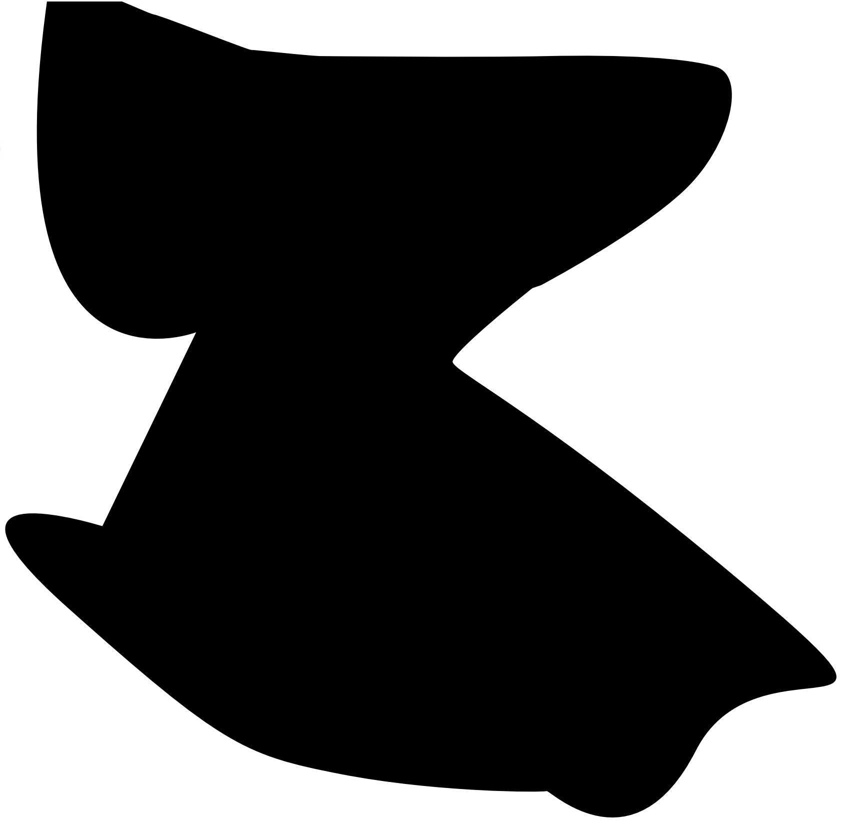
L5 5800

feb 1/3 2022

shelat

divide & conquer



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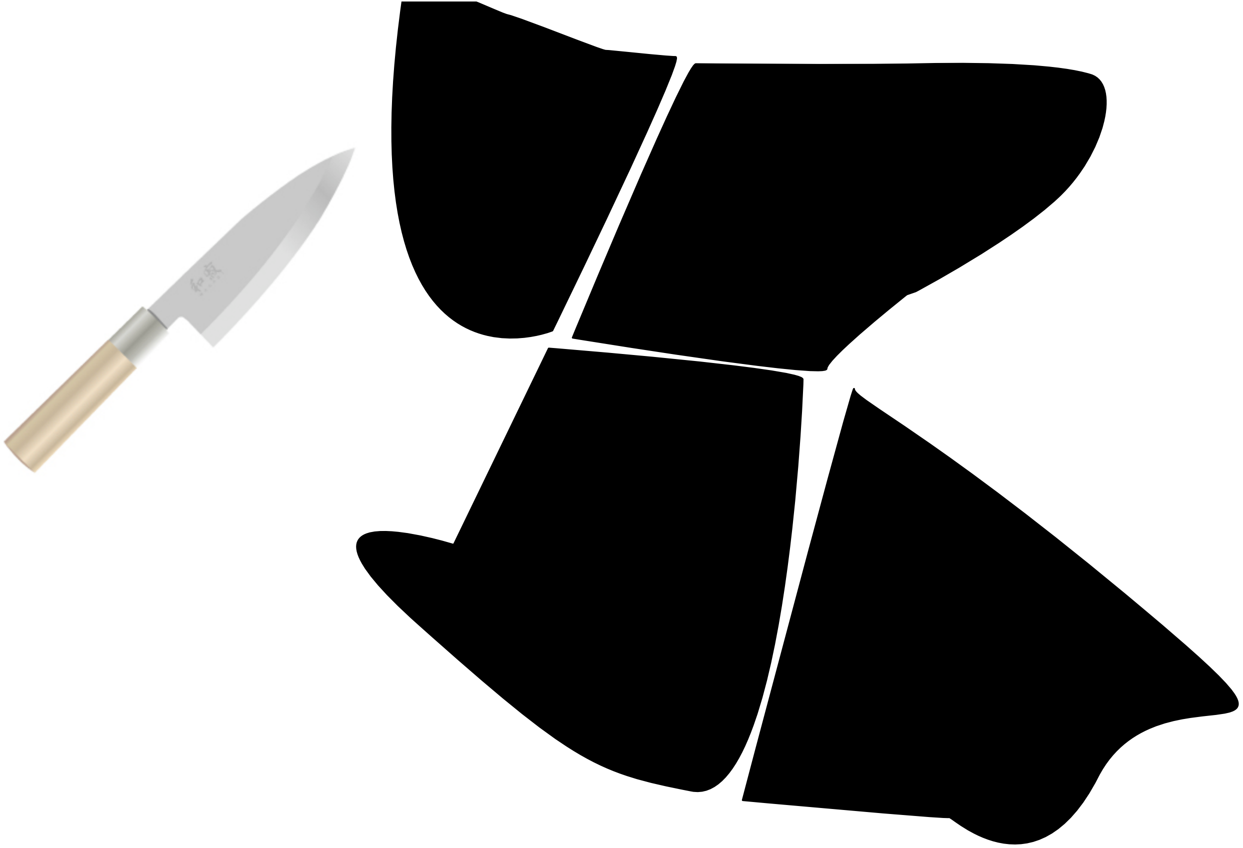

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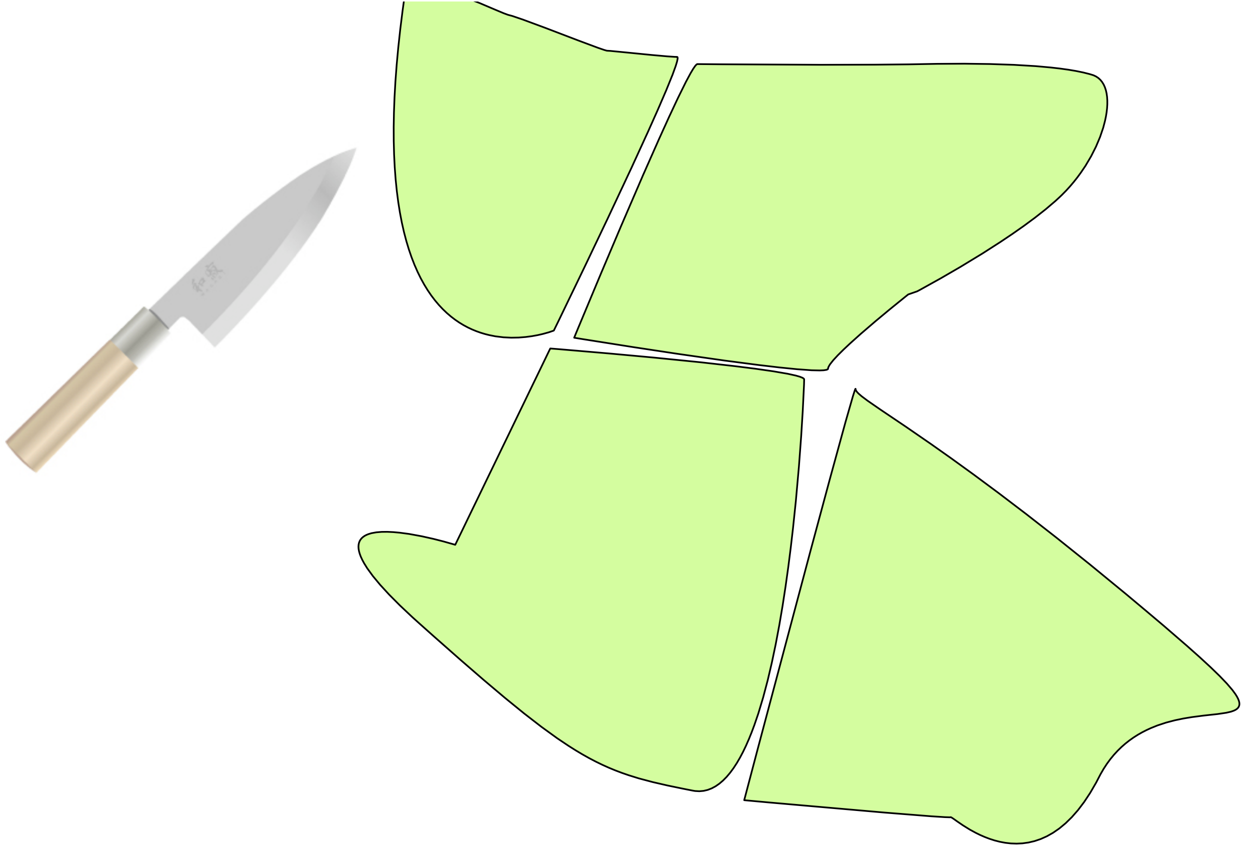

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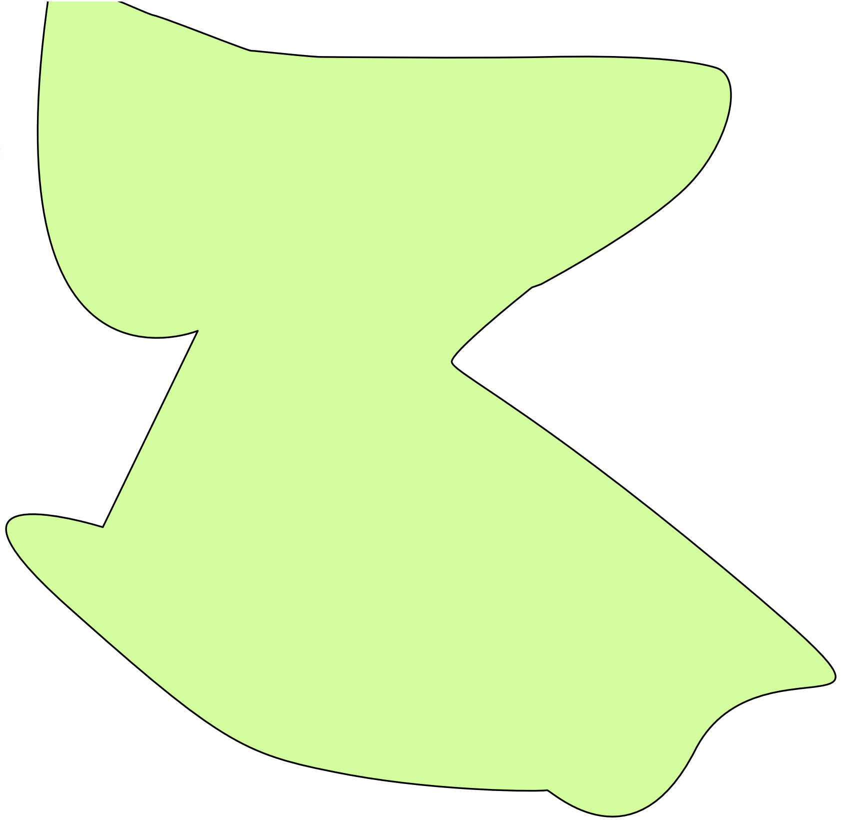

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Examples we will discuss

- Mergesort
- Arbitrage
- Closest Pair of points
- Matrix multiplication / Karatsuba
- MEDIAN - algorithm
- FFT

Merge

```
merge-sort $(A, p, r)$
  if $p < r$
    $q \leftarrow \lfloor (p + r)/2 \rfloor$
    merge-sort $(A, p, q)$
    merge-sort $(A, q + 1, r)$
    merge$(A, p, q, r)$
```

```
MERGE($A[1 .. n], m$):
  $i \leftarrow 1$;  $j \leftarrow m + 1$
  for $k \leftarrow 1$ to $n$
    if $j > n$
      $B[k] \leftarrow A[i]$;  $i \leftarrow i + 1$
    else if $i > m$
      $B[k] \leftarrow A[j]$;  $j \leftarrow j + 1$
    else if $A[i] < A[j]$
      $B[k] \leftarrow A[i]$;  $i \leftarrow i + 1$
    else
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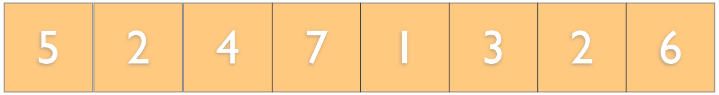
  for $k \leftarrow 1$ to $n$
    $A[k] \leftarrow B[k]$
```

jeff erickson



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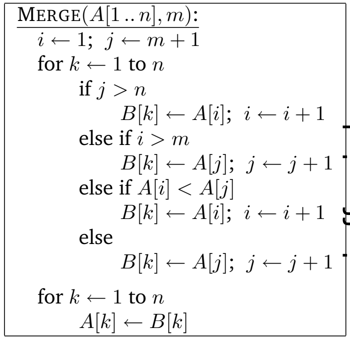

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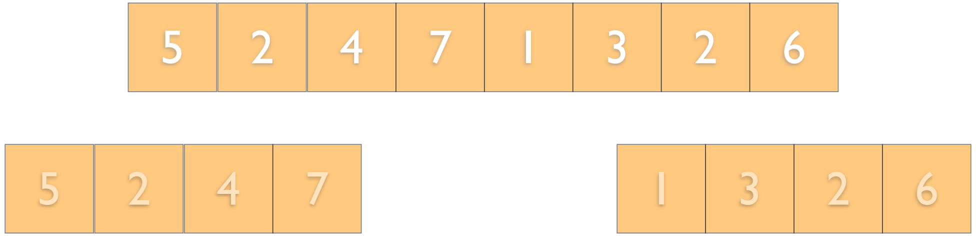
jeff erickson



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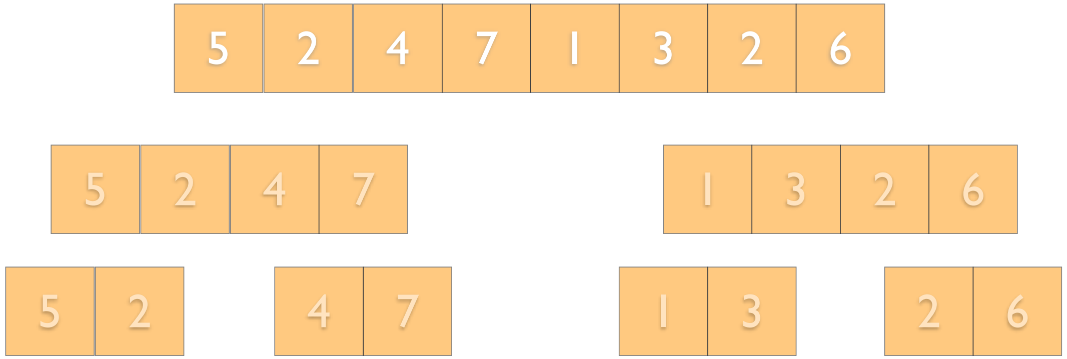
jeff erickson



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    for $k \leftarrow 1$ to $n$
        $A[k] \leftarrow B[k]$
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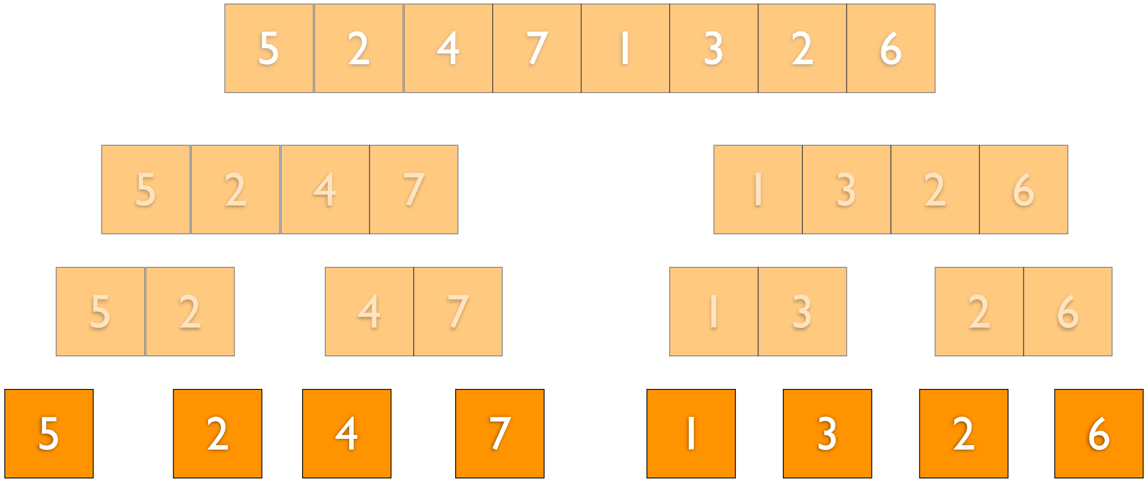
jeff erickson



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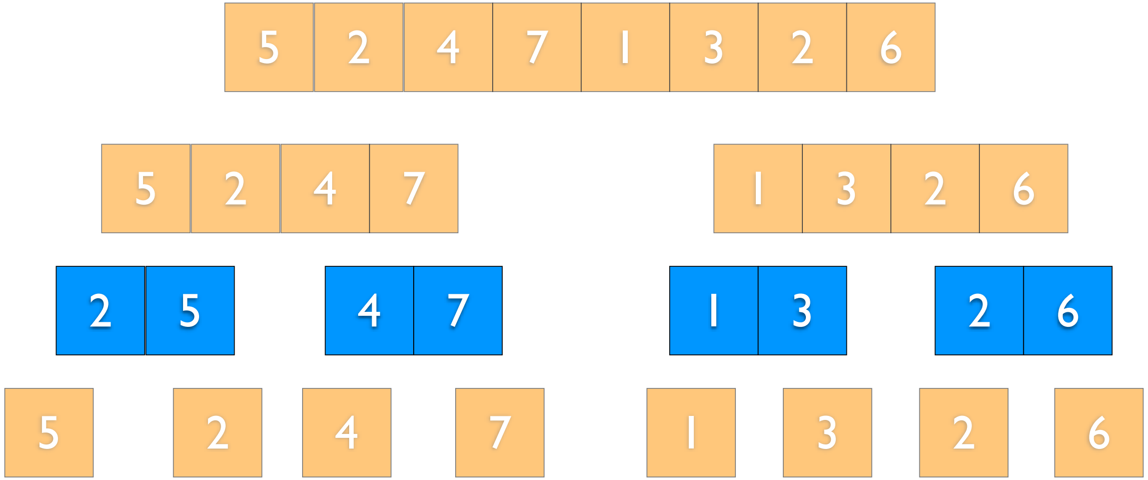

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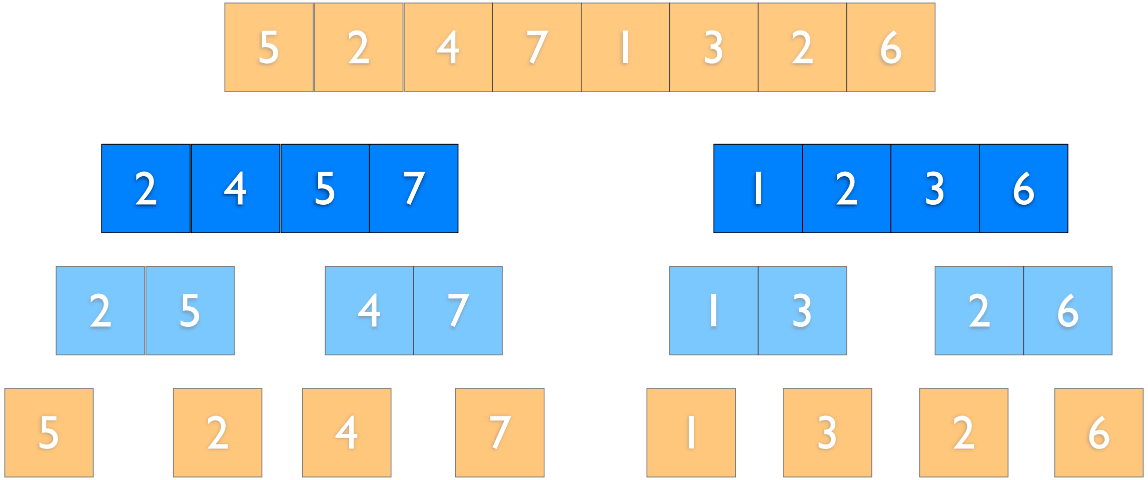

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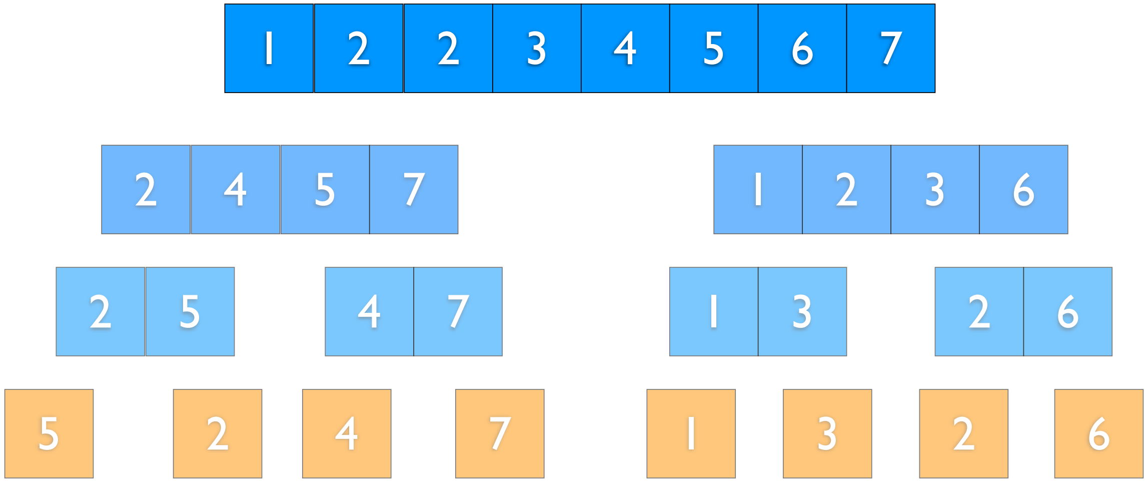

