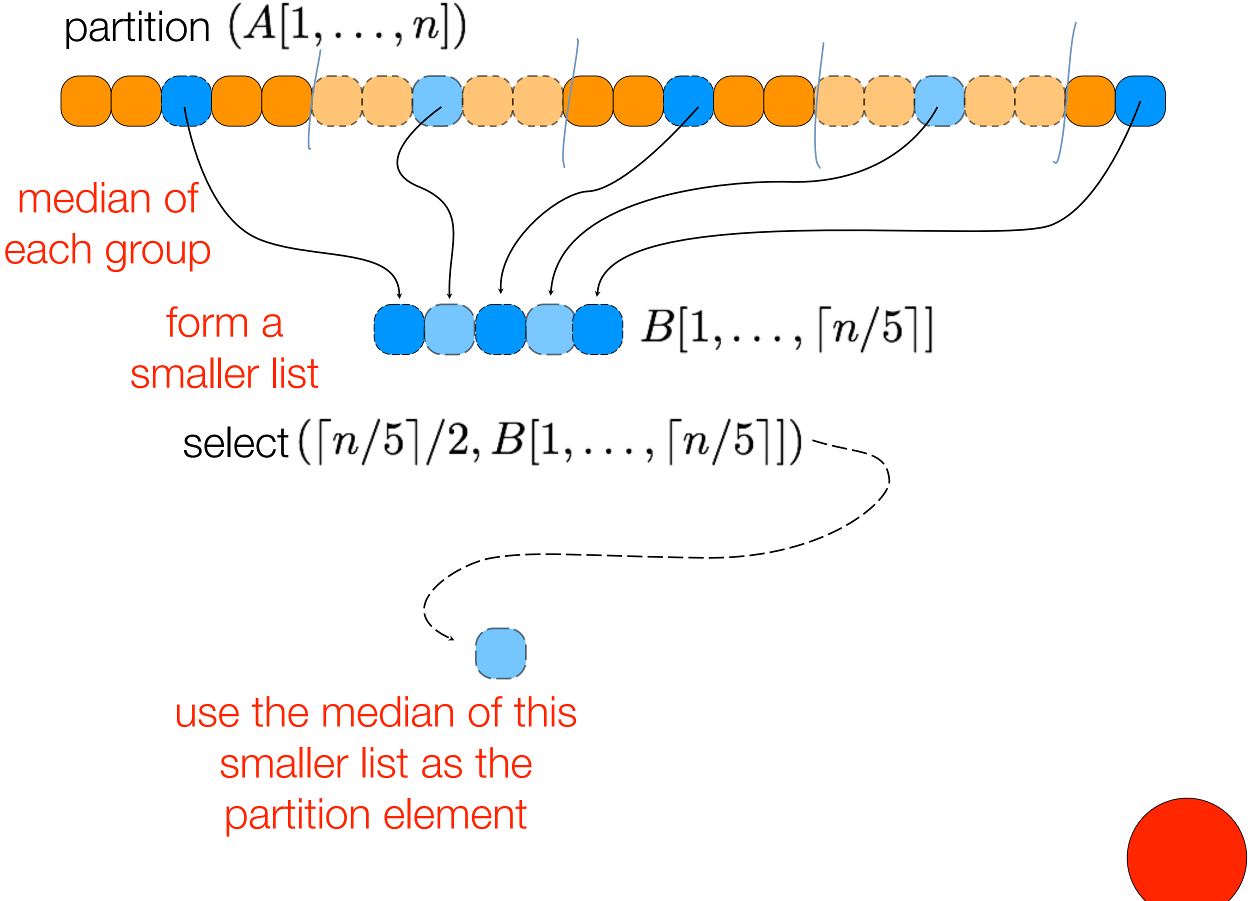
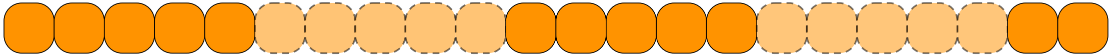




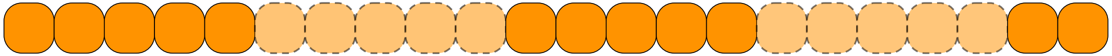
partition $(A[1,\ldots,n])$
median of each group
form a smaller list
$B[1,\ldots,\lceil n/5\rceil]$
select $(\lceil n/5\rceil/2, B[1,\ldots,\lceil n/5\rceil])$
use the median of this smaller list as the partition element

partition $(A[1,\ldots,n])$



1.
2.
3.
4.
5.

