MATHEMATICAL PRIMER FOR NORTHEASTERN CS5800

SHELAT

1. Logarithms and Exponentials

The logarithm of x with respect to base b, i.e. $\log_b(x)$, is the value y such that $b^y = x$. In this sense, taking logarithms is the inverse of raising to a power. We will almost always use logarithms that are base 2. Thus, when you see $\log(x)$, it typically means $\log_2(x)$, and sometimes we also abbreviate as $\lg(x)$ or $\lg x$. This Wikipedia article here has more details.

Here are some notational conventions that we use:

(1) $\log^k(n) = (\log n)^k$
(2) $\log\log n = \log(\log n)$

Here are some key facts that we will use in the course:

(1) This identity follows by the definition: $b^{\log_b x} = x$ and $\log_b(b^x) = x$
(2) The product inside a log can be separated into a sum: $\log_b(xy) = \log_b(x) + \log_b(y)$
(3) A corollary of the above: $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$
(4) $\log_b a = \frac{1}{\log_a b}$
(5) $a^0 = 1$, $a^1 = a$
(6) It is possible to move exponents around: $(a^m)^n = a^{mn} = (a^n)^m$
(7) $a^m a^n = a^{m+n}$
(8) $a^{-1} = \frac{1}{a}$
(9) $a^{\log_b c} = c^{\log_b a}$. This follows by first writing a as $c^{\log_c a}$.
(10) $\frac{x}{(1+x)} \leq \ln(1+x) \leq x \quad \forall x > -1$

2. Series

Here are some basic series identities; see Appendix A of CLRS for more details.

(1) Simple arithmetic series:

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n\,(n+1)}{2}$$

(2) Harmonic series

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \leq \log(n) + 1$$

(3) Geometric series

$$\sum_{k=0}^{n} a^k = 1 + a + a^2 + a^3 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}$$

3. Iterated Log function

The function $\log^*(n)$ (read as "log star of n") represents the "number of times" you have to apply the logarithm function to n in order to get a value that is less than 1. Formally,

$$\log^*(n) = \min\{i \geq 0 \ : \ \log^i(n) \leq 1\}$$

This function appears in the analysis of the Union-Find data structure. In practice, we can treat $\lg^*(n)$ to be less than 5 because $\log^*(2^{10000}) < 5$, and there are believed to be less than 2^{300} elementary particles in the entire universe.