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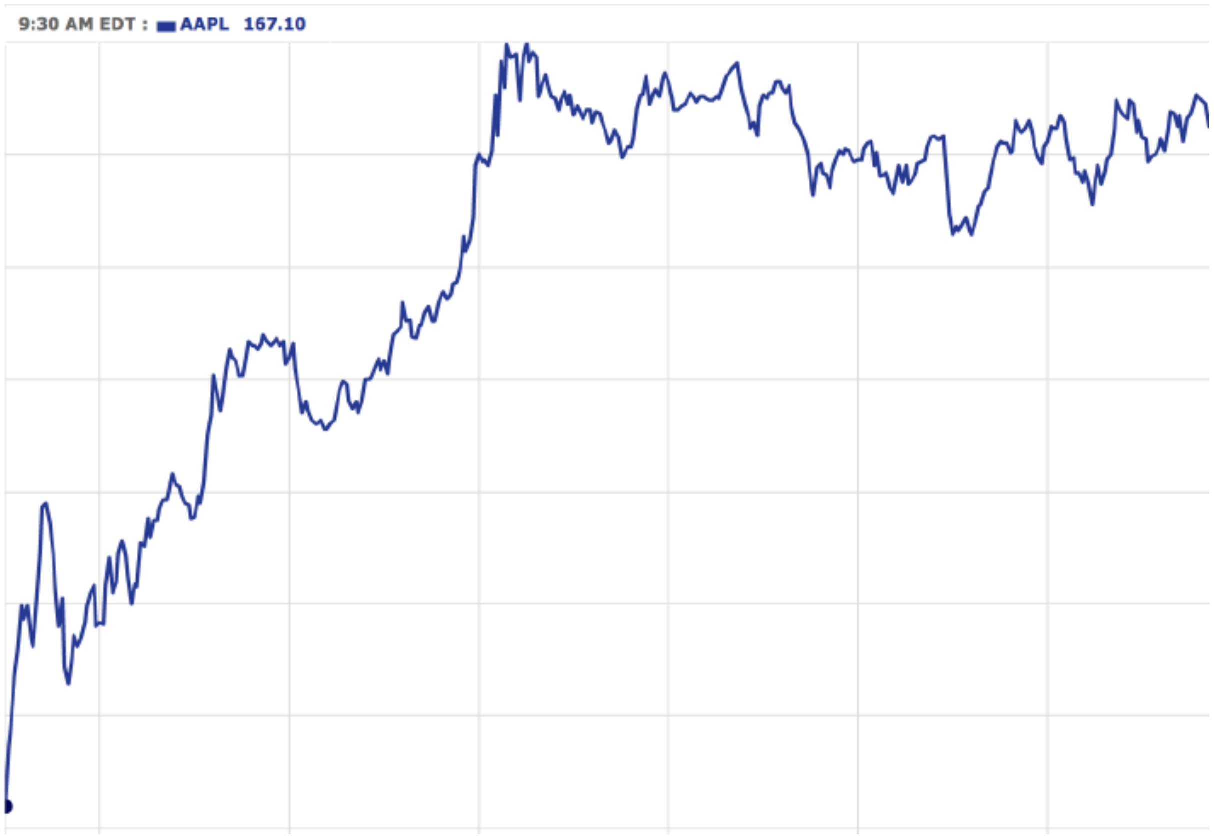
        merge$(A, p, q, r)$
```

$$T(n) = \underbrace{2T(n/2)}_{} + \Theta(n)$$

$$= \Theta(n \log n)$$

arbitrage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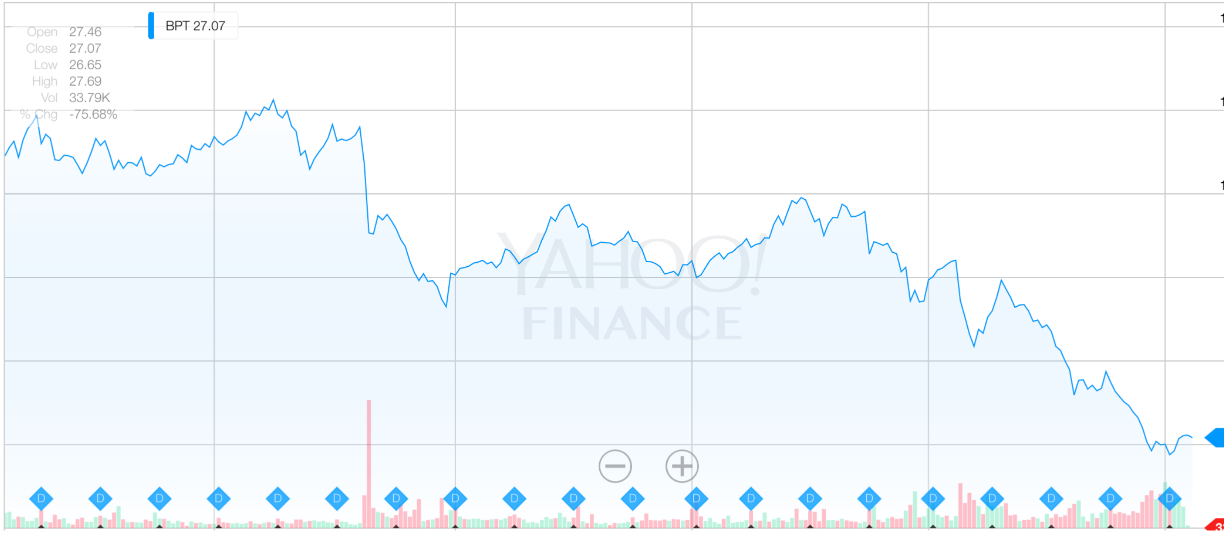
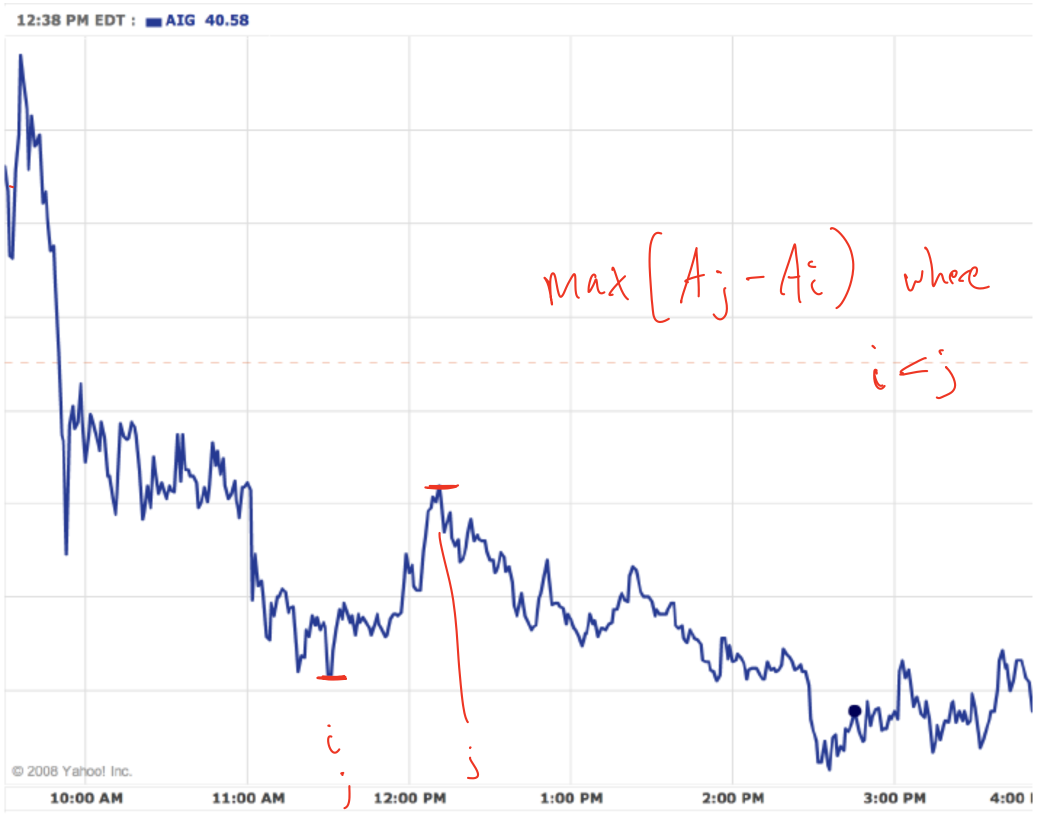
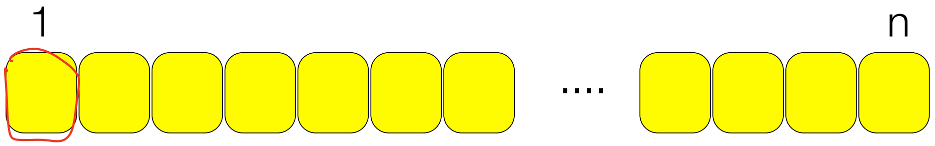

Open 27.46
Close 27.07
Low 26.65
High 27.69
Vol 33.79K
% Chg -75.68%
BPT 27.07
YAHOO! FINANCE
D



$\max(A_j - A_i)$ where

$i < j$

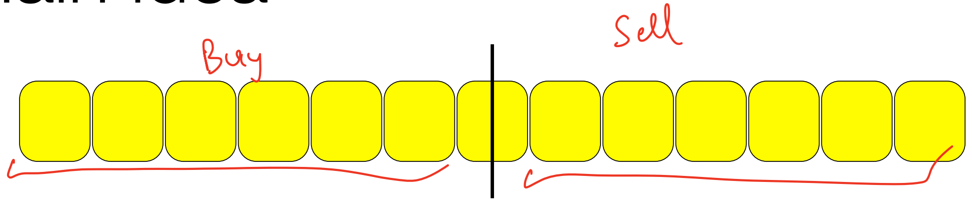
input: array of n numbers



goal: find the indicies i, j such that $i \leq j$ which maximizes $A_j - A_i$.

This is the best trade to make on this day.

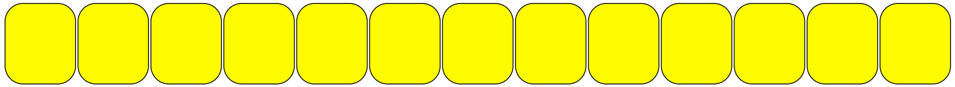
Main idea



Find the best arbitrage opportunity in LEFT and in RIGHT.

Then look for opportunities when you buy on the left and sell on the right.

first attempt



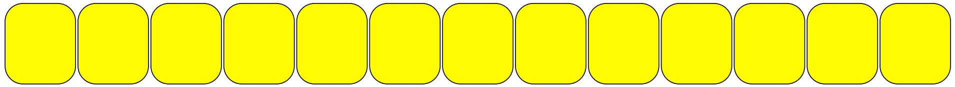
```
arbit(A[1...n])
```

first attempt

```
arbit(A[1...n])
    base case if |A|<=2
    lg = arbit(left(A))        - $T(n/2)$
    rg = arbit(right(A))       -
    minl = min(left(A))        ] $\Theta(n)$
    maxr = max(right(A))       ]
    return max{maxr-minl,lg,rg}
                A#   Morning  M  A
```

$$T(n) = 2T(\tfrac{n}{2}) + \Theta(n) = \Theta(n \log n)$$

first attempt: time $\Theta(n \log n)$



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    return max{maxr-minl,lg,rg}
```

T(n) = $\underline{2T(n/2)} + \underline{\Theta(n)}$

better approach

These are the steps that are taking $\Theta(n) time$

better approach

Can we find a solution that has $T(n) = \underline{2T(n/2) + O(1)}$?

These are the steps that are taking $\Theta(n) time$

better approach

Can we find a solution that has T(n) = 2T(n/2) + O(1) ?

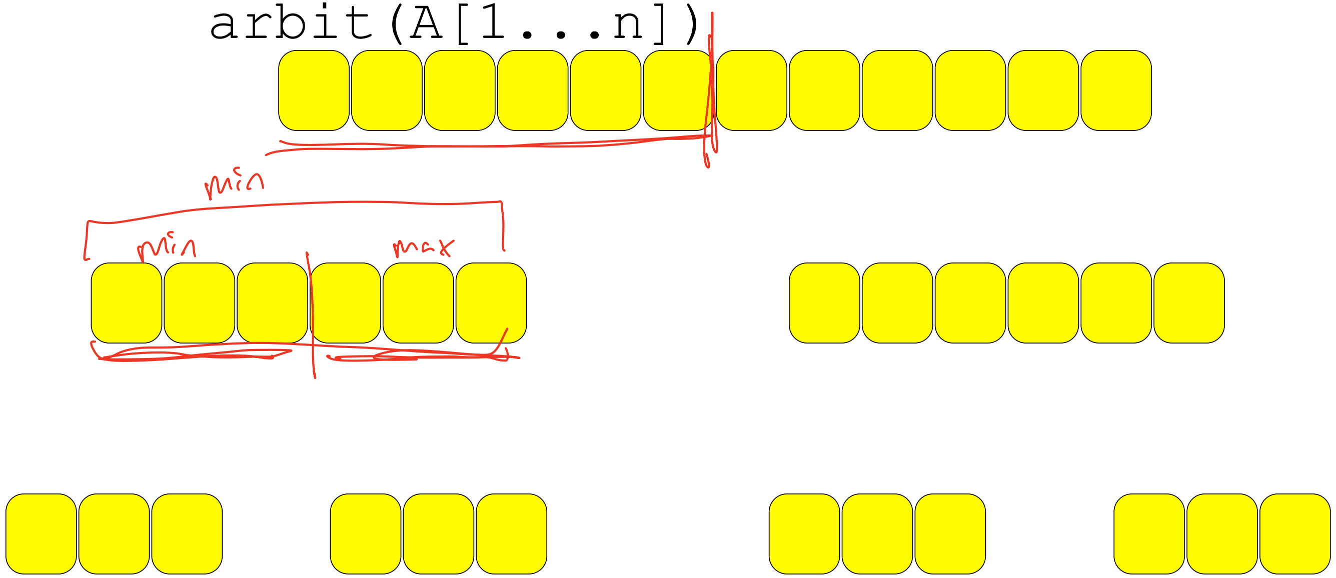
```
minl = min(left(A))
maxr = max(right(A))

return max{maxr-minl,lg,rg}
```

These are the steps that are taking $\Theta(n) time$

first attempt

`arbit(A[1...n])`



second attempt

```
arbit2(A[1…n])                    // Returns {best trade,min,max}
  base case if |A|<=2

lg, lmin, lmax ← arbit2(left(A))        T(n/2)
rg, rmin, rmax ← arbit2(right(A))       T(n/2)
mid = rmax - lmin                        1

return  max{lg, rg, mid},                1
        min{lmin, rmin},                 1
        max{lmax, rmax}                  1
```

second attempt

```
arbit2(A[1...n])                    // Returns {best trade,min,max}
    base case if |A|<=2, ...
    (lg,minl,maxl) = arbit2(left(A))
    (rg,minr,maxr) = arbit2(right(A))
    return max{maxr-minl,lg,rg},
           min{minl, minr},
           max{maxl, maxr}
```

$$T(n) = 2T(\tfrac{n}{2}) + \Theta(1) \Rightarrow T(n) = \Theta(n)$$

by Masters case $\underline{1}$.

second attempt

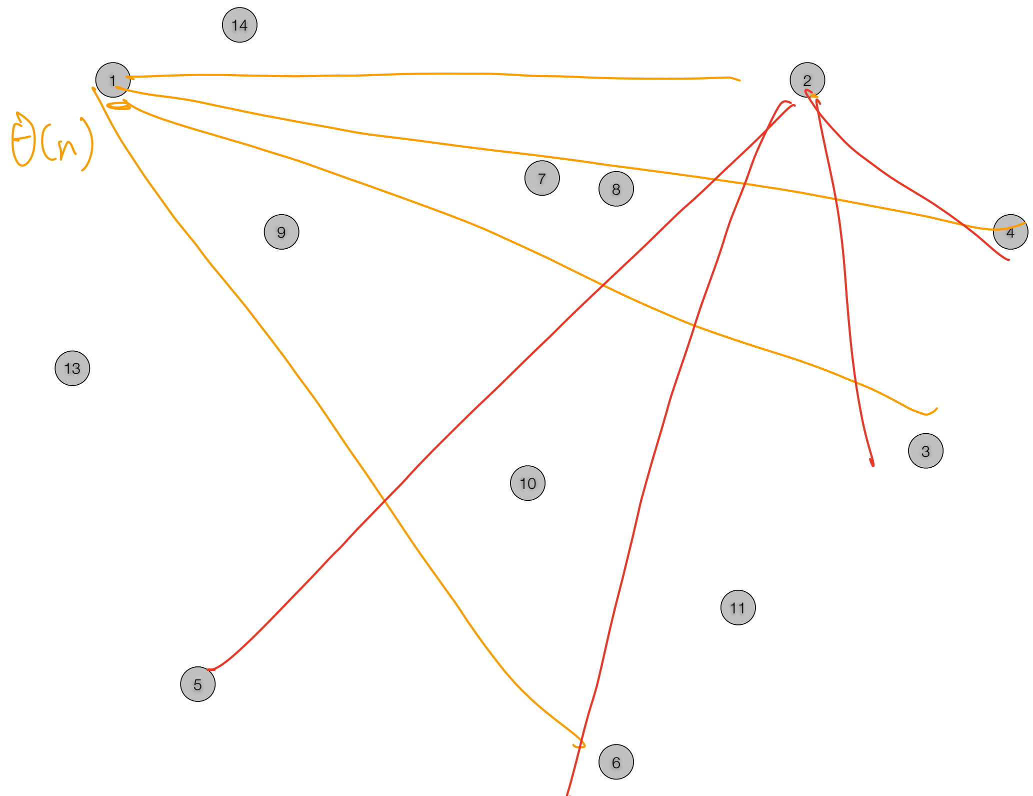
```
arbit2(A[1...n])                    // Returns {best trade,min,max}
    base case if |A|<=2, …
    (lg,minl,maxl) = arbit2(left(A))
    (rg,minr,maxr) = arbit2(right(A))
    return max{maxr-minl,lg,rg},
           min{minl, minr},
           max{maxl, maxr}
```

New runtime is $T(n) = 2T(n/2) + \Theta(1) = \Theta(n)$

closest pair

of points

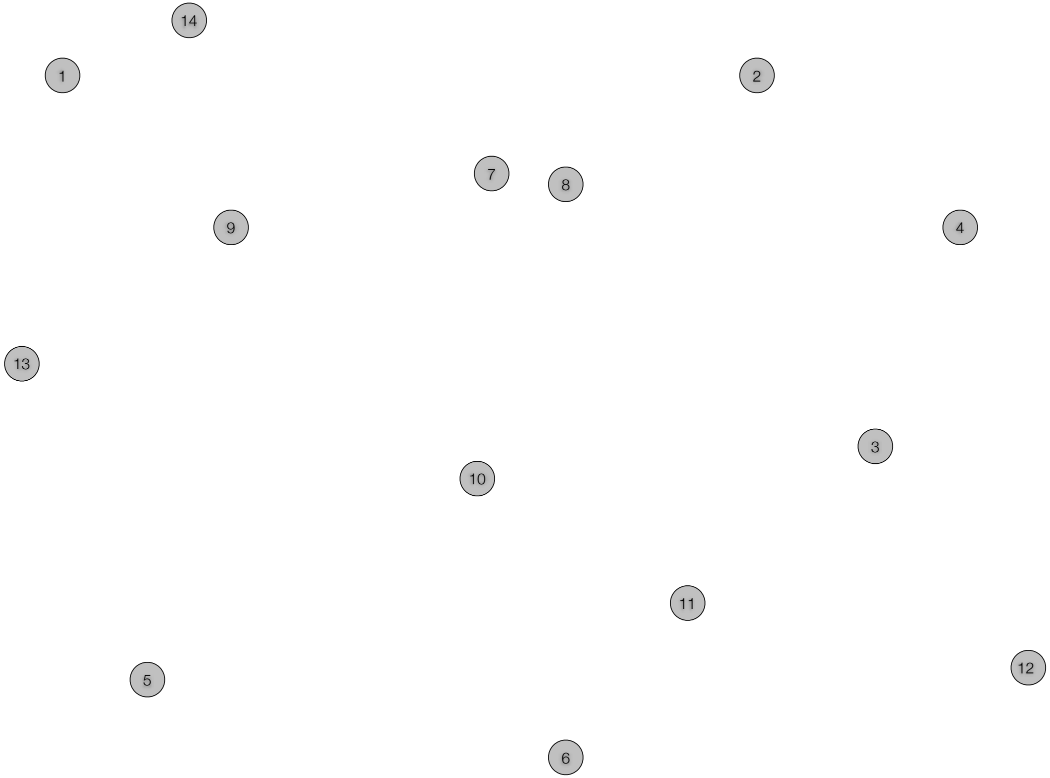
Simple brute force approach takes $\underline{\Theta(n^2)}$



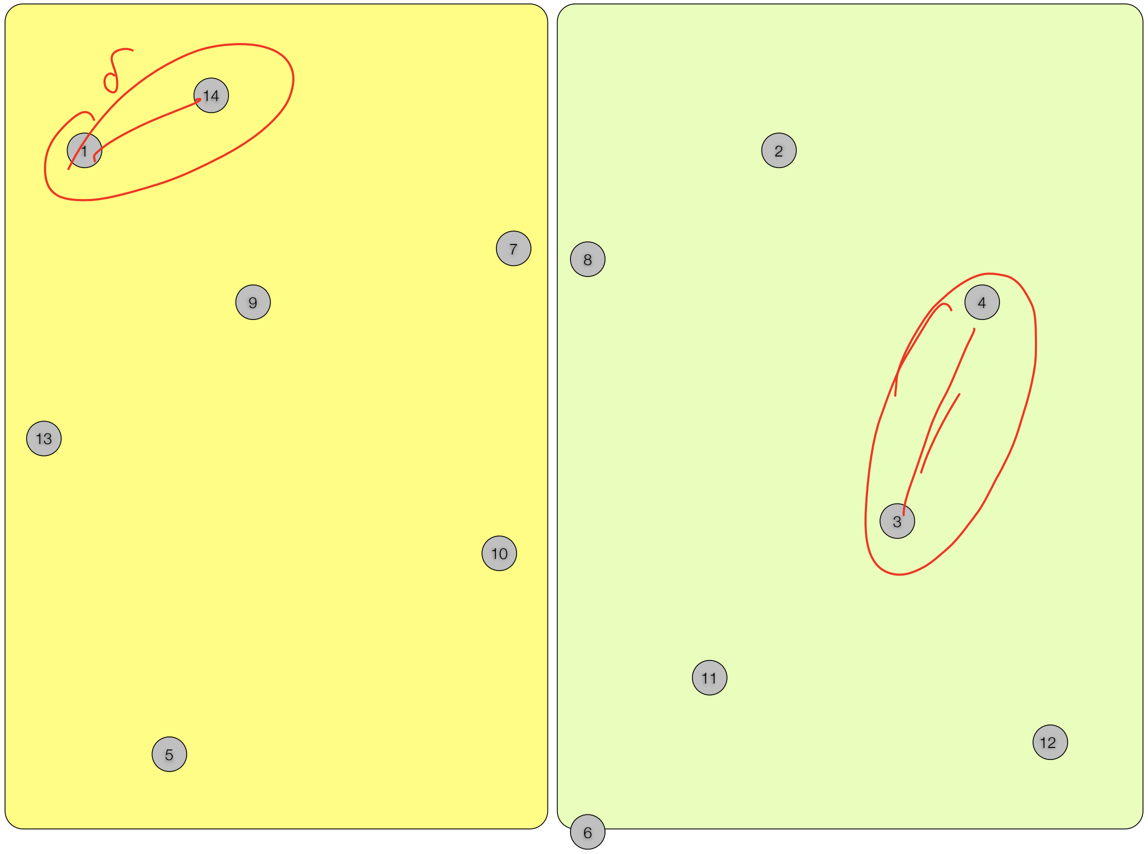
Assume all points have distinct x & y coordinates.