partition $(A[1, \ldots, n])$



divide list into groups of 5 elements

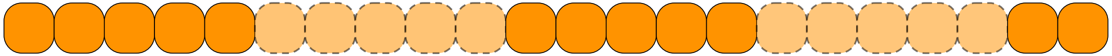
find median of each small list

gather all medians

call select(...) on this sublist to find median

return the result

partition $(A[1,\ldots,n])$



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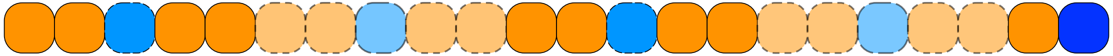
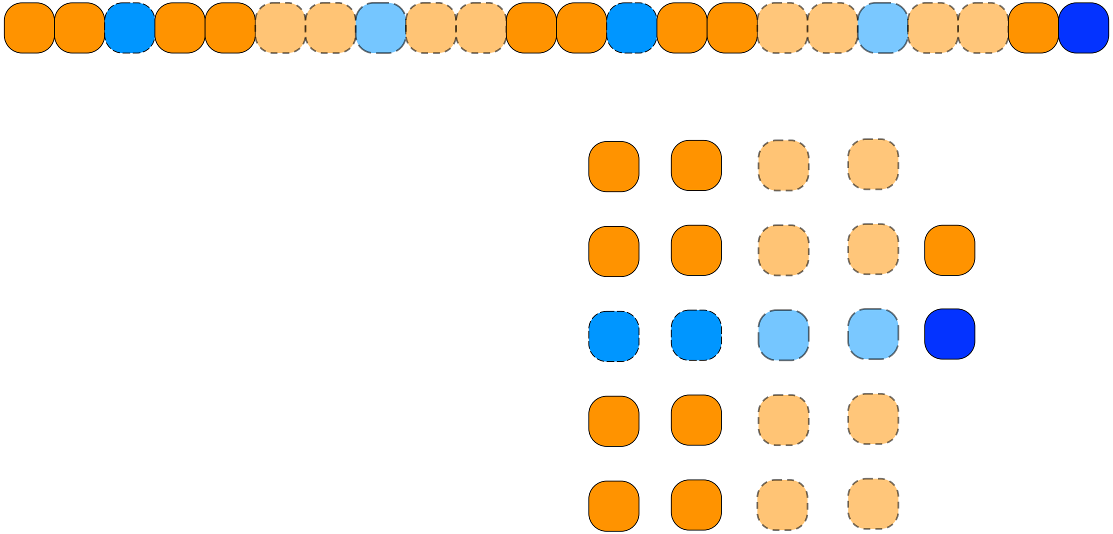
$$P(n) = S(\lceil n/5 \rceil) + O(n)$$