solve the large problem by

solving smaller problems
and combining solutions

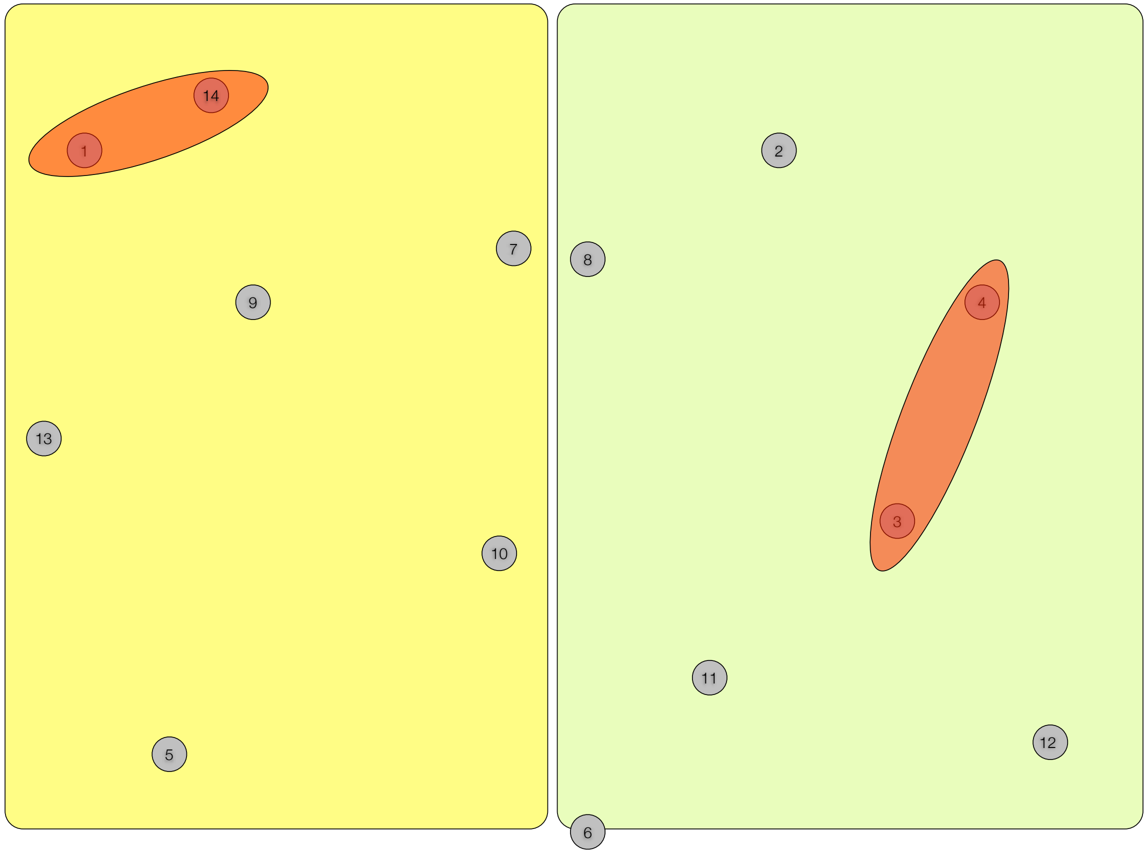

14
1
2
7
8
9
4
13
3
10
11
12
5
6

Divide & Conquer



Divide & Conquer

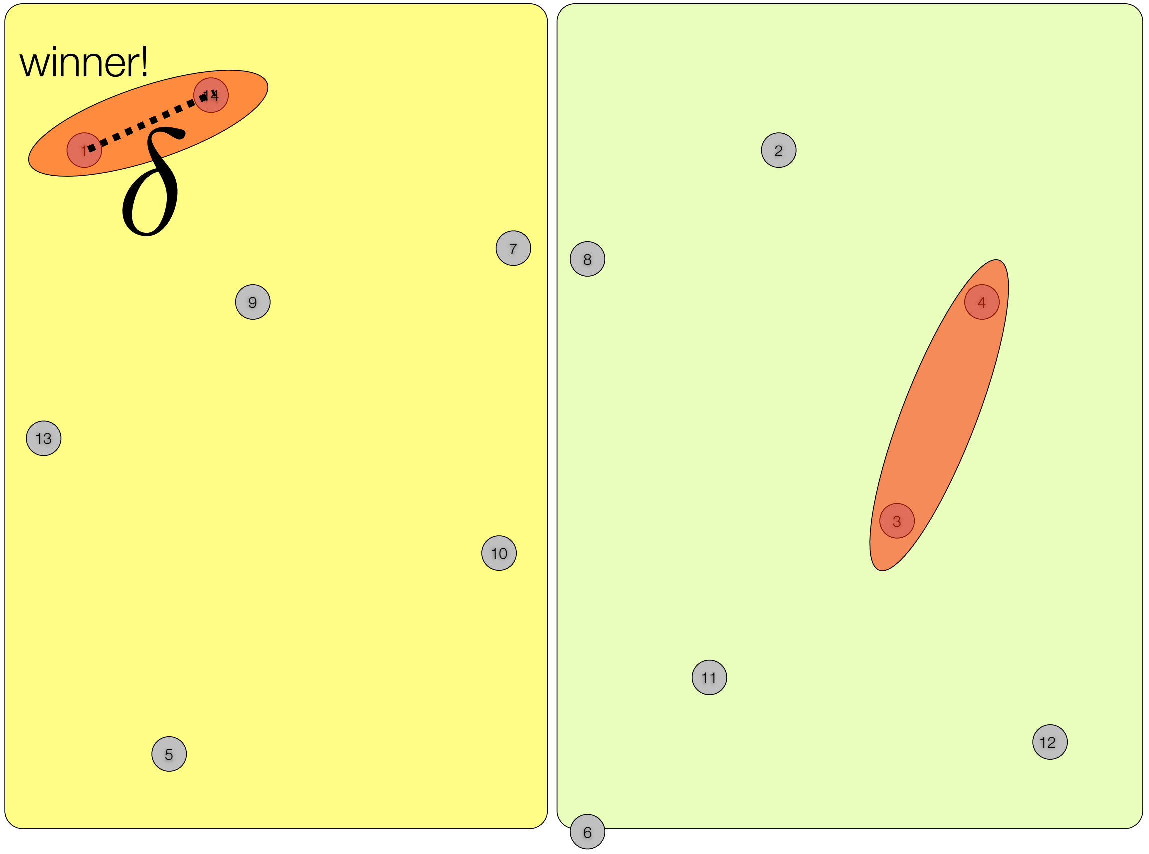
Find closest pair on the left half.



Find closest pair on the right half.

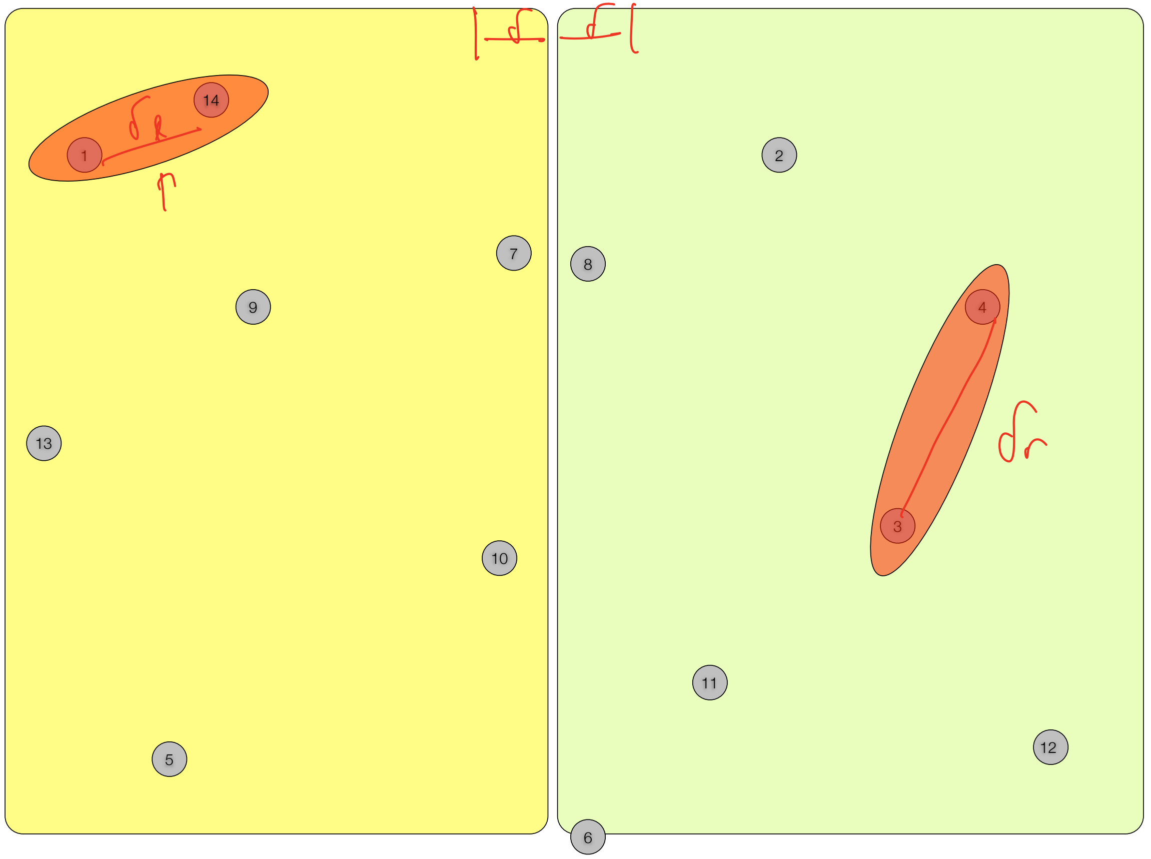
Divide & Conquer

Find closest pair on the left half.



Find closest pair on the right half.

Divide & Conquer

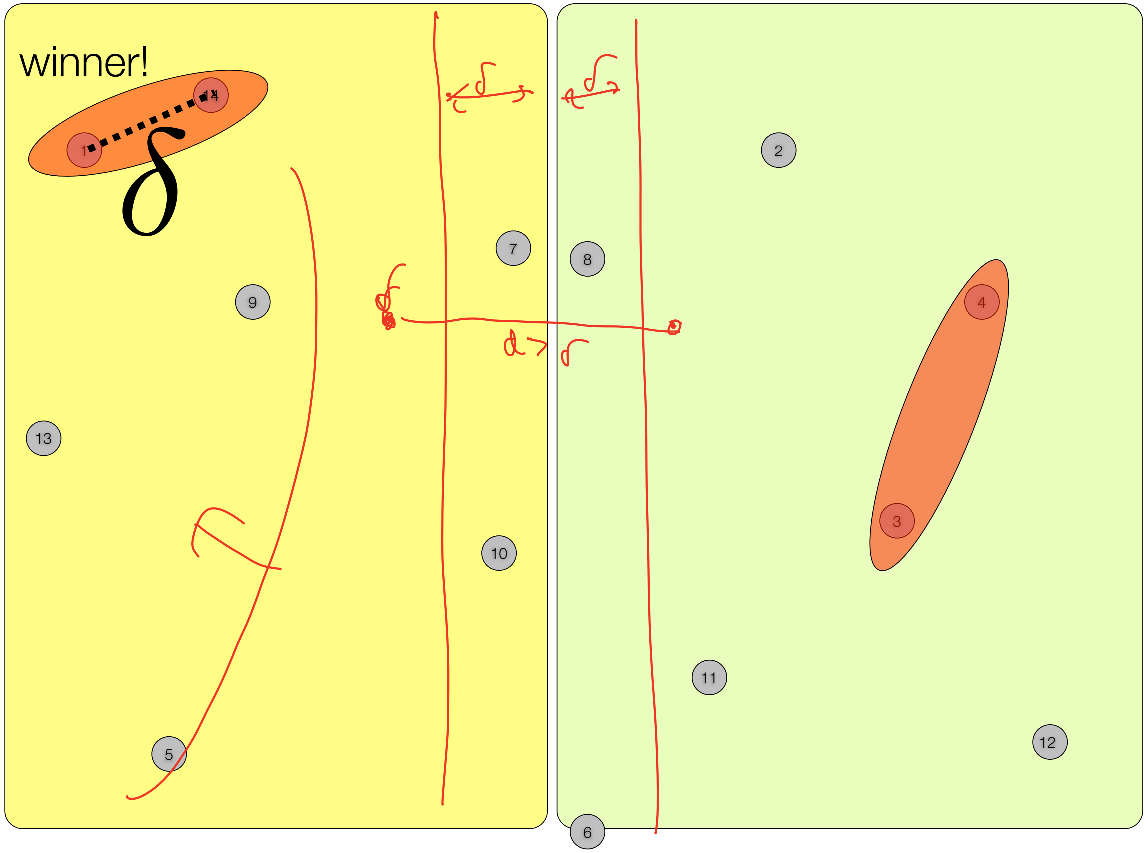


Now look for pairs between the left and right that are closer.

$\delta = \min(\delta_\ell, \delta_r)$

Divide & Conquer

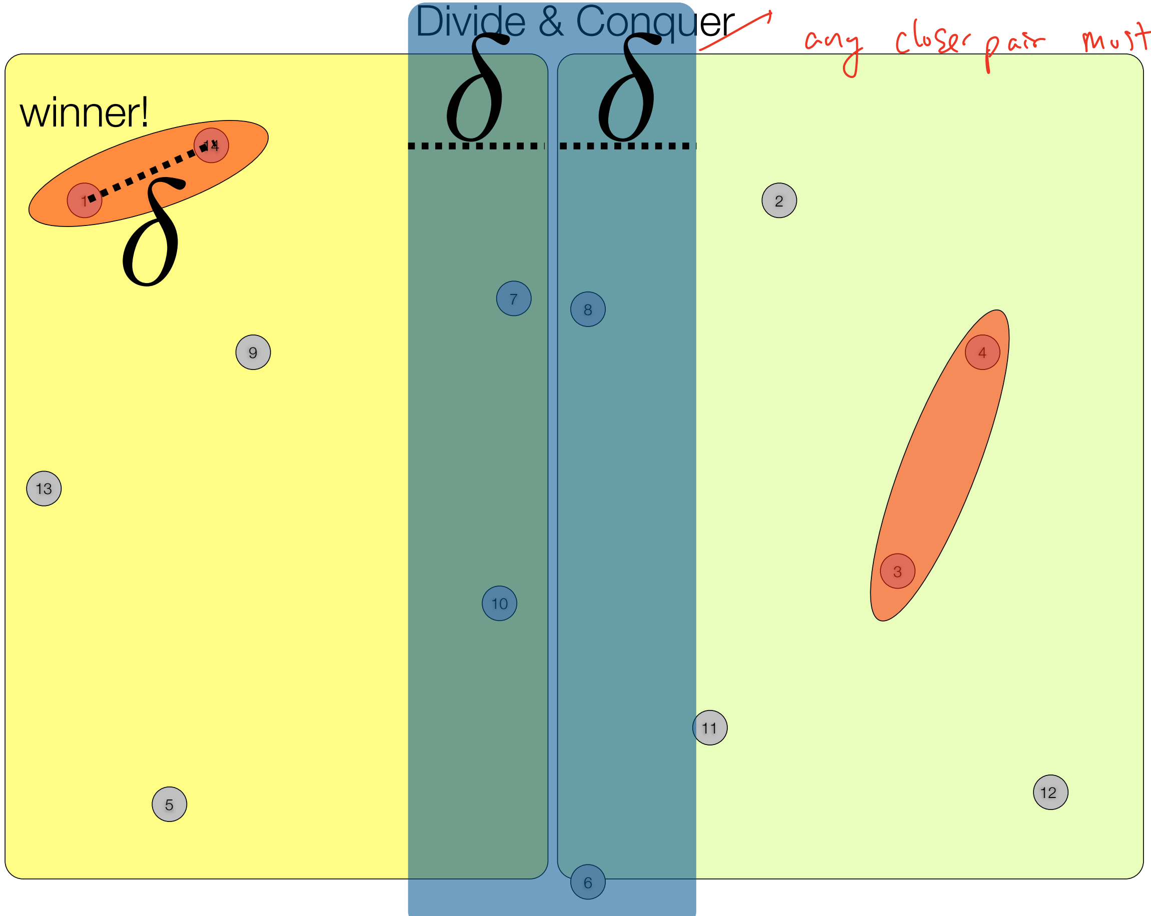
Now look for pairs between the left and right that are closer.



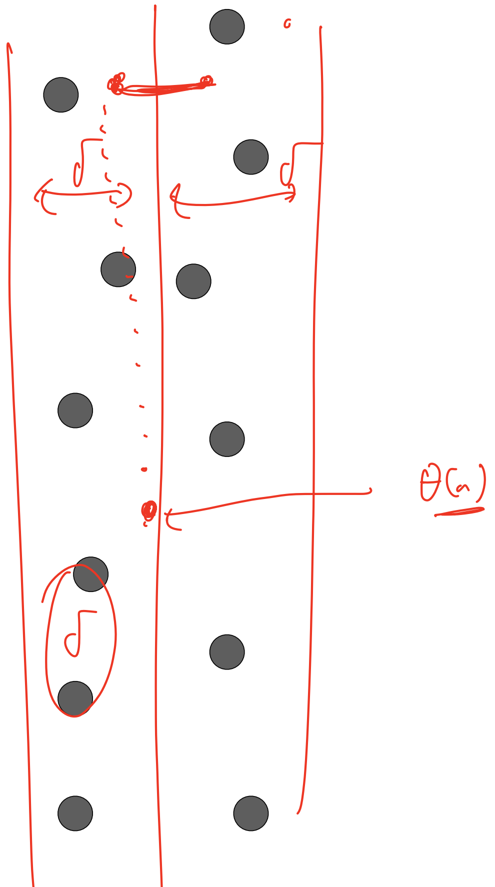
Divide & Conquer

any closer pair must be in this region.

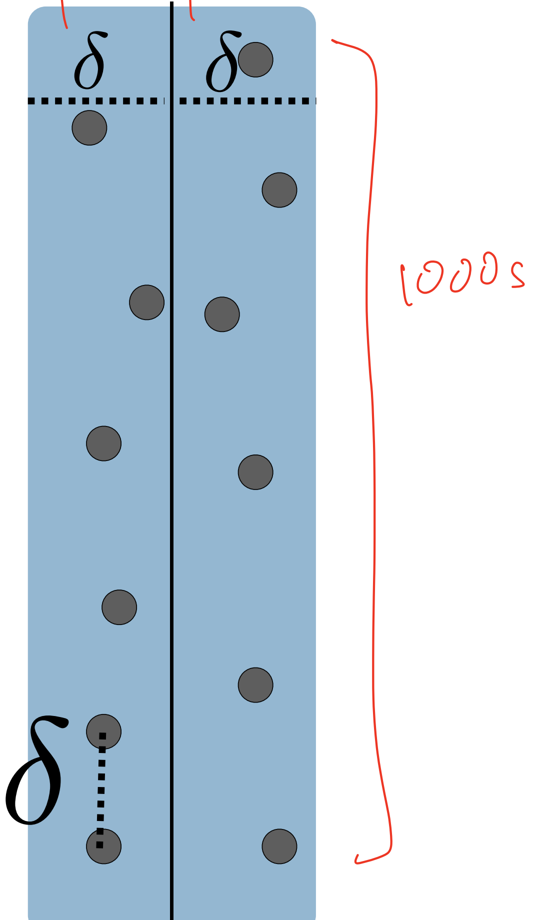
Now look for pairs between the left and right that are closer.



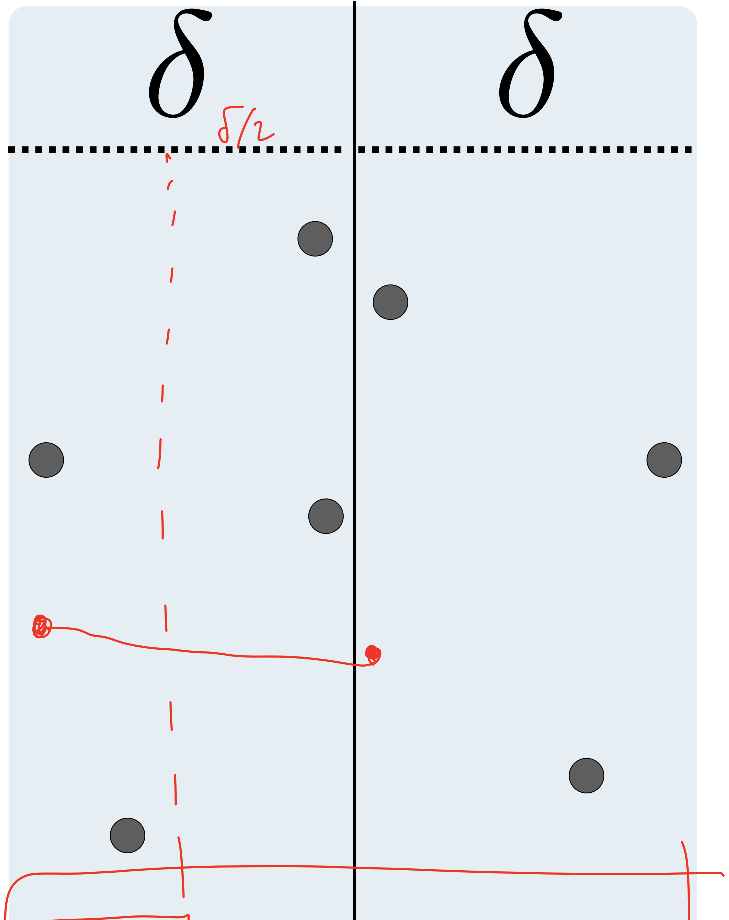
What if the input points are like this?

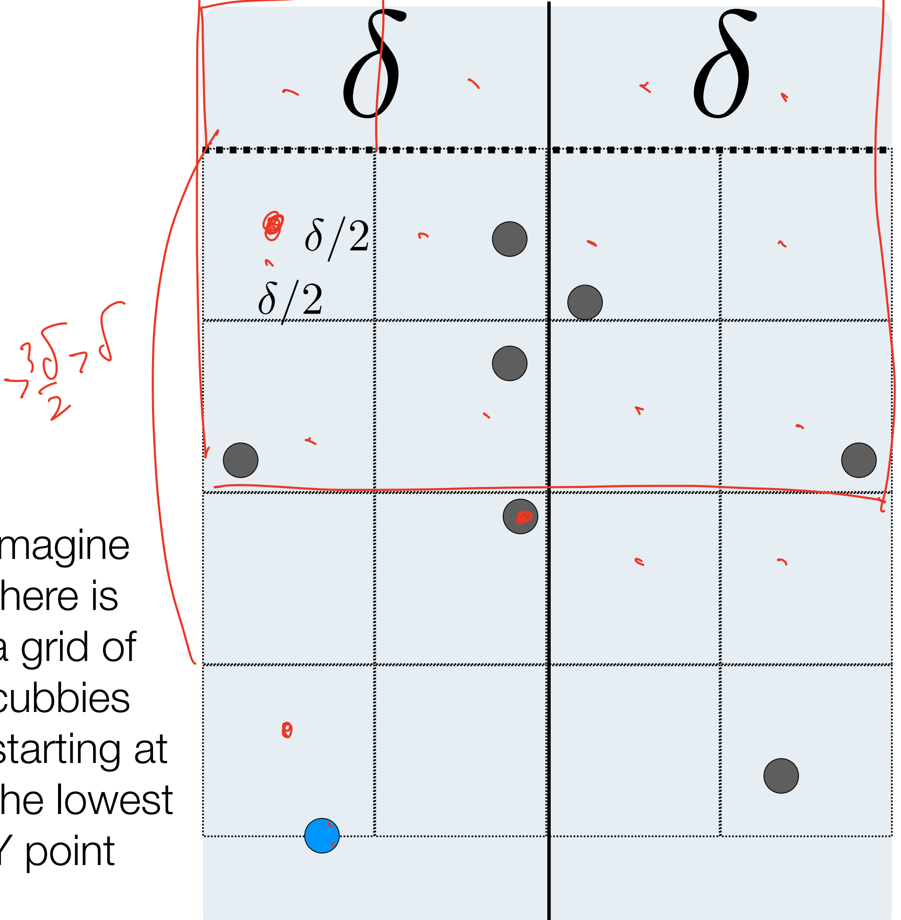


Then all of the points are within δ of the middle. If we need to check all of the points, we are back to $O(n^2)$

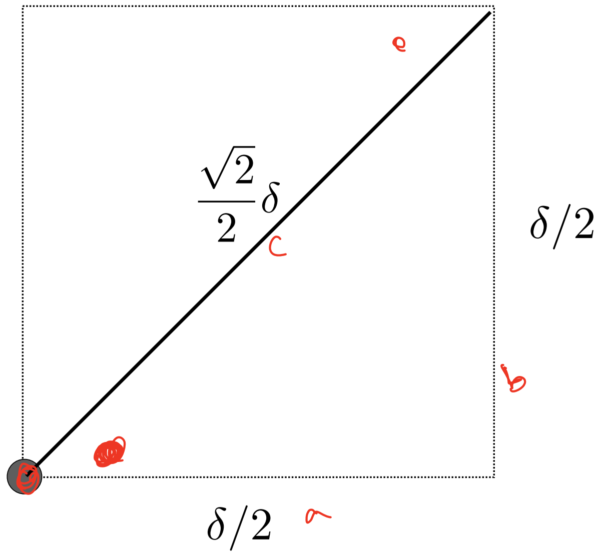


But we have extra information! The only candidates for closest pair are within δ of each other. How can we use this info?





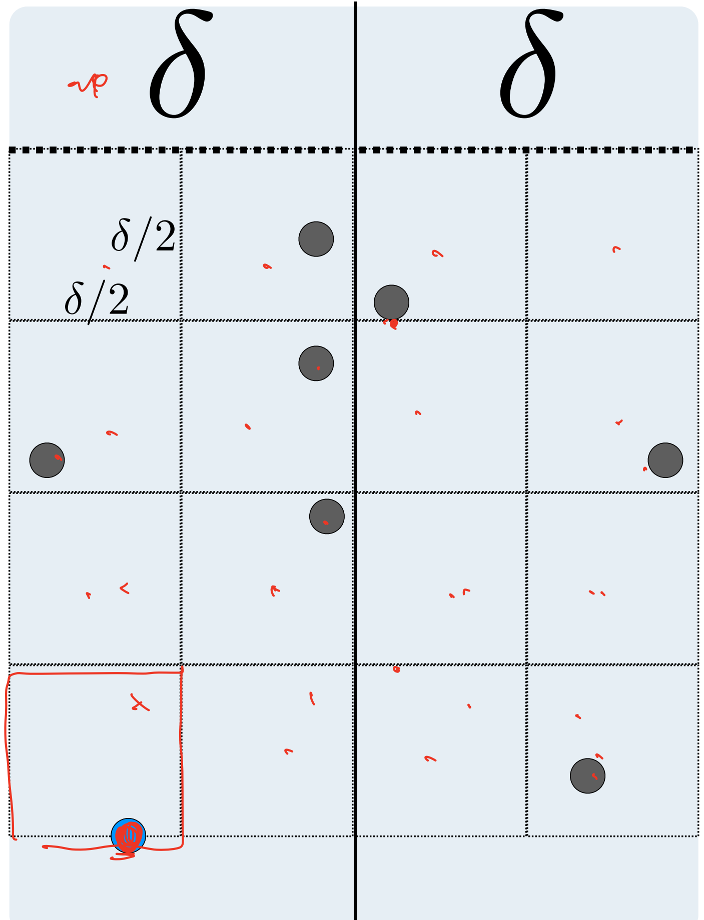
Imagine there is a grid of cubbies starting at the lowest Y point



A grid this size has a diagonal that is smaller than delta. That means each grid box can only have 1 point in it.

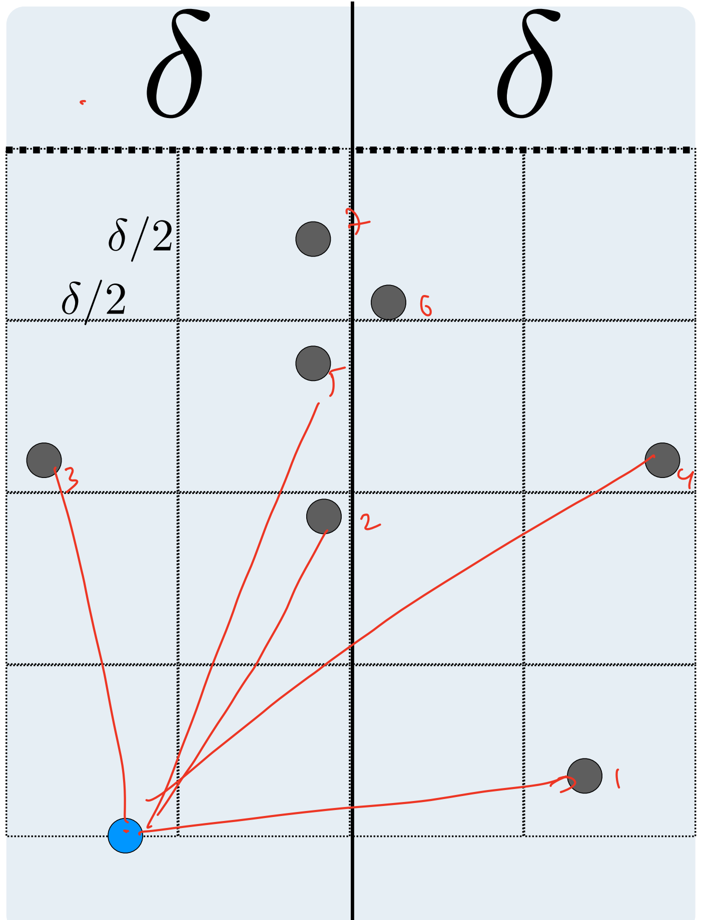
If there was another, then the closest pair on the left or right would have been this pair.

$$\left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta}{2}\right)^2 = \sqrt{\frac{2\delta^2}{4}} = \frac{\sqrt{2}\cdot\delta}{2} < \delta$$



FACT: At most 1 point in each cubby

Claim: If there is another point closer than δ, then it must be among the next 15 points sorted by y-coordinate.



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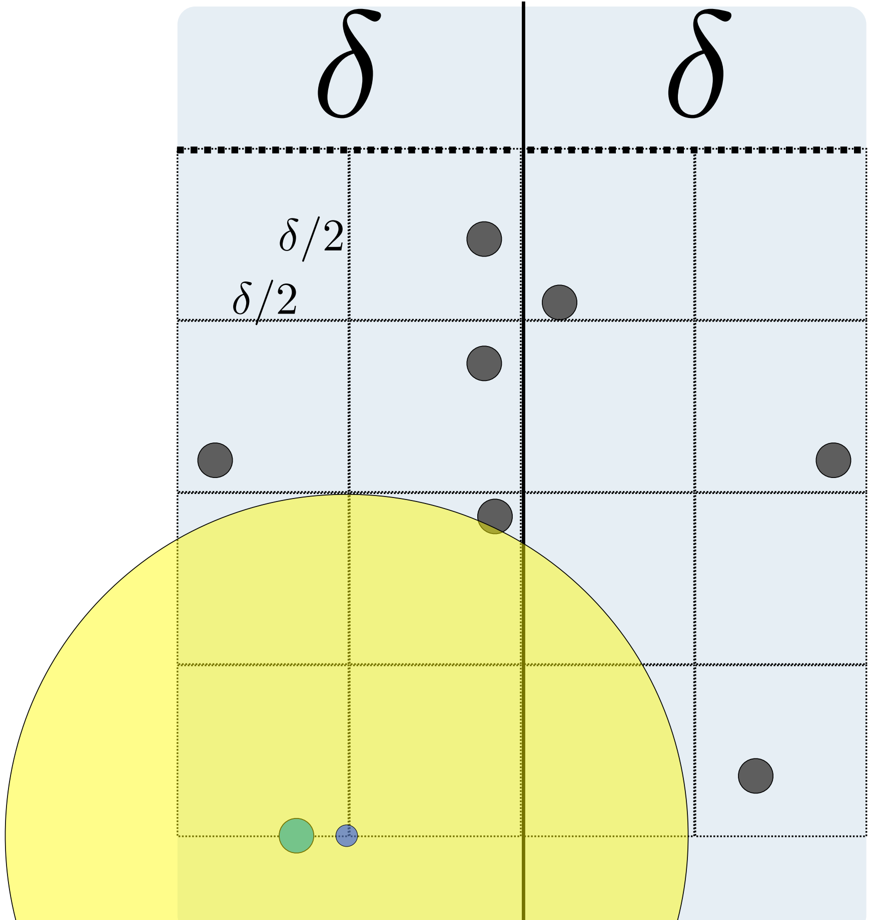
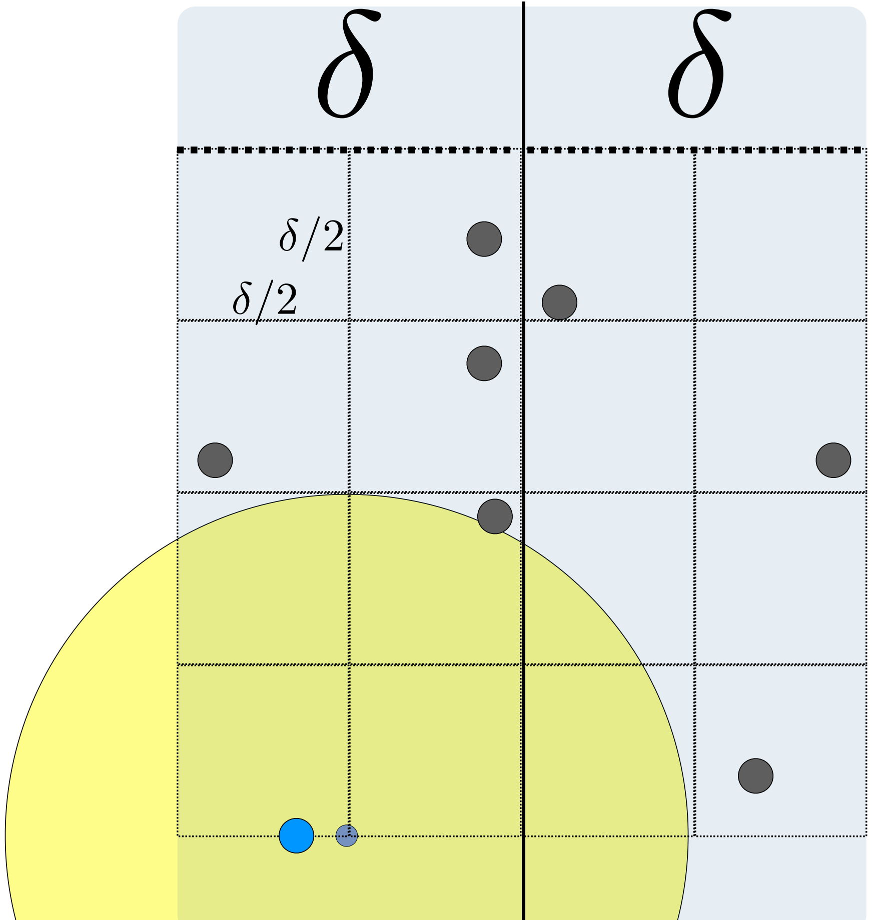


FACT: <=1 point per cubby



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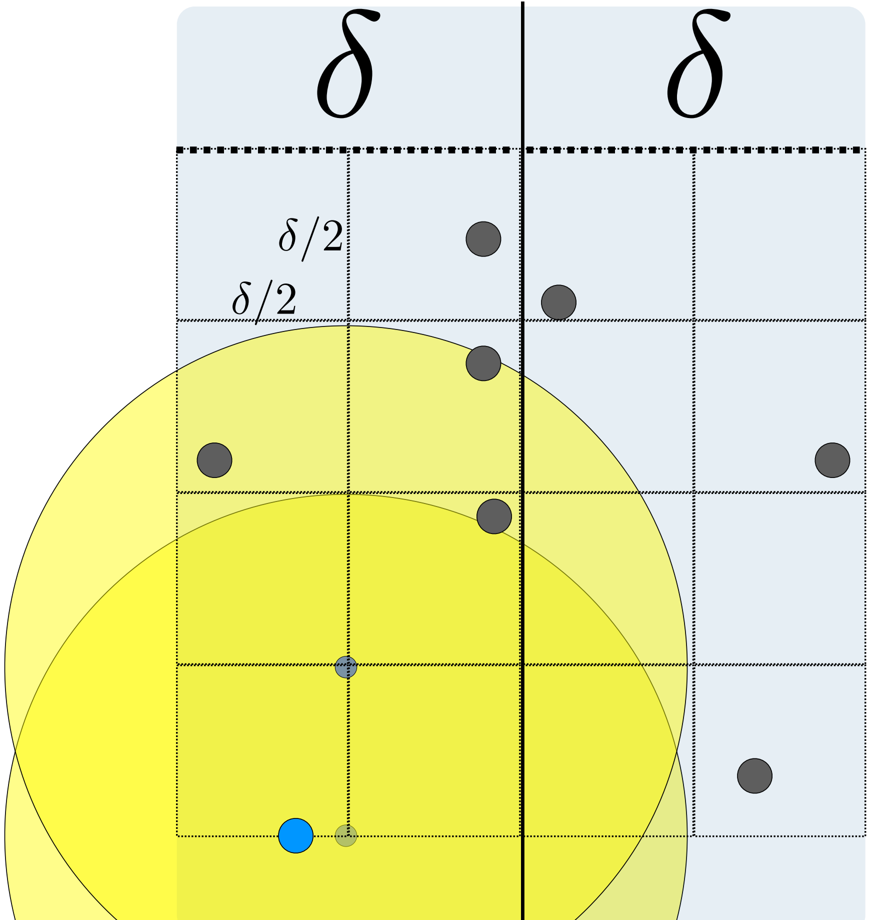

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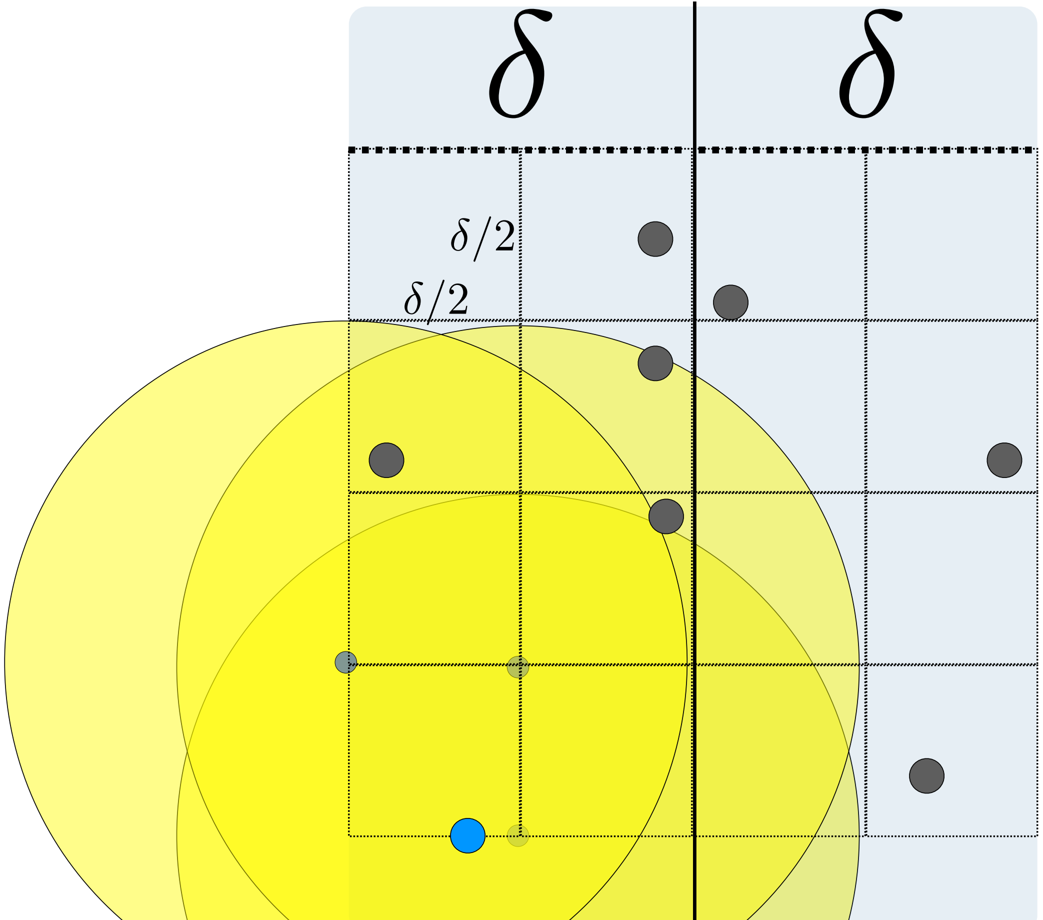

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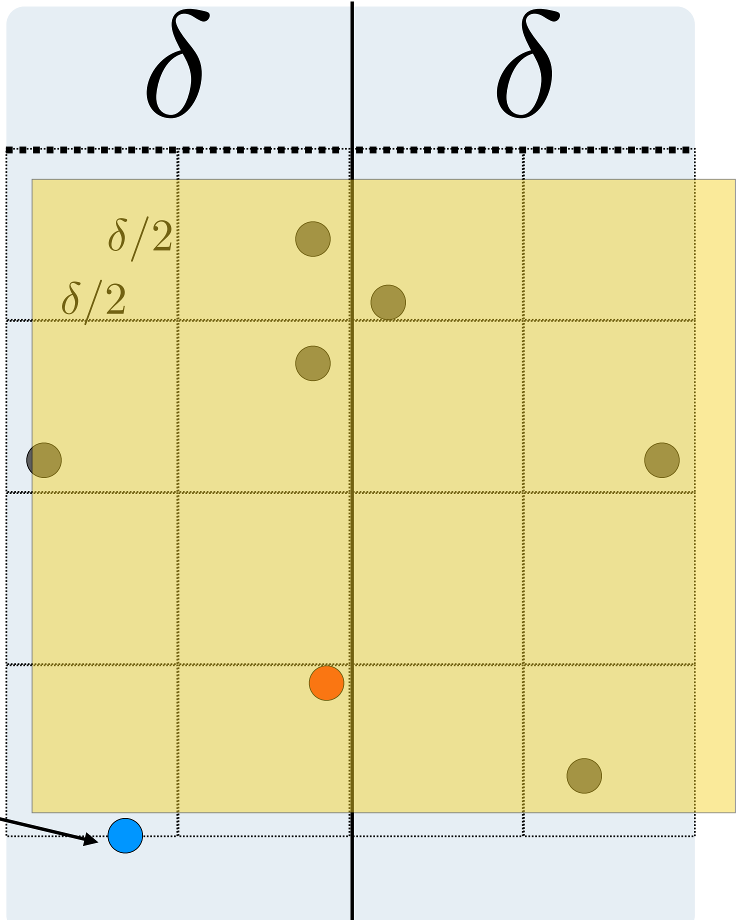
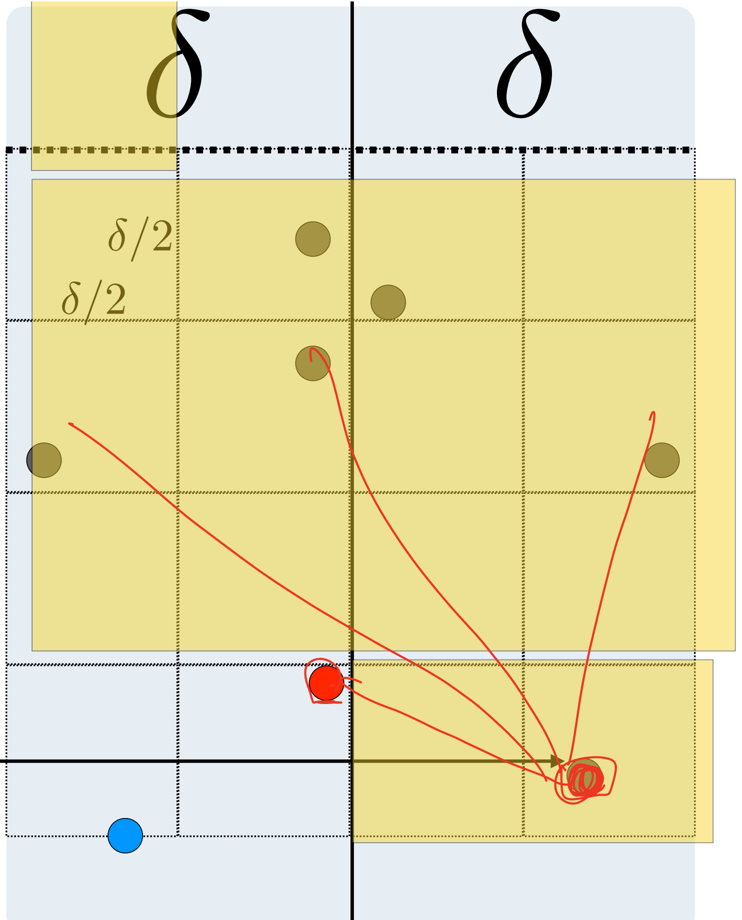


Visit its by y-order

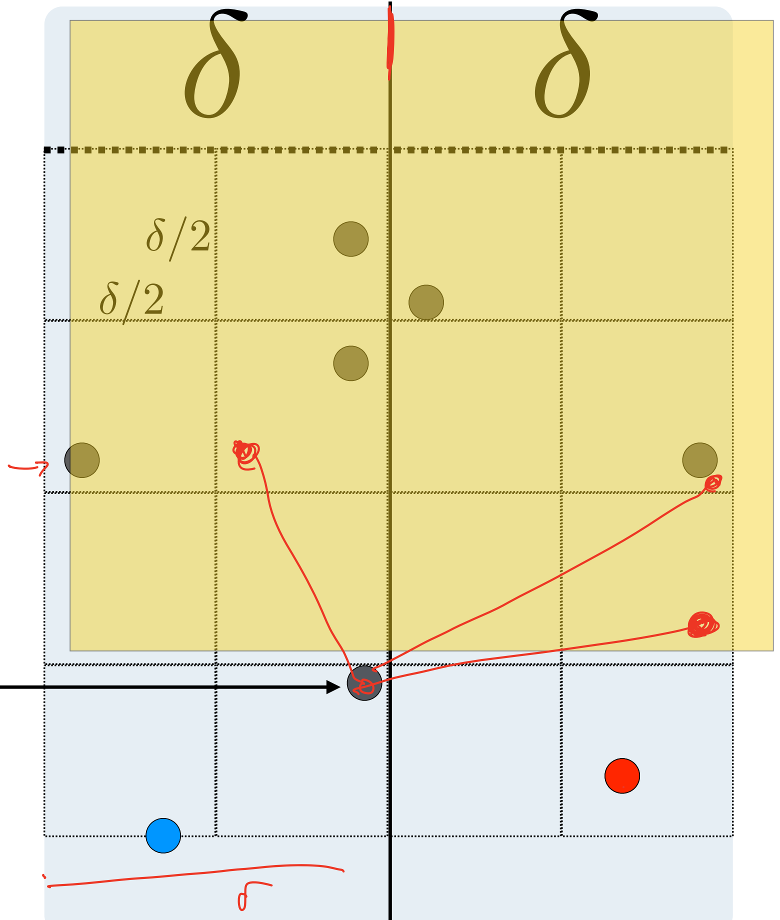
Start

Check the next 15 points



Check the next 15 points

Next



Check the next 15 points

Closest(P)

)

```
Closest(P)                     // returns the minimum distance delta
                               // and the closest pair Romeo, Juliet
    Base Case: If <8 points, brute force.
    1. Let q be the "middle-element" of points
    2. Divide P into Left, Right according to q
    3. delta,r,j = MIN(Closest(Left) ,  Closest(Right) )
    4. Mohawk = { Scan P, add pts that are <delta from q.x }
    5. For each point p in Mohawk (in y-order):   from bottom to top.
          Compute distance between p and its next 15 neighbors
          Update delta,r,j if any pair (~~x~~ p,y) is < delta
    6. Return (delta,r,j)
```

$\Theta(n)$

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

```
Closest(P)                              // returns the minimum distance delta
                                        // and the closest pair Romeo, Juliet
    Base Case: If $\overset{<2}{\underline{<8}}$ points, brute force.
    1. Let q be the "middle-element" of points
    2. Divide P into Left, Right according to q
    3. delta,r,j = MIN(Closest(Left) ,   Closest(Right)  )

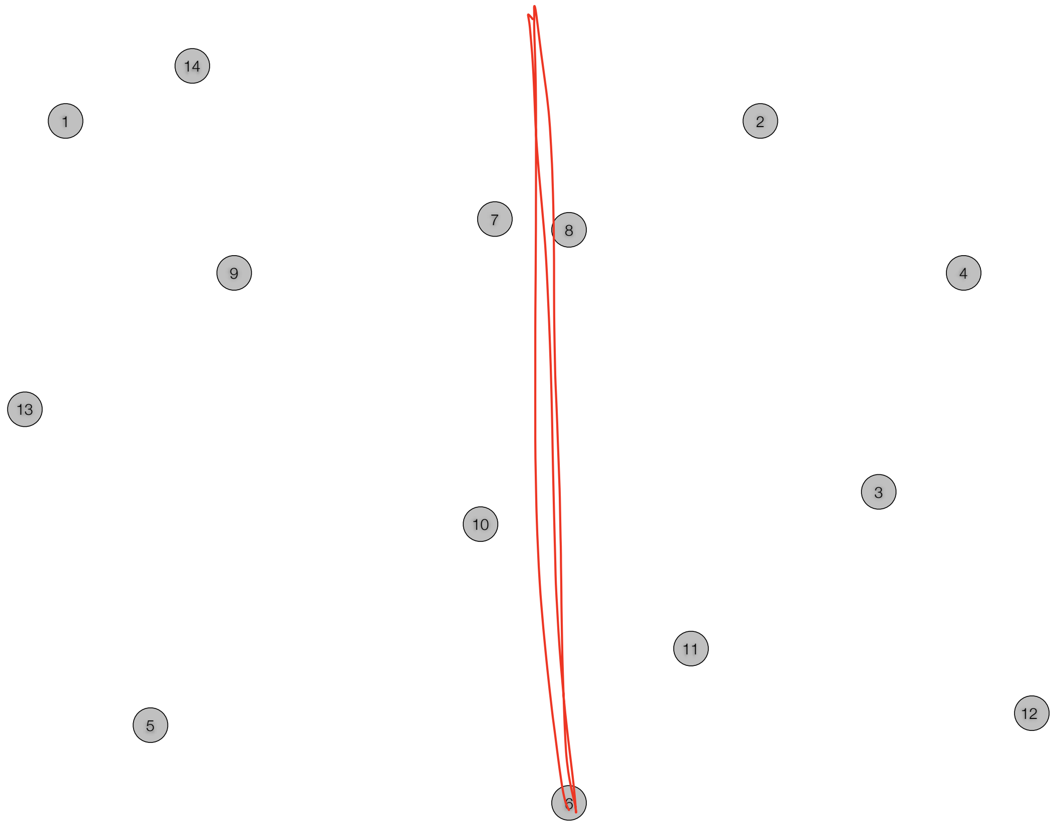
    4. Mohawk = { Scan P, add pts that are <delta from q.x }

    5. For each point p in Mohawk (in y-order):
          Compute distance between p and its next 15 neighbors
          Update delta,r,j if any pair (x,y) is < delta

    6. Return (delta,r,j)
```