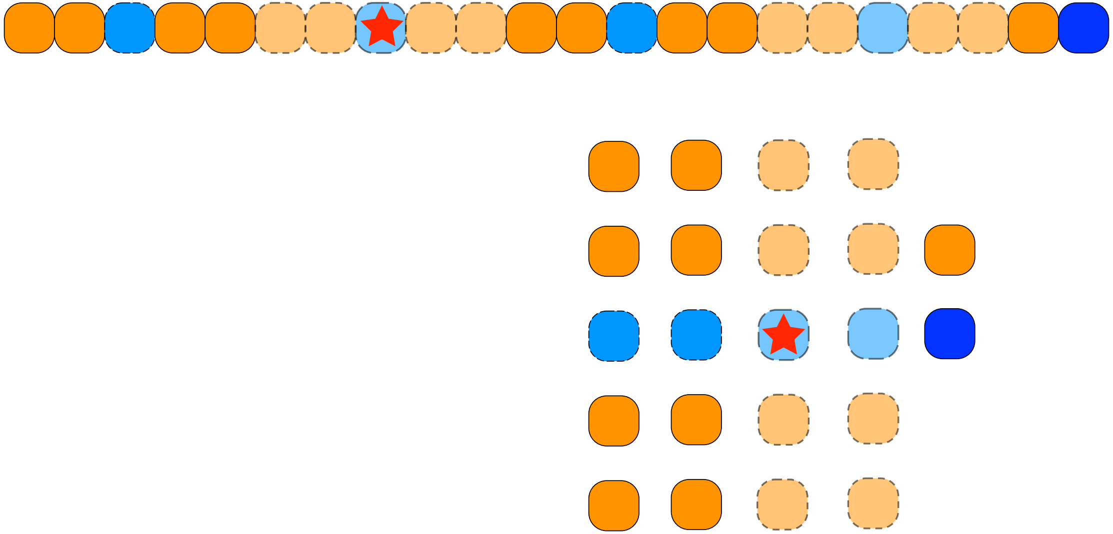
a nice property of our partition



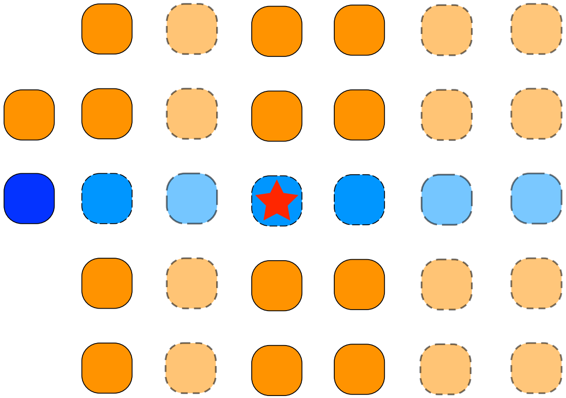
a nice property of our partition



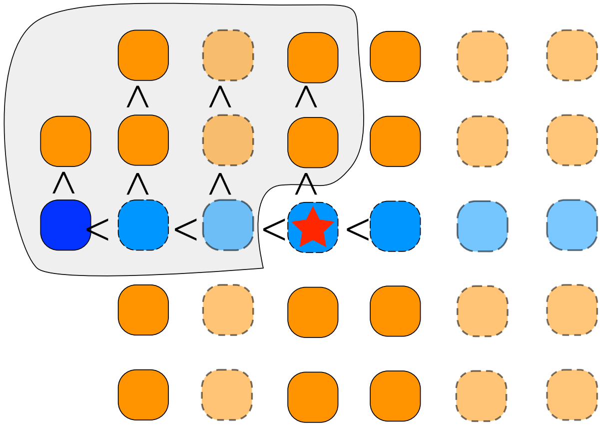
a nice property of our partition



SWITCH TO A BIGGER EXAMPLE



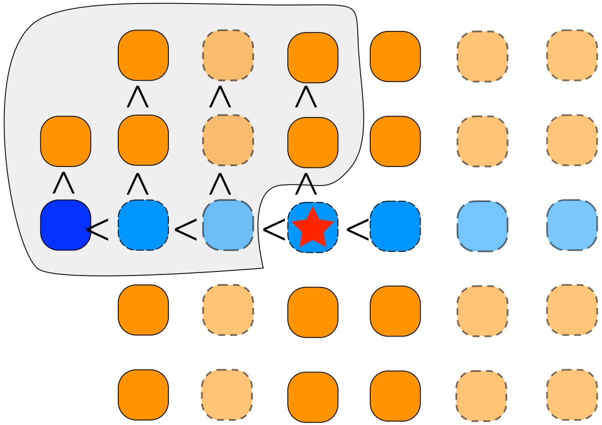
a nice property of our partition



a nice property of our partition

$$3\left(\left\lceil \frac{1}{2}\lceil n/5 \rceil \right\rceil - 2\right)$$

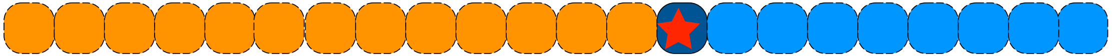
$$\geq \frac{3n}{10} - 6$$

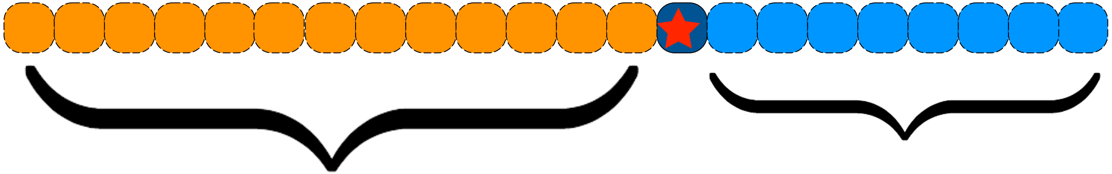


this implies there are
at most $\dfrac{7n}{10} + 6$ numbers

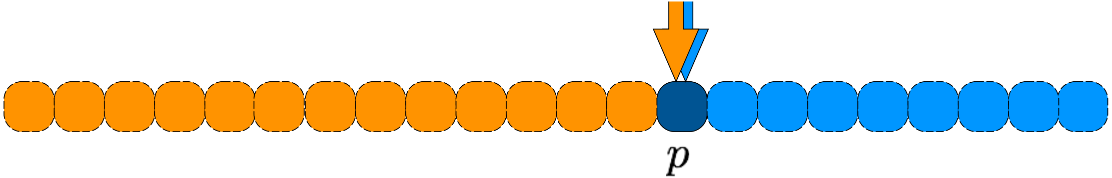
larger than ★
/smaller

a nice property of our partition

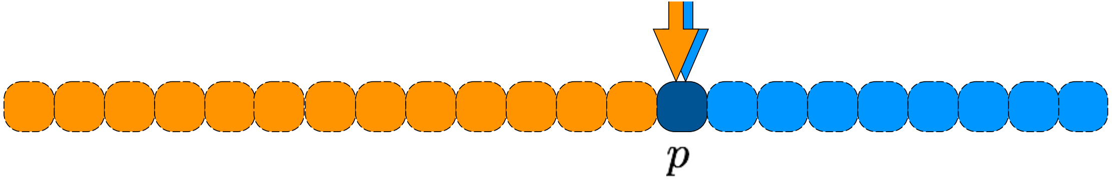