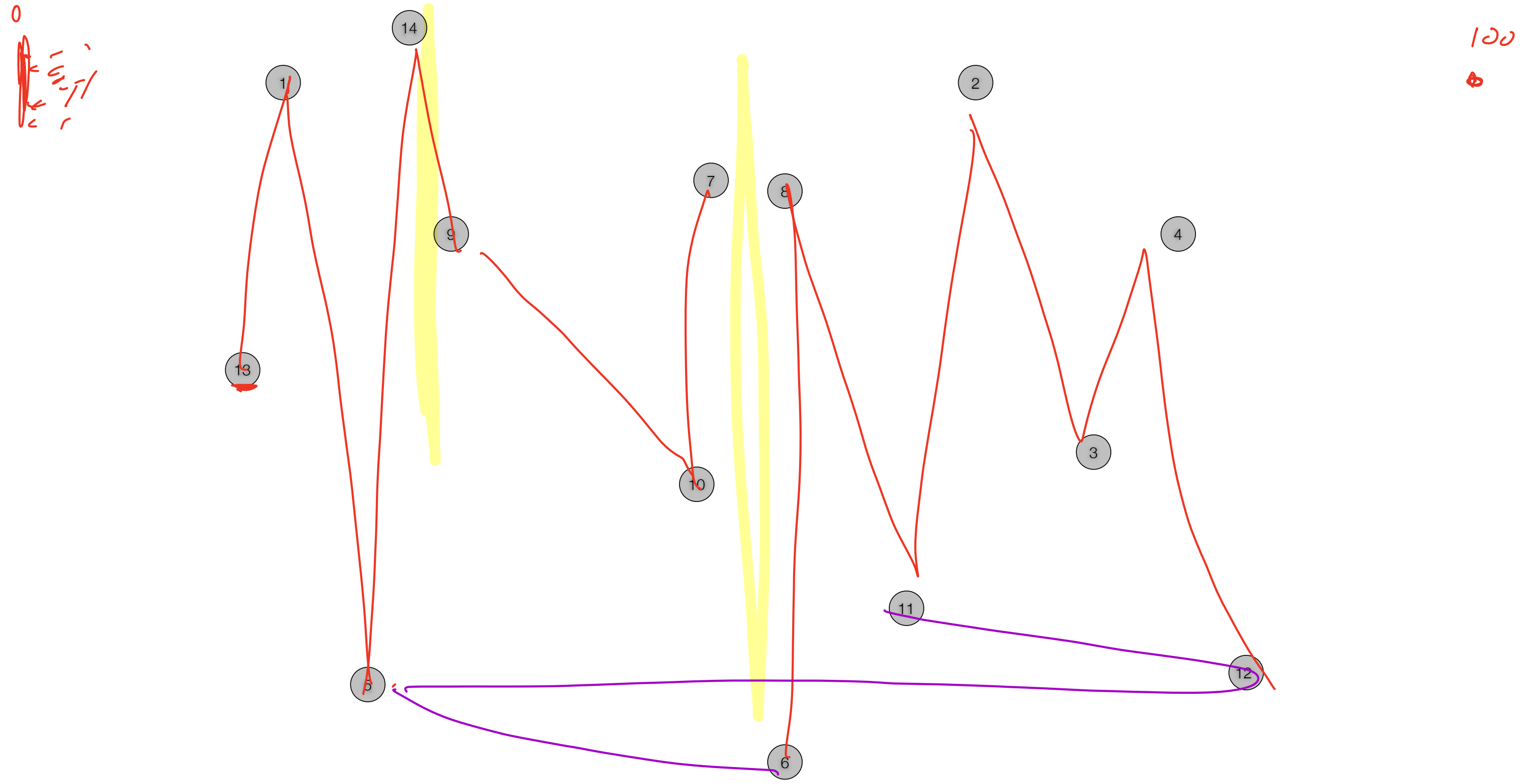
Can be reduced to 7!

Details: How to do step 1?



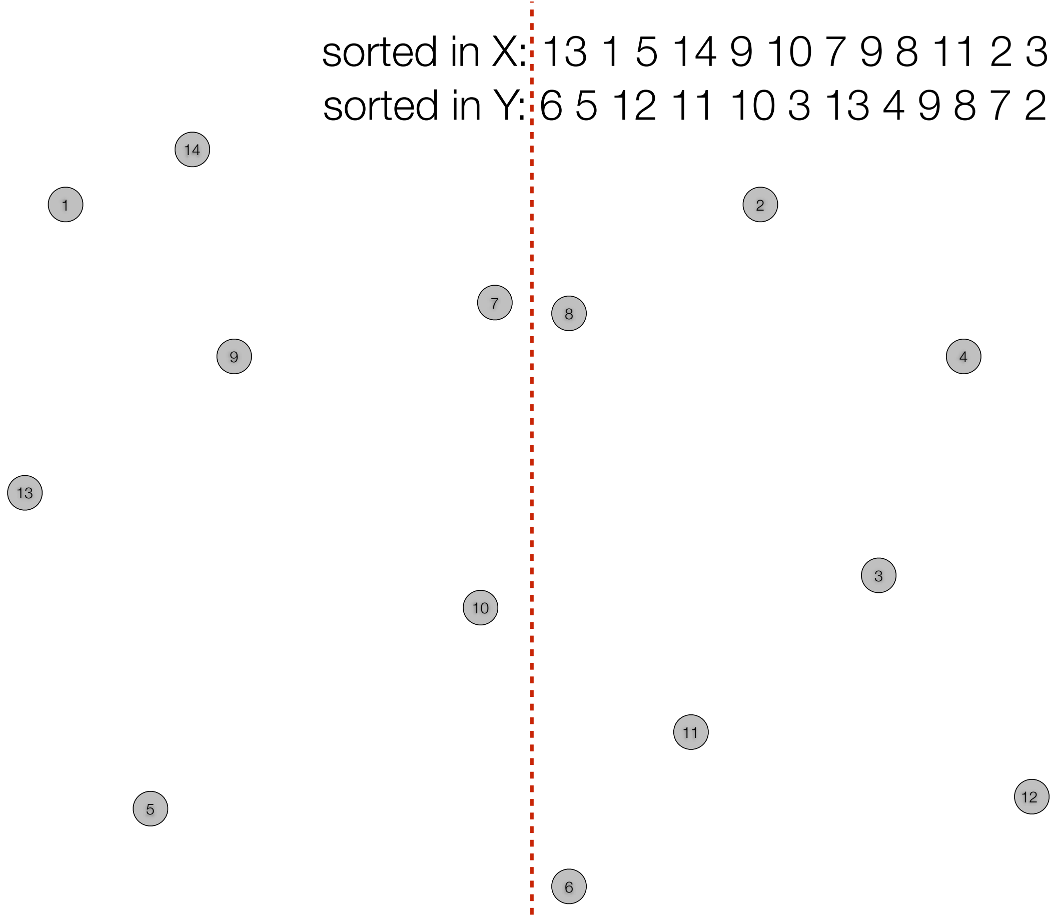
Points sorted in X: 13 1 5 14 9 10 7 6 8 11 2 3 4 12

Points sorted in Y: 6 5 12 11 10 3 13 4 9 8 7 2 1 14



sorted in X: 13 1 5 14 9 10 7 9 8 11 2 3 4 12

sorted in Y: 6 5 12 11 10 3 13 4 9 8 7 2 1 14



```
ClosestPair(P)
    Compute Sorted-in-X list SX        Θ(nlogn)
    Compute Sorted-in-Y list SY        Θ(nlogn)
    Closest(P,SX,SY)        Θ(nlogn)
```

$\Theta(n\log n)$

```
Closest(P,SX,SY)
    Let q be the middle-element of SX
    Divide P into Left, Right according to q
    delta,r,j = MIN(Closest(Left, LX, LY)   Closest(Right, RX, RY))

    Mohawk = { Scan SY, add pts that are delta from q.x }
    For each point p in Mohawk (in order): by Sy from bottom to top.
        Compute distance between p and its next 15 neighbors
        Update delta,r,j if any pair (x,y) is < delta

    Return (delta,r,j)
```

```
Closest(P,SX,SY)
    Let q be the middle-element of SX
    Divide P into Left, Right according to q
    delta,r,j = MIN(Closest(Left, LX, LY)   Closest(Right, RX, RY))

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        Compute distance between p and its next 15 neighbors
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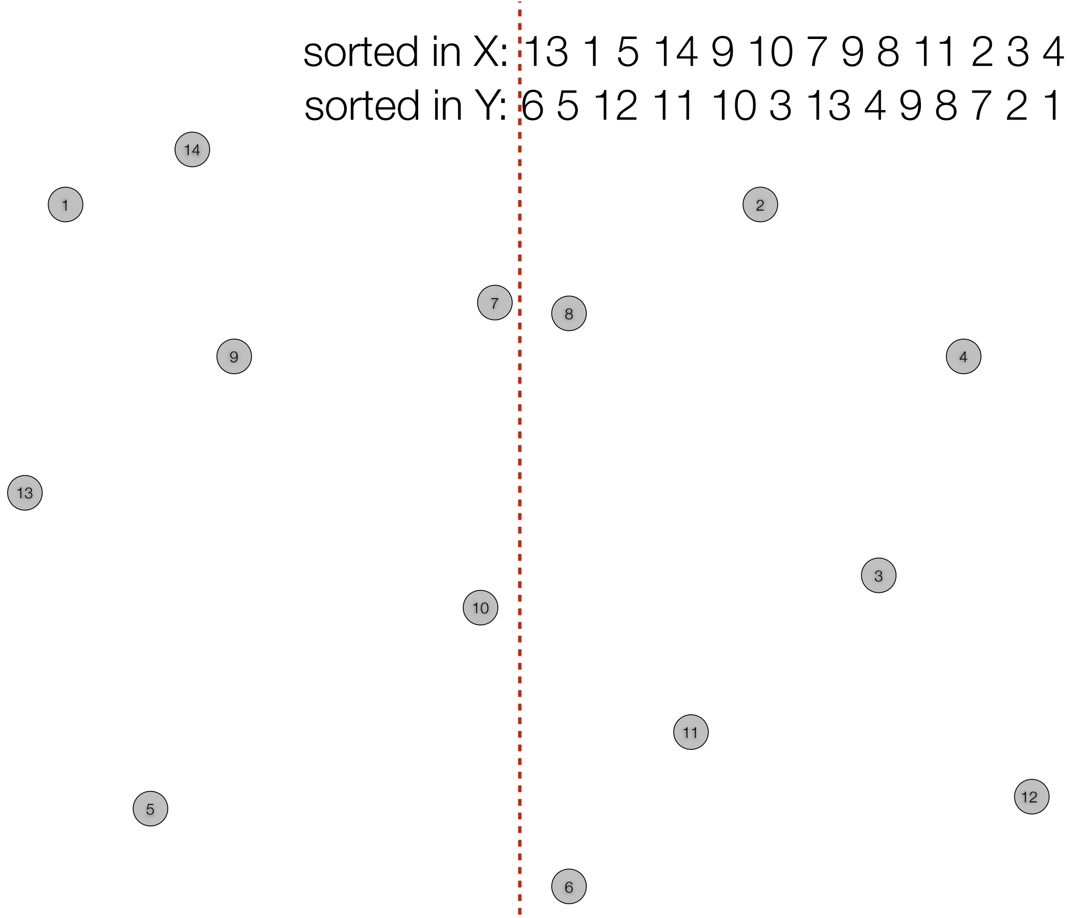
    Return (delta,r,j)
```

Can be reduced to 7!

sorted in X: 13 1 5 14 9 10 7 9 8 11 2 3 4 12

sorted in Y: 6 5 12 11 10 3 13 4 9 8 7 2 1 14



```
Closest(P,SX,SY)
    Let q be the middle-element of SX
    Divide P into Left, Right according to q. Scan to get LY, RY.
    delta,r,j = MIN(Closest(Left, LX, LY)   Closest(Right, RX, RY))

    Mohawk = { Scan SY, add pts that are delta from q.x }

    For each point p in Mohawk (in order):
        Compute distance between p and its next 15 neighbors
        Update delta,r,j if any pair (x,y) is < delta

    Return (delta,r,j)
```

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    Return (delta,r,j)
```

Can be reduced to 7!

sorted in X: 13 1 5 14 9 10 7 9 8 11 2 3 4 12

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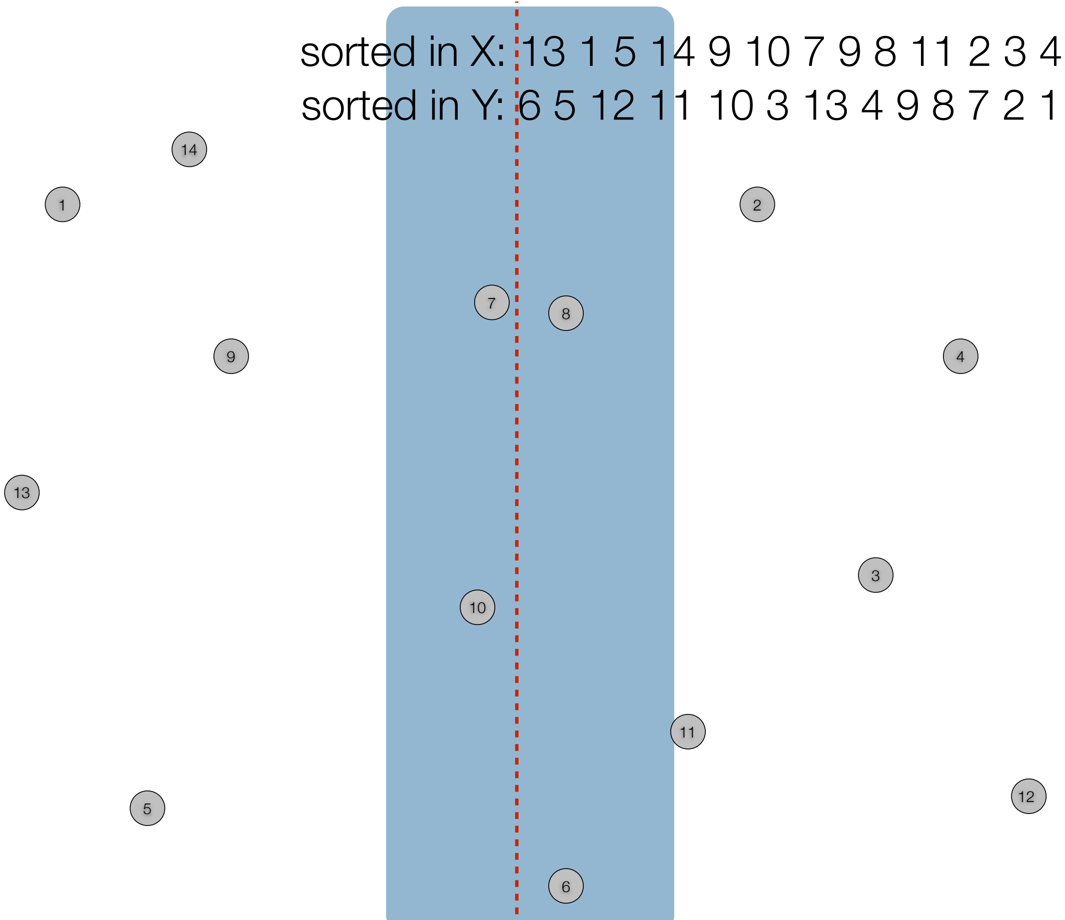

```
Closest(P,SX,SY)

    Let q be the middle-element of SX

    Divide P into Left, Right according to q. Scan to get LY, RY.

    delta,r,j = MIN(Closest(Left, LX, LY)   Closest(Right, RX, RY))


    Mohawk = { Scan SY, add pts that are delta from q.x }

    For each point p in Mohawk (in order):
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    Return (delta,r,j)
```

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Can be reduced to 7!

Running time for Closest pair algorithm

$$T(n) =$$

Running time for Closest pair algorithm

$$T(n) =$$

$$T(n) = 2T(n/2) + \Theta(n) = \Theta(n \log n)$$

@author Robert Sedgewick
@author Kevin Wayne

```
public ClosestPair(Point2D[] points) {
    int N = points.length;
    if (N <= 1) return;

    // sort by x-coordinate (breaking ties by y-coordinate)
    Point2D[] pointsByX = new Point2D[N];
    for (int i = 0; i < N; i++)
        pointsByX[i] = points[i];
    Arrays.sort(pointsByX, Point2D.X_ORDER);

    // check for coincident points
    for (int i = 0; i < N-1; i++) {
        if (pointsByX[i].equals(pointsByX[i+1])) {
            bestDistance = 0.0;
            best1 = pointsByX[i];
            best2 = pointsByX[i+1];
            return;
        }
    }

    // sort by y-coordinate (but not yet sorted)
    Point2D[] pointsByY = new Point2D[N];
    for (int i = 0; i < N; i++)
        pointsByY[i] = pointsByX[i];

    // auxiliary array
    Point2D[] aux = new Point2D[N];

    closest(pointsByX, pointsByY, aux, 0, N-1);
}
```

```
// find closest pair of points in pointsByX[lo..hi]
// precondition:  pointsByX[lo..hi] and pointsByY[lo..hi] are the same sequence of points, sorted by x,y-coord
private double closest(Point2D[] pointsByX, Point2D[] pointsByY, Point2D[] aux, int lo, int hi) {
    if (hi <= lo) return Double.POSITIVE_INFINITY;

    int mid = lo + (hi - lo) / 2;
    Point2D median = pointsByX[mid];

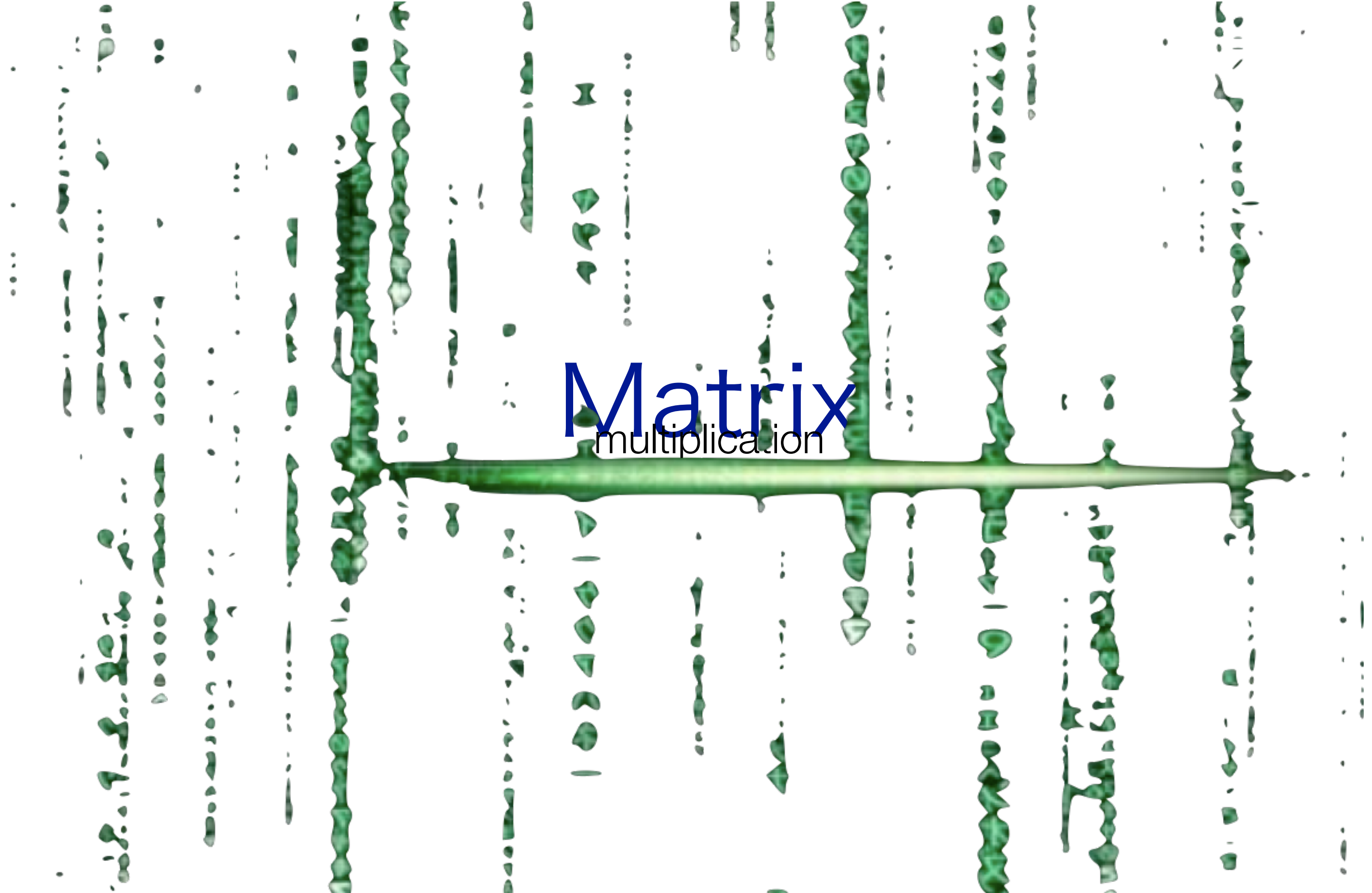
    // compute closest pair with both endpoints in left subarray or both in right subarray
    double delta1 = closest(pointsByX, pointsByY, aux, lo, mid);
    double delta2 = closest(pointsByX, pointsByY, aux, mid+1, hi);
    double delta = Math.min(delta1, delta2);

    // merge back so that pointsByY[lo..hi] are sorted by y-coordinate
    merge(pointsByY, aux, lo, mid, hi);

    // aux[0..M-1] = sequence of points closer than delta, sorted by y-coordinate
    int M = 0;
    for (int i = lo; i <= hi; i++) {
        if (Math.abs(pointsByY[i].x() - median.x()) < delta)
            aux[M++] = pointsByY[i];
    }

    // compare each point to its neighbors with y-coordinate closer than delta
    for (int i = 0; i < M; i++) {
        // a geometric packing argument shows that this loop iterates at most 7 times
        for (int j = i+1; (j < M) && (aux[j].y() - aux[i].y() < delta); j++) {
            double distance = aux[i].distanceTo(aux[j]);
            if (distance < delta) {
                delta = distance;
                if (distance < bestDistance) {
                    bestDistance = delta;
                    best1 = aux[i];
                    best2 = aux[j];
                    // StdOut.println("better distance = " + delta + " from " + best1 + " to " + best2);
                }
            }
        }
    }
    return delta;
}
```

http://algs4.cs.princeton.edu/99hull/ClosestPair.java.html



Matrix

multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \bigstar \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} \underline{5\cdot 1 + 2\cdot 7} & 6\cdot 1 + 2\cdot 8 \\ 3\cdot 5 + 4\cdot 7 & 6\cdot 3 + 4\cdot 8 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 20 \\ 43 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \star \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$\begin{array}{cc} & n \\ n & \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \end{array} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & & & \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & & & \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{bmatrix}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & & & \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & \boxed{c_{2,2}} & \cdots & c_{2,n} \\ \vdots & & & \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{bmatrix}$$

$\Theta(n^2)$ entries.

$$\underline{c_{i,j}} = \sum_{k=1}^{n} a_{\underline{i,}k} \cdot b_{k,j} \qquad \Theta(n)$$

standard matmult takes $n^2 \cdot n = \Theta(n^3)$ operations.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & & & \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & & & \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{bmatrix}$$

$n/2$

$n/2$

how can we do this operation more efficiently ??

$$\underset{\frac{n}{2}\times\frac{n}{2}}{}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} =$$

$$\begin{bmatrix} A\cdot E + BG & AF + BH \\ C\cdot E + DG & CF + DH \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

n/2
n/2
$\frac{n^2}{4}$

$$= \begin{bmatrix} \overset{1}{AE} + \overset{2}{BG} & \overset{3}{AF} + \overset{4}{BH} \\ \overset{5}{CE} + \overset{6}{DG} & \overset{7}{CF} + \overset{8}{DH} \end{bmatrix}$$

$$T(n) = \underline{8}\, T\left(\frac{n}{\underline{2}}\right) + \Theta(n^2)$$

By masters $\Theta(n^{\log_2 8}) = \Theta(n^3)$ case 1

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

$$T(n) = 8T(n/2) + \Theta(n^2)$$

$$\Theta(n^3)$$

$$= \begin{bmatrix} R{=}\,AE + BG & S{=}\,\underline{AF + BH} \\ T{=}\,\underline{CE + DG} & U{=}\,CF + DH \end{bmatrix}$$

[Strassen]

$$\left\{\begin{aligned}
\underline{P_1} &= A(F - H) \\
P_2 &= (A + B)H \\
P_3 &= \underline{(C + D)E} \\
P_4 &= \underline{D(G - E)} \\
P_5 &= (A + \underline{D)(E + H)} \\
P_6 &= (B - D)(G + H) \\
P_7 &= (A - C)(E + F)
\end{aligned}\right.$$

$$S = P_1 + P_2$$

$$A(F-H) + (A+B)\cdot H = \underline{AF} - \cancel{AH} + \cancel{AH} + \underline{BH}$$

$$T = P_3 + P_4 = \underline{CE} + \cancel{DE} + \underline{DG} - \cancel{DE}$$

$$R = P_5 + P_4 - P_2 + P_6$$

$$= AE + \cancel{AH} + \cancel{DE} + \cancel{DH} \qquad = AE + BG$$

$$+ \cancel{DG} - \cancel{DE}$$

$$- \cancel{AH} - \cancel{BH}$$

$$+ BG + \cancel{BH} - \cancel{DG} - \cancel{DH}$$

$$R = P_5 + P_4 - P_2 + P_6 \qquad S = P_1 + P_2$$

$$= \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

$$T = P_3 + P_4 \qquad \underline{U = P_5 + P_1 - P_3 - P_7}$$

[strassen]

$$P_1 = A(F - H)$$

$$P_2 = (A + B)H$$

$$P_3 = (C + D)E$$

$$P_4 = D(G - E)$$

$$P_5 = (A + D)(E + H)$$

$$P_6 = (B - D)(G + H)$$

$$P_7 = (A - C)(E + F)$$

$$M(n) = 7M\left(\frac{n}{2}\right) + \overset{18}{\Theta(n^2)}$$

$$= \Theta\left(n^{\log_2 7}\right)$$

$$= \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

$R = P_5 + P_4 - P_2 + P_6$

$S = P_1 + P_2$

$T = P_3 + P_4$

$U = P_5 + P_1 - P_3 - P_7$

[strassen]

$P_1 = A(F - H)$

$P_2 = (A + B)H$

$P_3 = (C + D)E$

$P_4 = D(G - E)$

$P_5 = (A + D)(E + H)$

$P_6 = (B - D)(G + H)$

$P_7 = (A - C)(E + F)$

$$M(n) = 7M(n/2) + 18n^2$$

$$= \Theta(n^{\log_2 7})$$

$n^{\log_2 7} =$ $\leftarrow \sim 2.807$

taking this idea further

3x3 matricies [Laderman'75] in 23 multe

$$\begin{bmatrix} A & B & C \\ D & E & F \\ H & I & J \end{bmatrix} \begin{bmatrix} K & L & M \\ N & O & P \\ Q & R & S \end{bmatrix}$$

$$L(n) = 23\,L\left(\frac{n}{3}\right) + \Theta(n^2)$$

$$= \Theta\left(n^{\log_3 23}\right)$$

Strassen

$$n^{\log_2 7} \sim n^{2.807}$$

$$n^{\log_3 23} \sim n^{2.85}$$

(worse!!)

1978 victor pan method

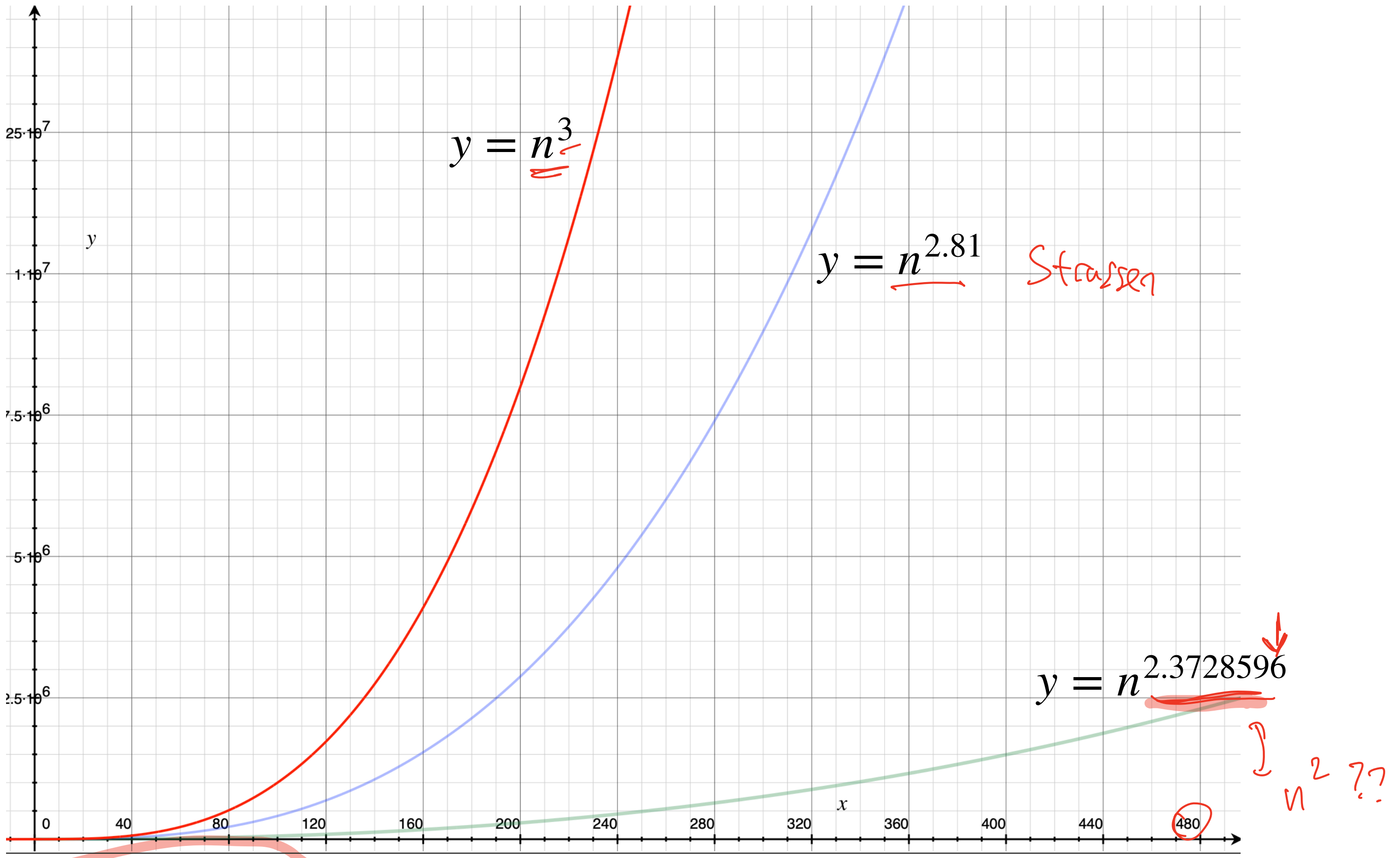
70x70 matrix using 143640 mults

what is the recurrence:

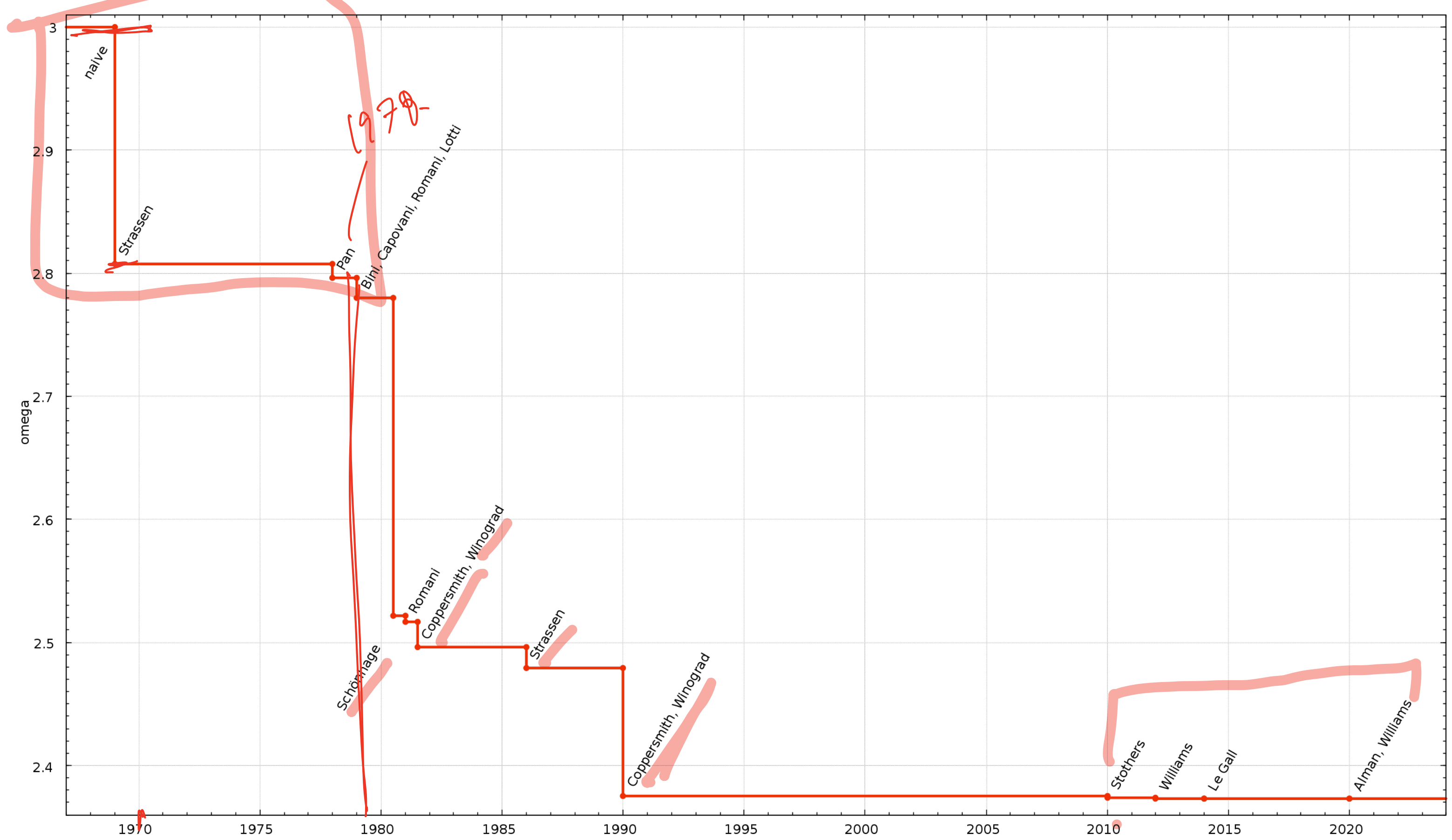
$$T(n) = 143640\, T\left(\frac{n}{70}\right) + \Theta(n^2)$$

$$n^{\log_{70} 143640} \sim n^{2.795}$$

(Improvement !!)

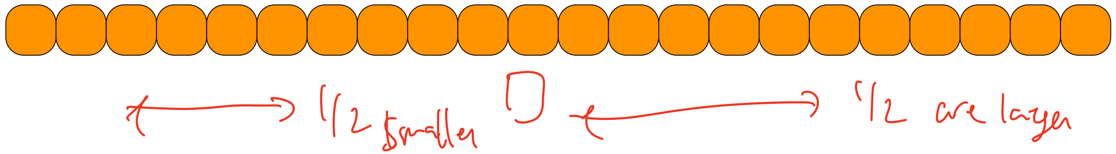


$y = n^3$
$y = n^{2.81}$
Strassen
$y = n^{2.3728596}$
n^2 ??
$25\cdot10^7$
$1\cdot10^7$
$7.5\cdot10^6$
$5\cdot10^6$
$2.5\cdot10^6$
y
x
0
40
80
120
160
200
240
280
320
360
400
440
480

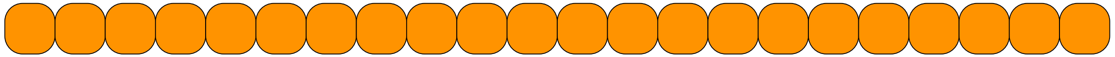


https://en.wikipedia.org/wiki/File:Bound_on_matrix_multiplication_omega_over_time.svg

MEDIAN



problem: given a list of **n** elements, find the element of rank **n**/2. (half are larger, half are smaller)

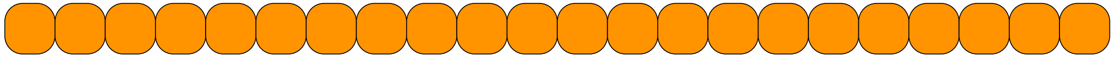


problem: given a list of **n** elements, find the element of rank n/2. (half are larger, half are smaller)

can generalize to **i**

first solution: sort and pluck.

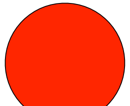
$$O(\underline{n \log n})$$

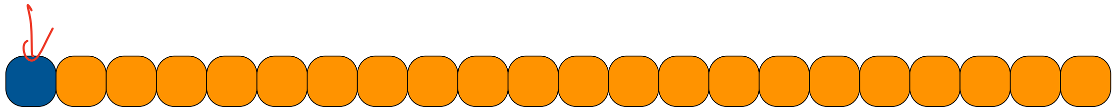


problem: given a list of **n** elements, find the element of rank **i**.

key insight:

we do not have to “fully” sort.
semi sort can suffice.



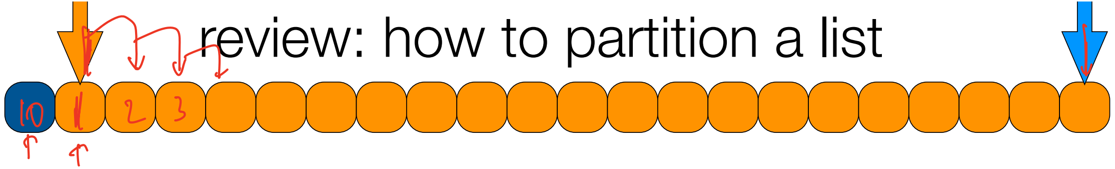


pick first element

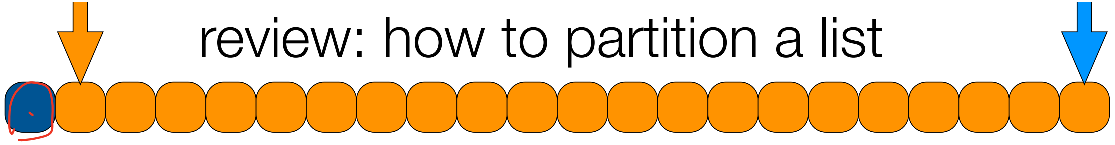
partition list about this one

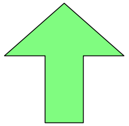
see where we stand

review: how to partition a list

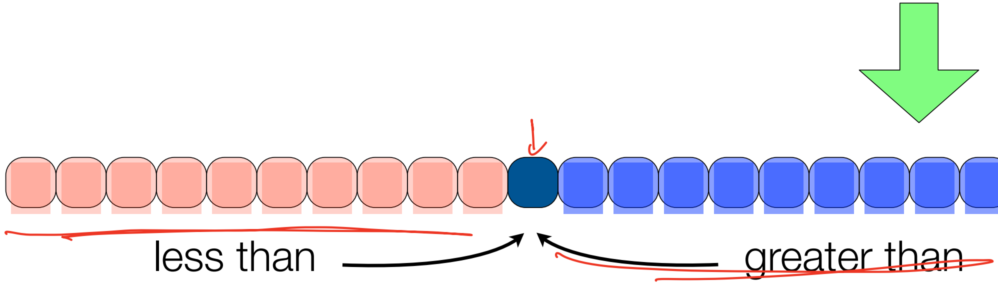


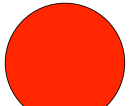
review: how to partition a list



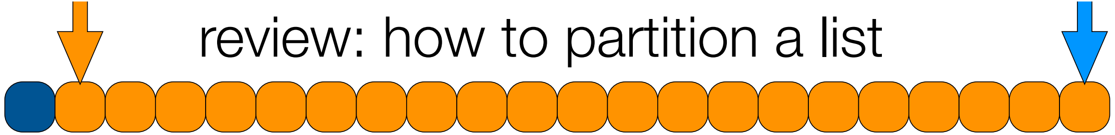


GOAL: start with THIS LIST and END with THAT LIST

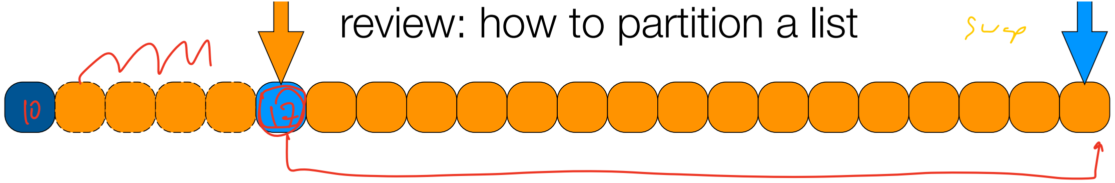




review: how to partition a list



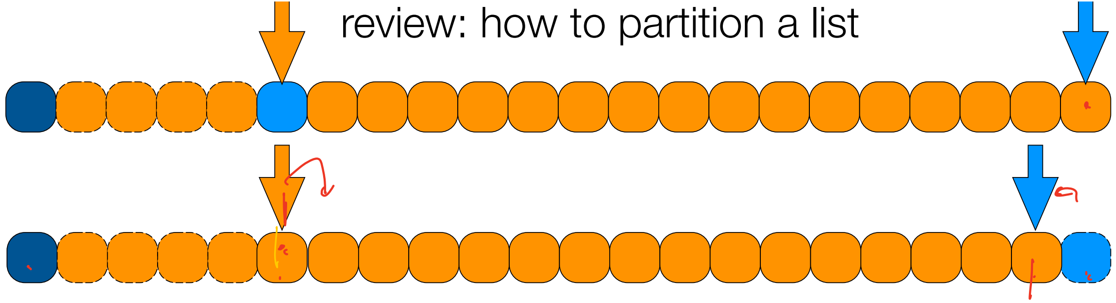
review: how to partition a list



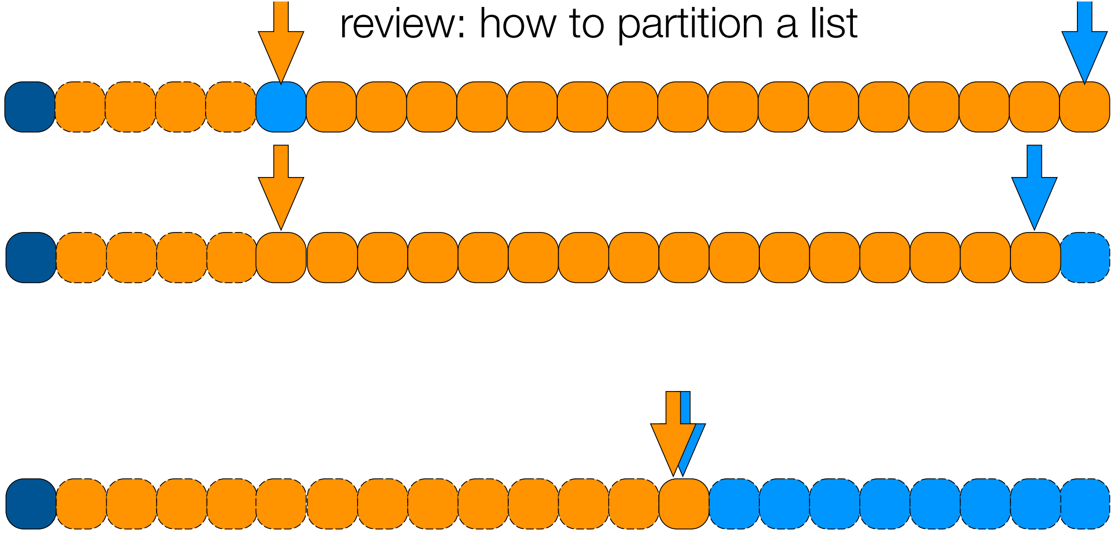
Since the orange ptr is larger than the pivot

swap elements with the blue & move the blue pointer

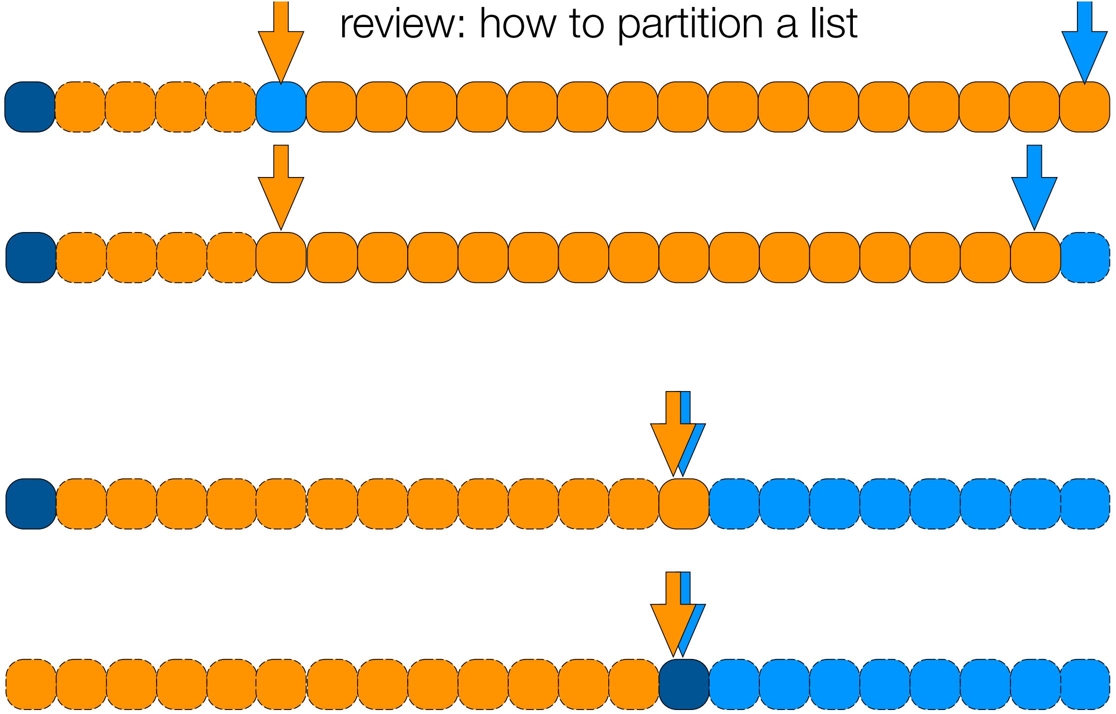
review: how to partition a list



review: how to partition a list

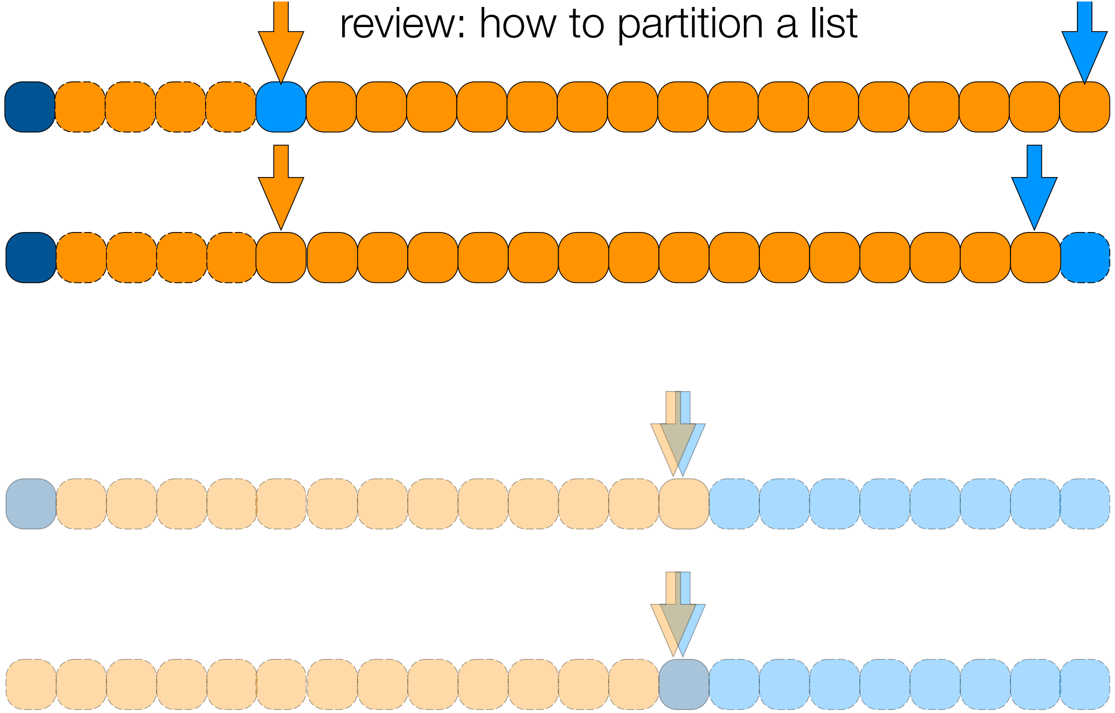


review: how to partition a list

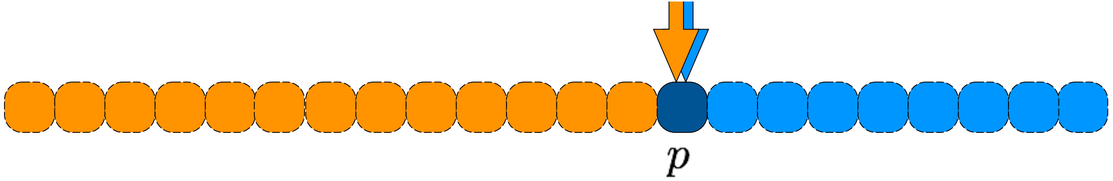


$\Theta(n)$ time, partitioned the array

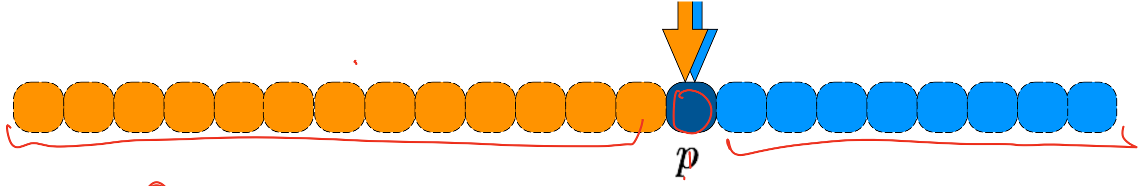
review: how to partition a list



partitioning a list about an element takes linear time.



select $(i, A[1, \ldots, n])$