$$\leq \frac{7n}{10} + 6 \qquad \leq \frac{7n}{10} + 6$$



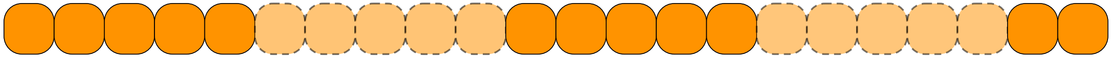
select $(i, A[1, \ldots, n])$



select $(i, A[1, \ldots, n])$

handle base case for small list
else pivot = FindPartitionValue(A,n)
partition list about pivot
if pivot is position **i**, return pivot
else if pivot is in position > **i** select $(i, A[1, \ldots, p-1])$
else select $((i-p-1), A[p+1, \ldots, n])$

FindPartition $(A[1,\ldots,n])$



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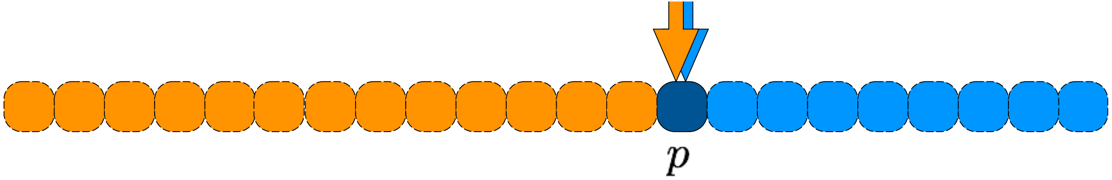

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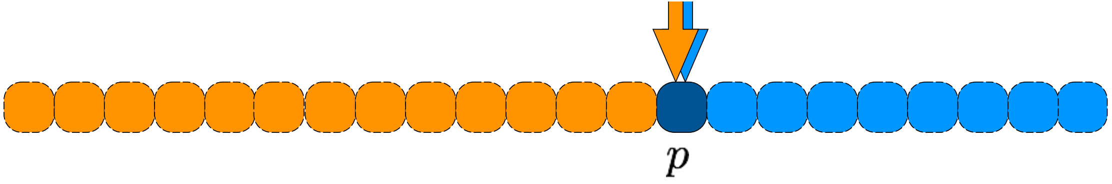
$$S(n) = S(\lceil n/5 \rceil) + O(n) + S(7n/10 + 6)$$



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$$S(n) = S(\lceil n/5 \rceil) + O(n) + S(7n/10 + 6)$$

$$\Theta(n)$$