```
select $(i, A[1,\ldots,n])$
    handle base case of 1 element.
    partition list about first element
    if pivot p is position i, return pivot
    else if pivot p is in position > i  select $(i, A[1,\ldots,p-1])$
    else  select $((i-p-1), A[p+1,\ldots,n])$
```

```
select $(i, A[1,\ldots,n])$
    handle base case.
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    if pivot is position i, return pivot
    else if pivot is in position > i    select $(i, A[1,\ldots,p-1])$
    else select $((i-p-1), A[p+1,\ldots,n])$
```

(Assume our partition always splits list into two eql parts)

In this lucky case.

$$S(n) = S\left(\frac{n}{2}\right) + \Theta(n) = \Theta(n)$$

select $(i, A[1, \ldots, n])$

Assume our partition always splits list into two eql parts

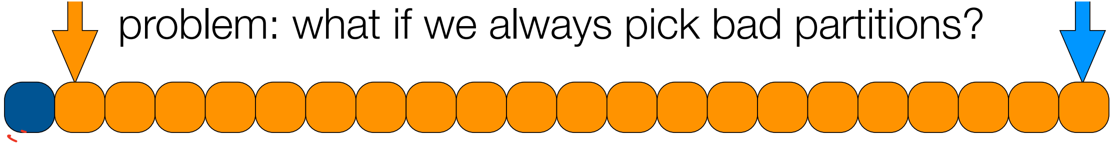
```
handle base case.
partition list about first element
if pivot is position i, return pivot
else if pivot is in position > i    select $(i, A[1, \ldots, p-1])$
else  select $((i-p-1), A[p+1, \ldots, n])$
```

→ how can we pick good partitions??

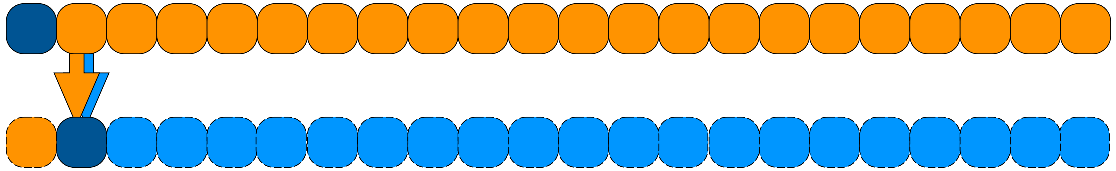
$$T(n) = T(n/2) + O(n)$$

$$\Theta(n)$$

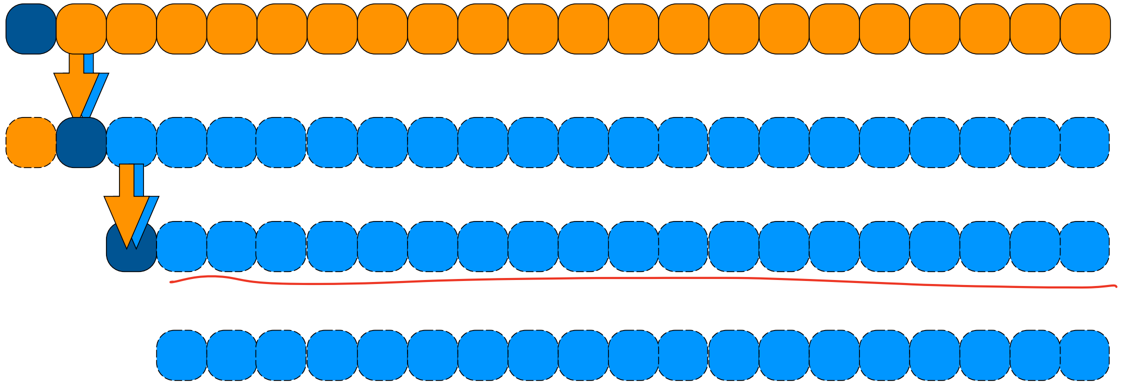
problem: what if we always pick bad partitions?

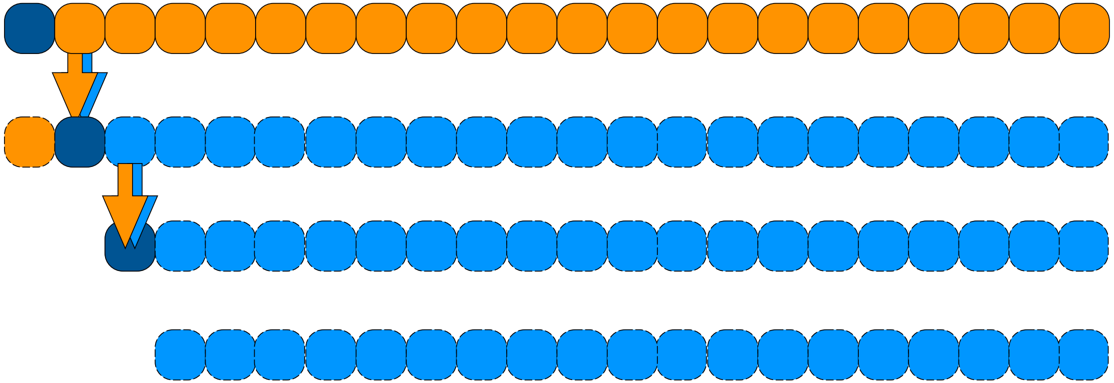


problem: what if we always pick bad partitions?

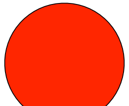


problem: what if we always pick bad partitions?





problem: what if we always pick bad partitions?



```
select $(i, A[1, \ldots, n])$
    handle base case.
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```

$$T(n) = T(n-1) + O(n)$$

$$\Theta(n^2)$$

Needed:

a good partition element

partition $(A[1,\ldots,n])$

Needed:

a good partition element

partition $(A[1,\ldots,n])$

produce an element where
30% smaller, 30% larger

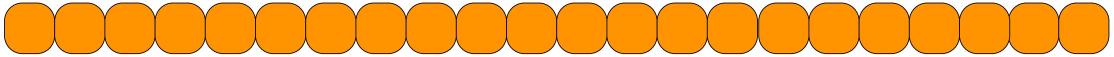
solution: bootstrap

image: gucci

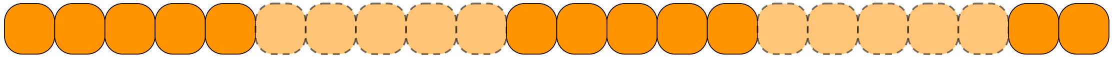
image: mark nason

image: d&g

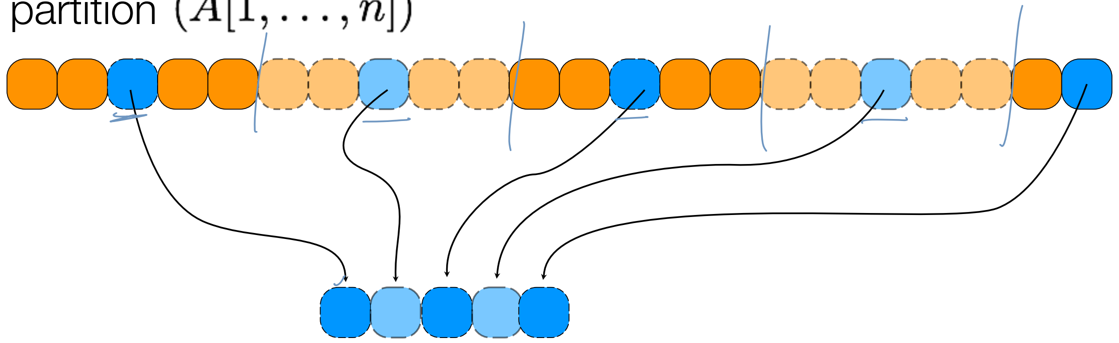
partition $(A[1, \ldots, n])$



partition $(A[1, \ldots, n])$



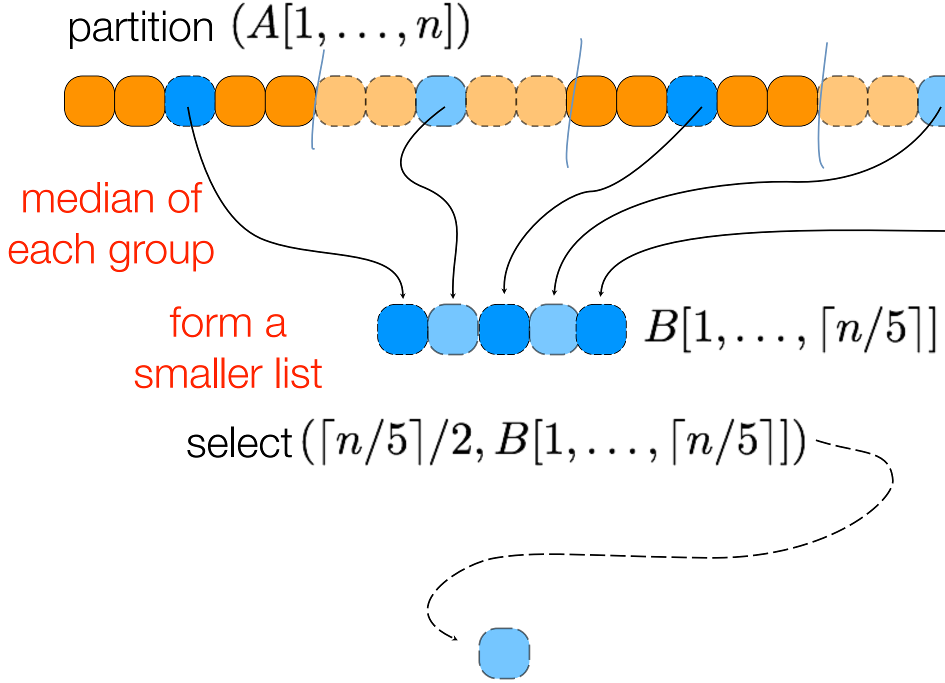
partition $(A[1, \ldots, n])$



divide list into groups of 5 elements

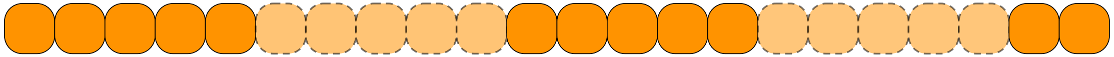
find median of each small list using brute force

gather all medians


partition $(A[1,\ldots,n])$
median of each group
form a smaller list
$B[1,\ldots,\lceil n/5\rceil]$
select $(\lceil n/5\rceil/2, B[1,\ldots,\lceil n/5\rceil])$

use the median of this smaller list as the partition element

partition $(A[1, \ldots, n])$



divide list into groups of 5 elements

find median of each small list using brute force

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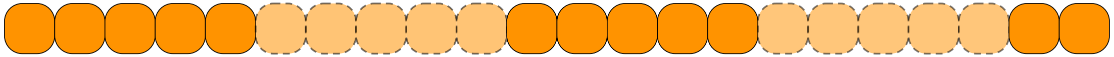
call select(...) on this sublist to find median

return the result

partition $(A[1,\ldots,n])$



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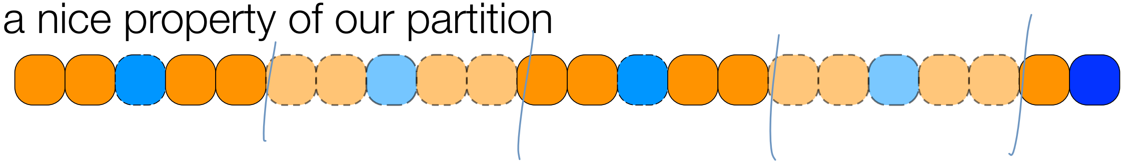
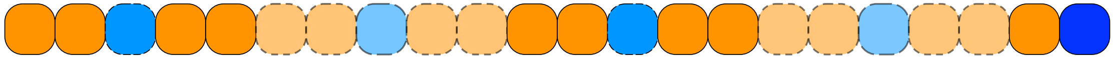
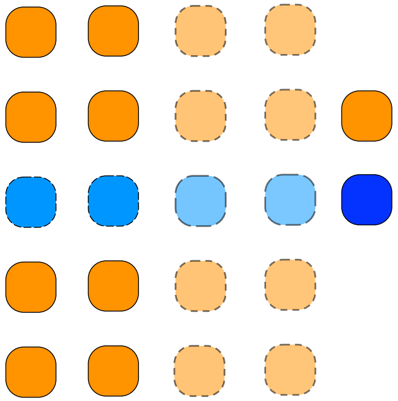
$$P(n) = S(\lceil n/5 \rceil) + O(n)$$

a nice property of our partition



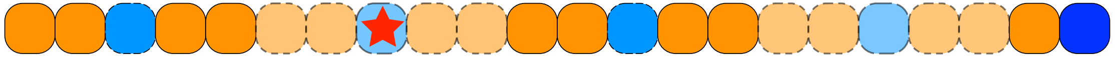
a nice property of our partition

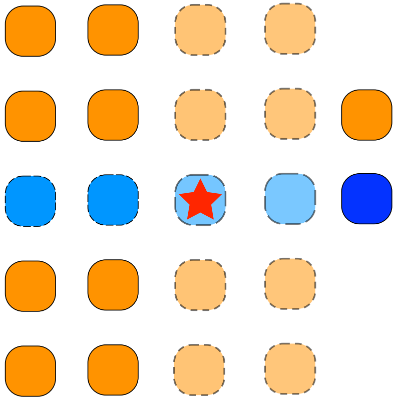




Imagine rearranging the elements by sorting each column and then also sorting the medians.

a nice property of our partition





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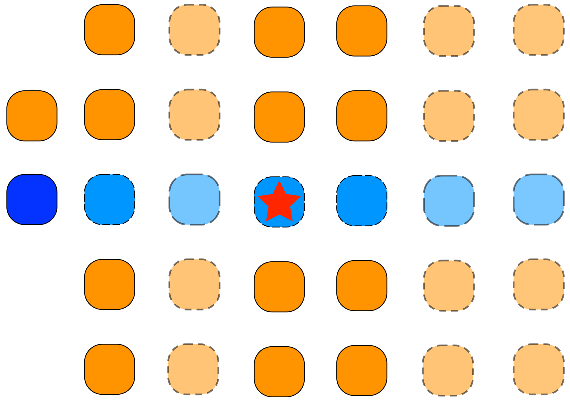
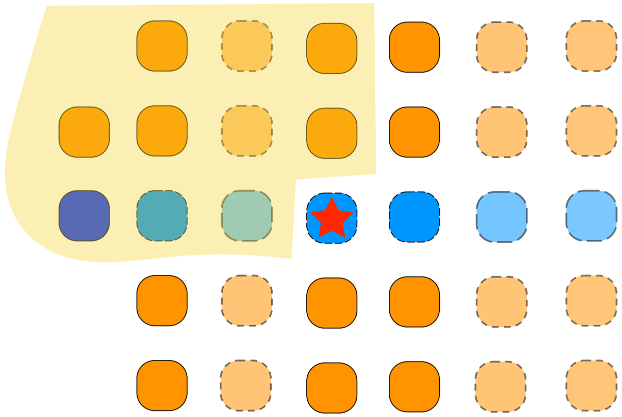
SWITCH TO A BIGGER EXAMPLE



SWITCH TO A BIGGER EXAMPLE

These yellow elements are all smaller than the median. How many are there?



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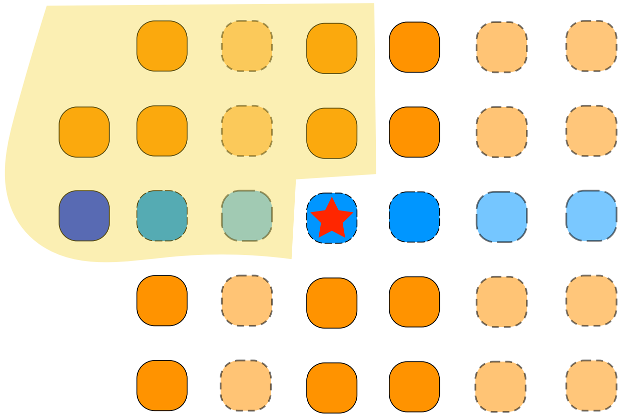


$$3\left(\left\lceil \frac{1}{2}\lceil n/5 \rceil \right\rceil - 2\right)$$

$$\geq \frac{3n}{10} - 6$$

There are $\lceil n/5 \rceil / 2$ columns. Ignoring the first and last, each column has 3 elements in it that are smaller than the median.

a nice property of our partition

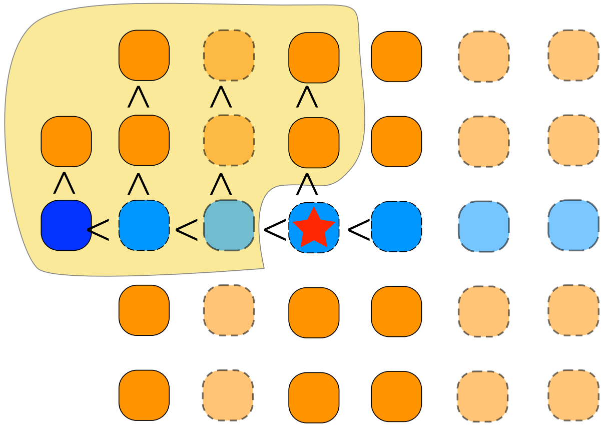
$$3\left(\left\lceil \frac{1}{2}\lceil n/5\rceil \right\rceil - 2\right)$$

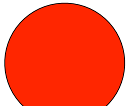
$$\geq \frac{3n}{10} - 6$$

this implies there are

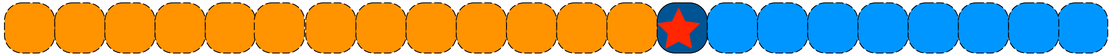
at most $\frac{7n}{10} + 6$ numbers

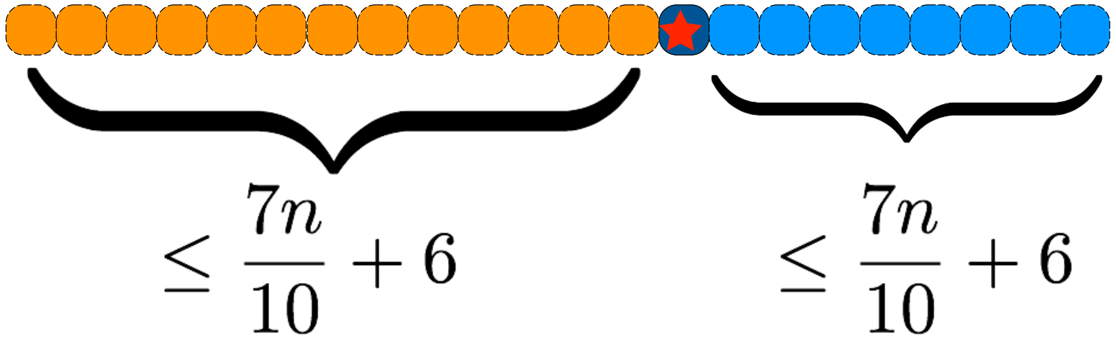
larger than ★
/smaller



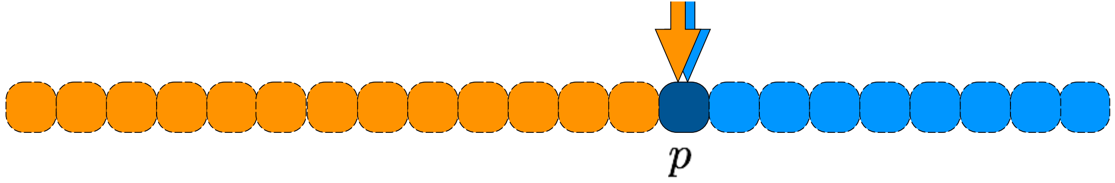


a nice property of our partition



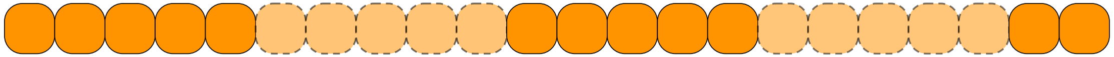


The median-of-medians is guaranteed to have a **linear fraction** of the input that is smaller and larger than it.



```
select $(i, A[1, \ldots, n])$
    handle base case for small list
    else pivot = FindPartitionValue(A,n)
    partition list about pivot
    if pivot is position **i**, return pivot
    else if pivot is in position > **i**    select $(i, A[1, \ldots, p-1])$
    else select $((i-p-1), A[p+1, \ldots, n])$
```

FindPartition $(A[1, \ldots, n])$



divide list into groups of 5 elements

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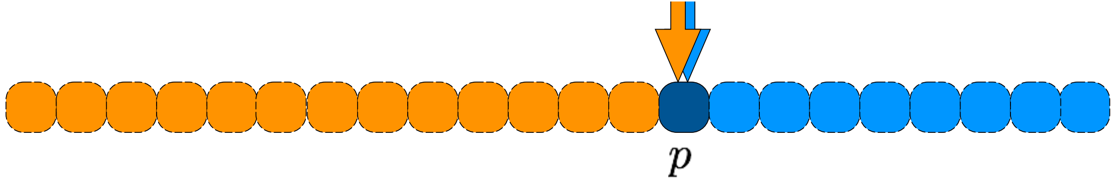
$$P(n) = S(\lceil n/5 \rceil) + O(n)$$



```
select $(i, A[1, \ldots, n])$
    handle base case for small list
    else pivot = FindPartitionValue(A,n)
    partition list about pivot
    if pivot is position i, return pivot
    else if pivot is in position > i    select $(i, A[1, \ldots, p-1])$
    else select $((i-p-1), A[p+1, \ldots, n])$
```

$$S(n) = S(\lceil n/5 \rceil) + \Theta(n) + S(\lceil 7n/10 + 6 \rceil)$$

$$\Theta(n)$$

You can use induction like in the homework problem.

How to get intuition for S(n)

Fast

Fourier Transform

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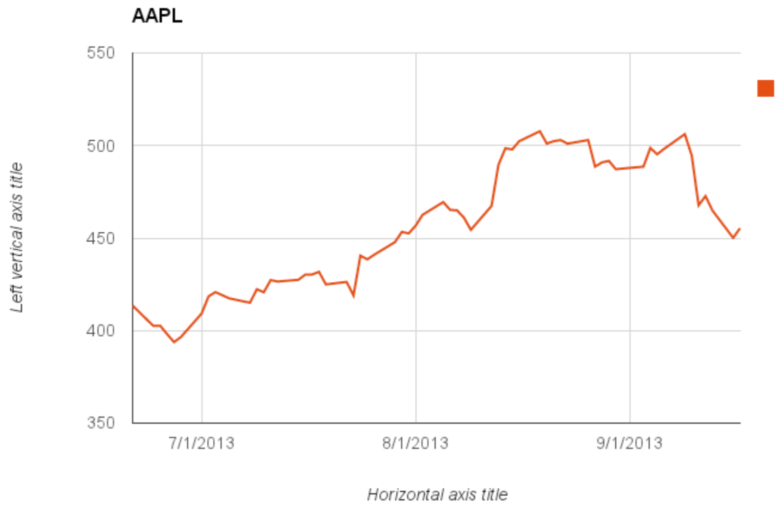
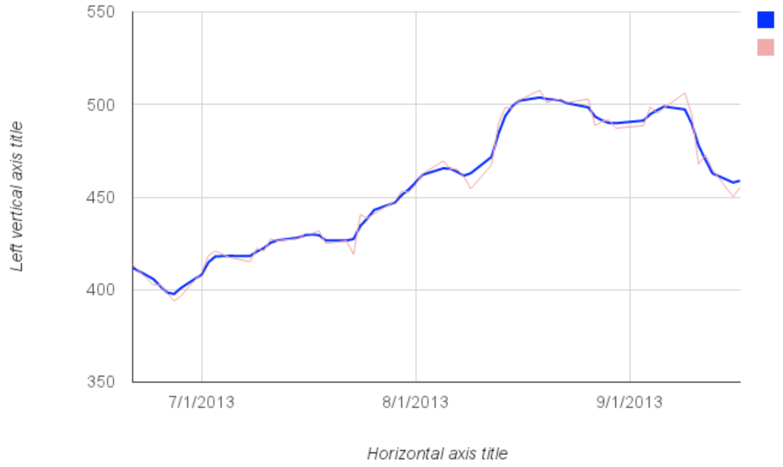

AAPL
550
500
450
400
350
7/1/2013
8/1/2013
9/1/2013
Left vertical axis title
Horizontal axis title

AAPL


550
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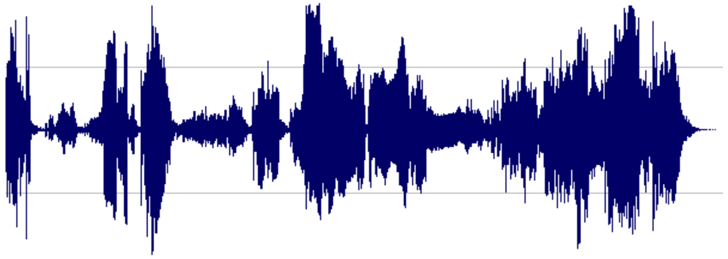
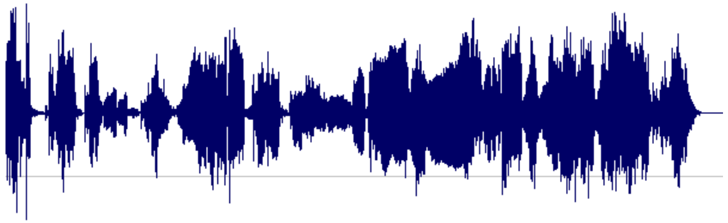
7/1/2013
8/1/2013
9/1/2013
Left vertical axis title
Horizontal axis title

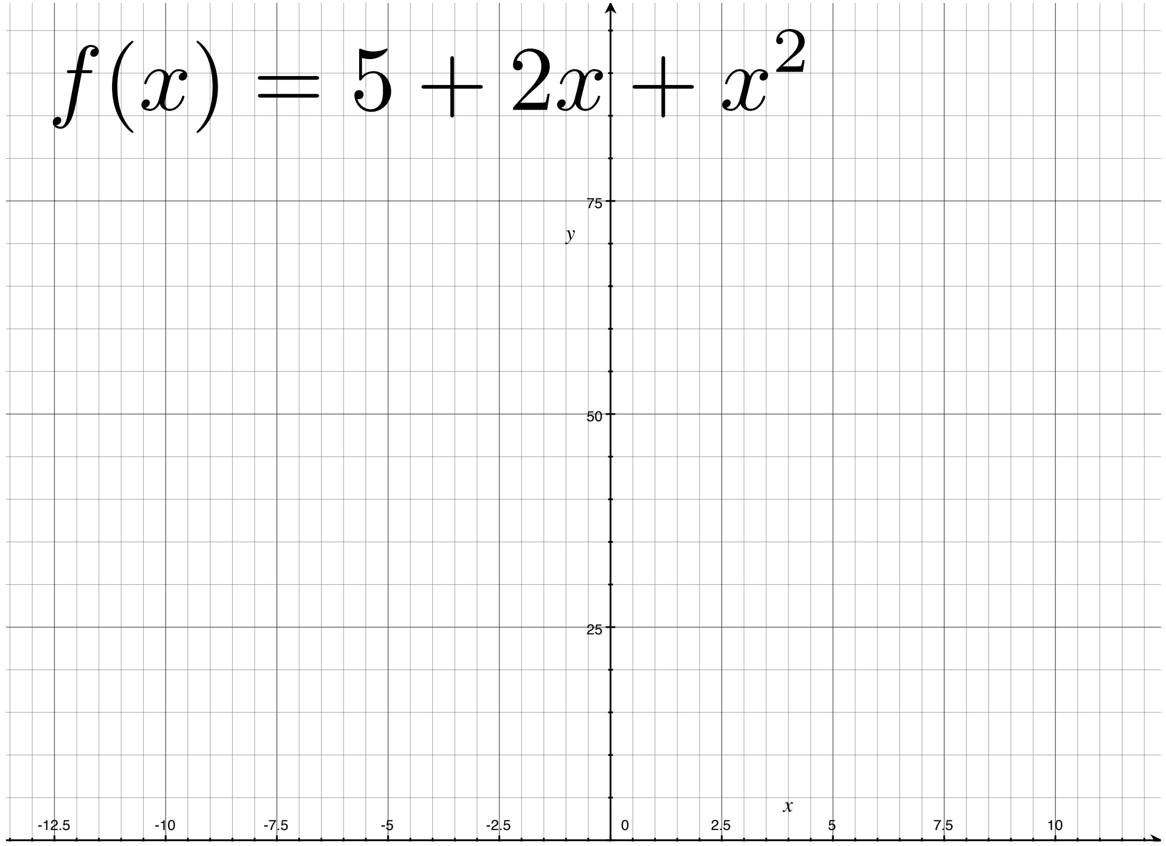
big ideas:

big ideas:

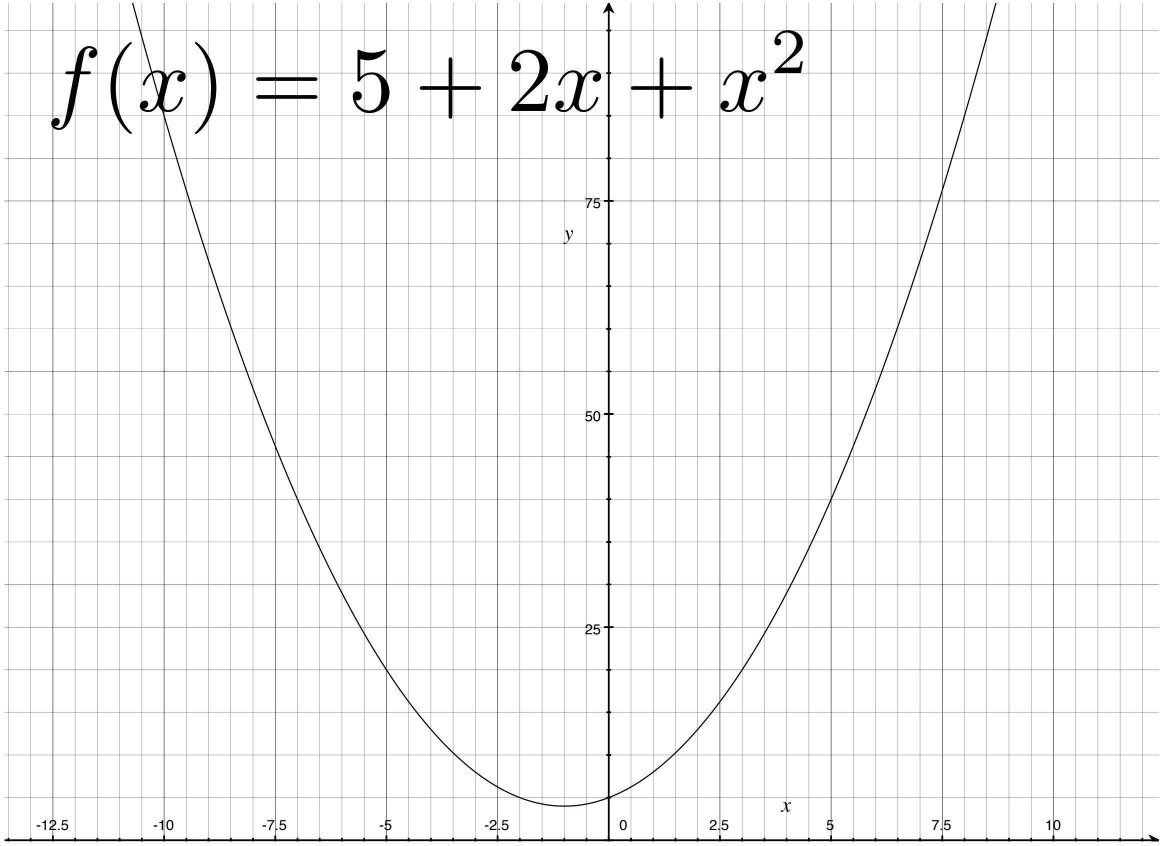
1. Changing representation from polynomial (coefficient form) into polynomial (point-wise form)

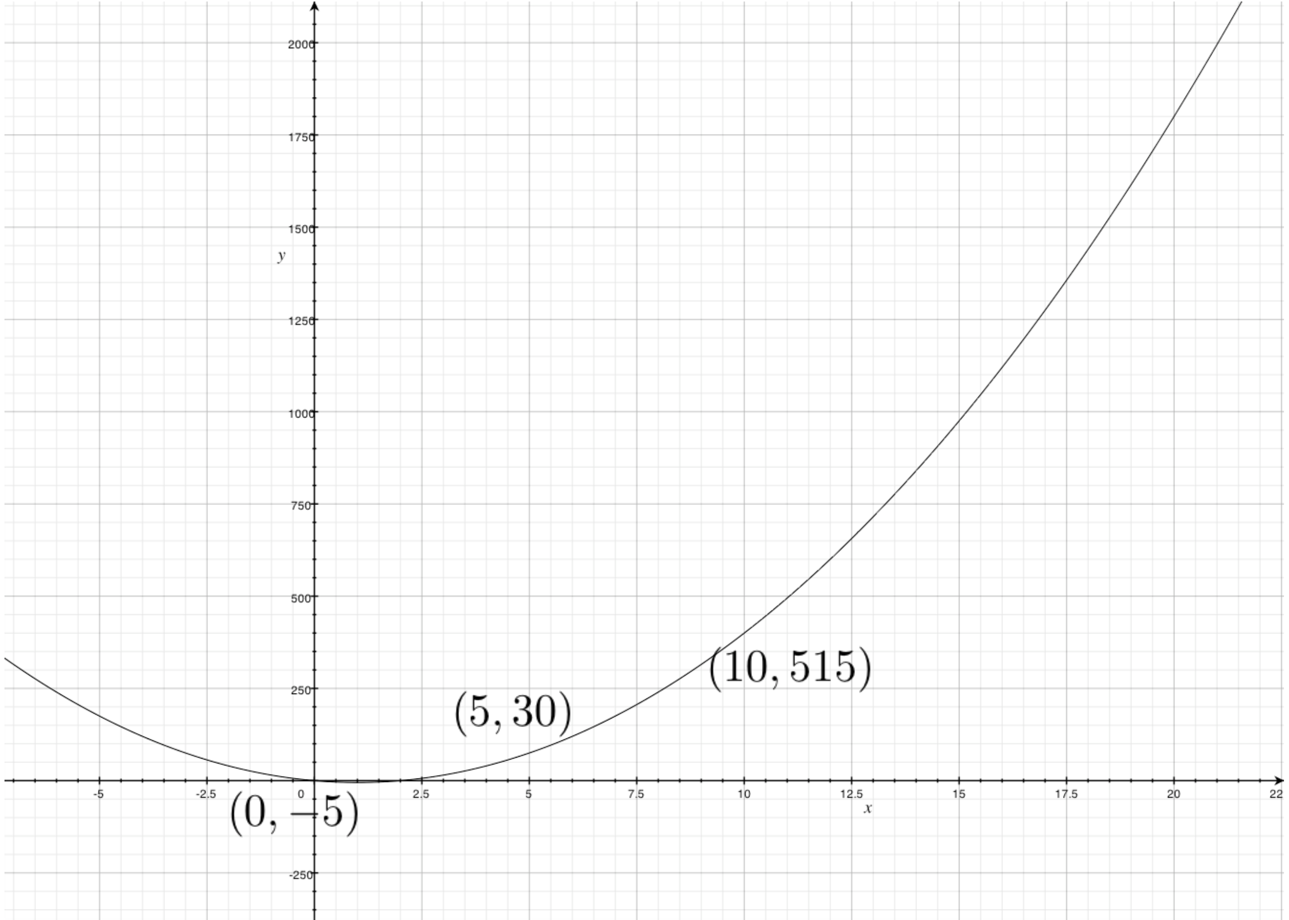
2. Clever divide and conquer

$$f(x) = 5 + 2x + x^2$$



$$f(x) = 5 + 2x + x^2$$





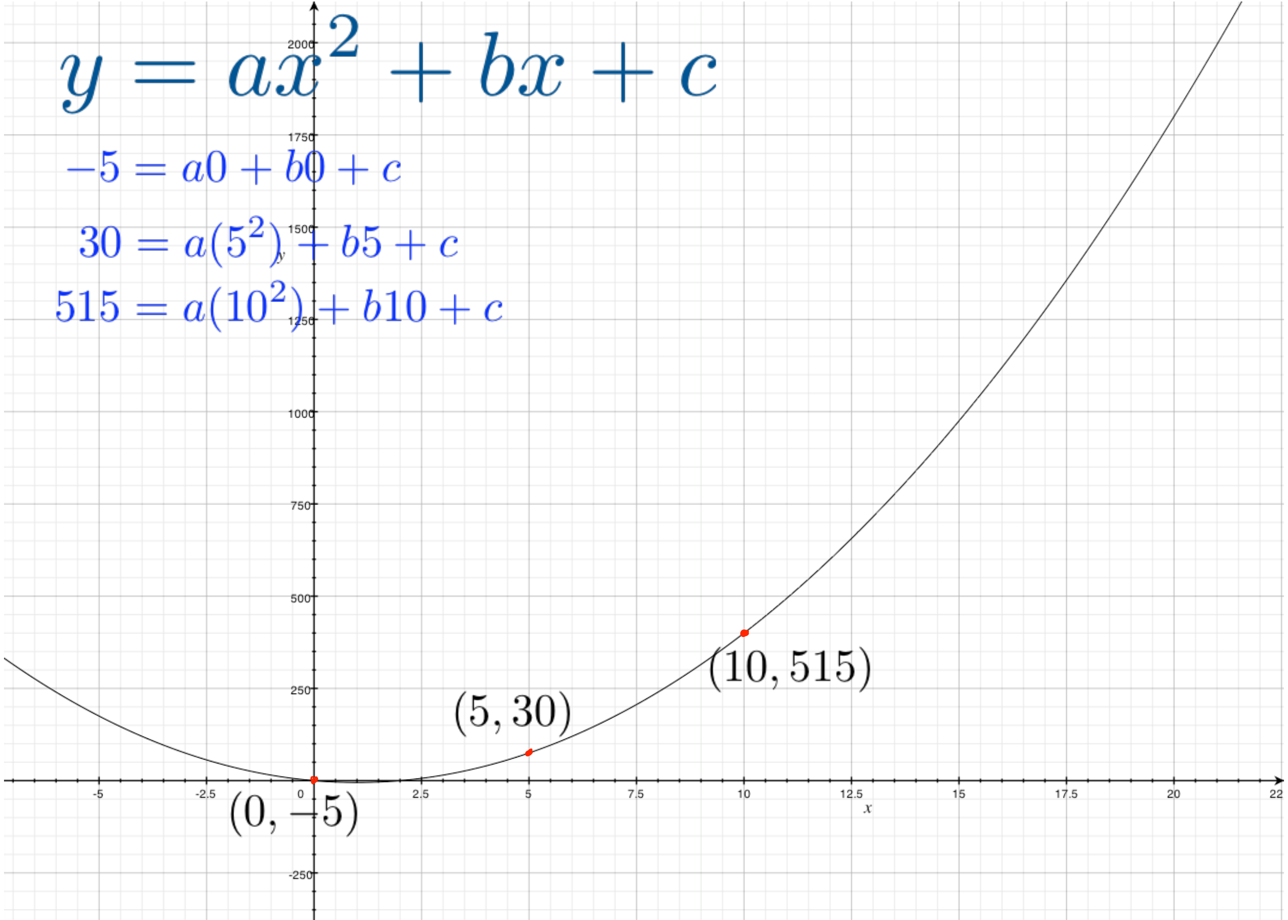
2000
1750
1500
y
1250
1000
750
500
250
-250
-5
-2.5
0
2.5
5
7.5
10
12.5
15
17.5
20
22
x
$(10, 515)$
$(5, 30)$
$(0, -5)$

$$y = ax^2 + bx + c$$

$$-5 = a0 + b0 + c$$

$$30 = a(5^2) + b5 + c$$

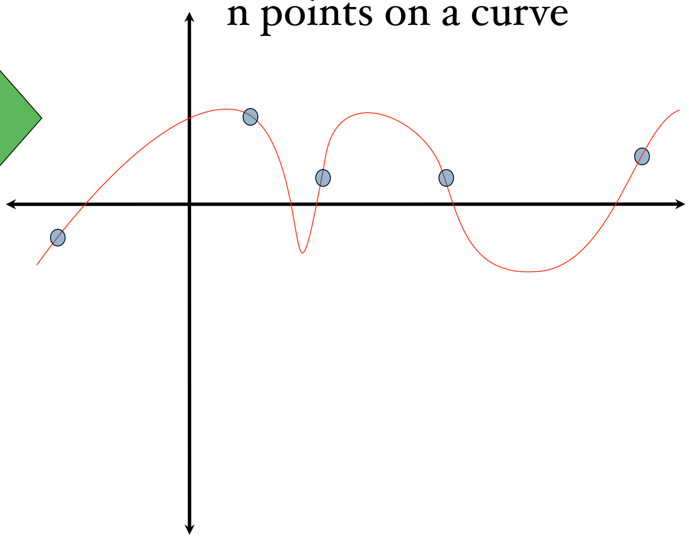
$$515 = a(10^2) + b10 + c$$



$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

degree $n - 1$
polynomial

$$A(x)$$



FFT

input: $a_0, a_1, a_2, \ldots, a_{n-1}$

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

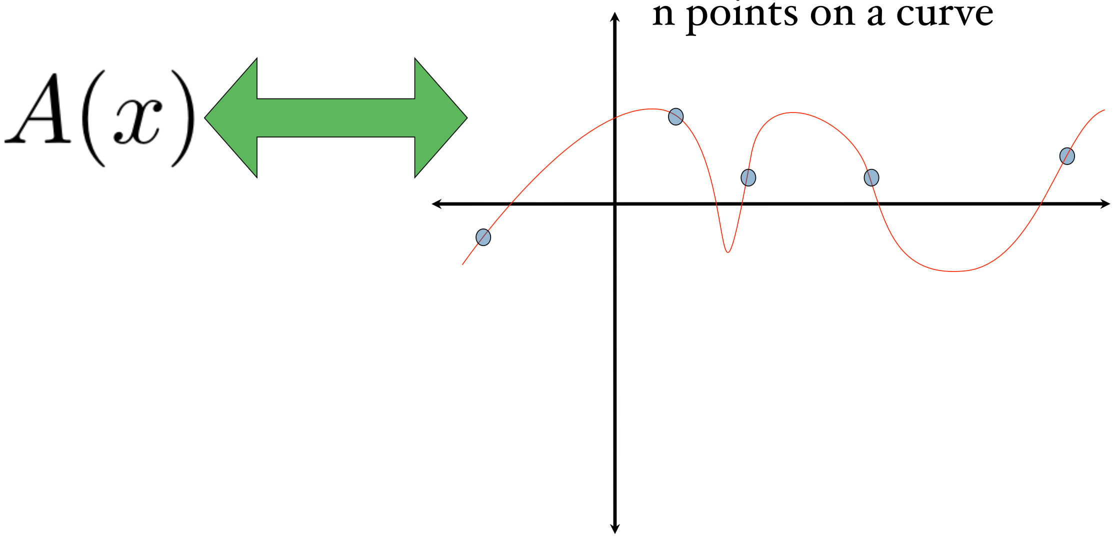
output:

FFT

input: $a_0, a_1, a_2, \ldots, a_{n-1}$

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

output: evaluate polynomial A at (any) n different points.



Later, we shall see that the same ideas for FFT can be used to implement Inverse-FFT.

Inverse FFT: Given n-points,

Later, we shall see that the same ideas for FFT can be used to implement Inverse-FFT.

Inverse FFT: Given n-points,

$$y_0, y_1, \cdots, y_{n-1}$$

find a degree n polynomial A such that

$$y_i = A(\omega_i)$$