2550 Intro to cybersecurity

Public key Crypto

abhi shelat

Recap

Revisit our model for Encryption

Symmetric key enc has 1 major drawback.

Bob

Carol

Dave

Alice

Evan

George

Francis

Symmetric key enc has 1 major drawback.

$k_{ba}, k_{bc}, k_{bd}, k_{be}, k_{bf}, k_{bg}$
Bob

$k_{ca}, k_{cb}, k_{cd}, k_{ce}, k_{cf}, k_{cg}$
Carol

$k_{da}, k_{db}, k_{dc}, k_{de}, k_{df}, k_{dg}$
Dave

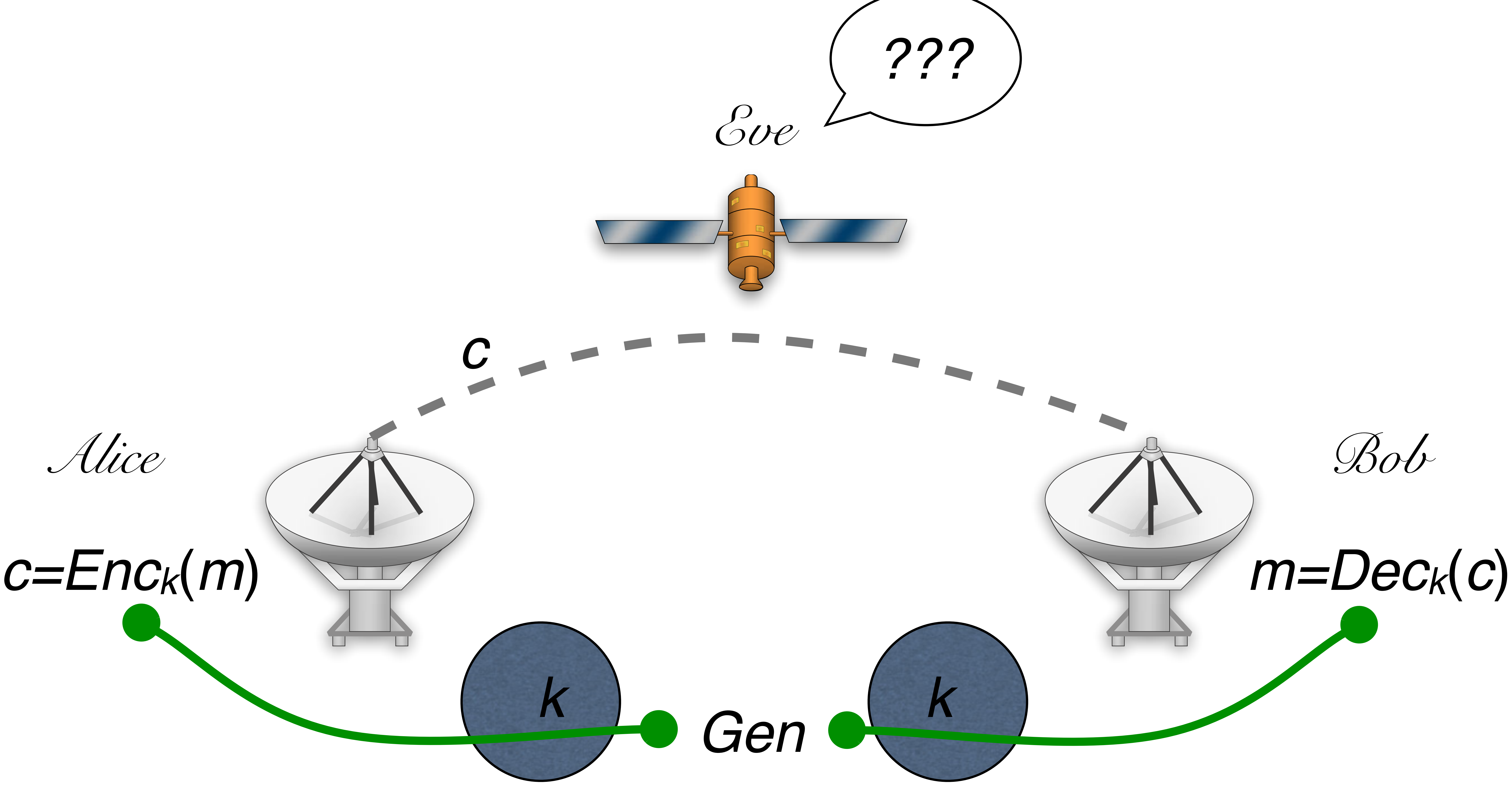
Alice
$k_{ab}, k_{ac}, k_{ad}, k_{ae}, k_{af}, k_{ag}$

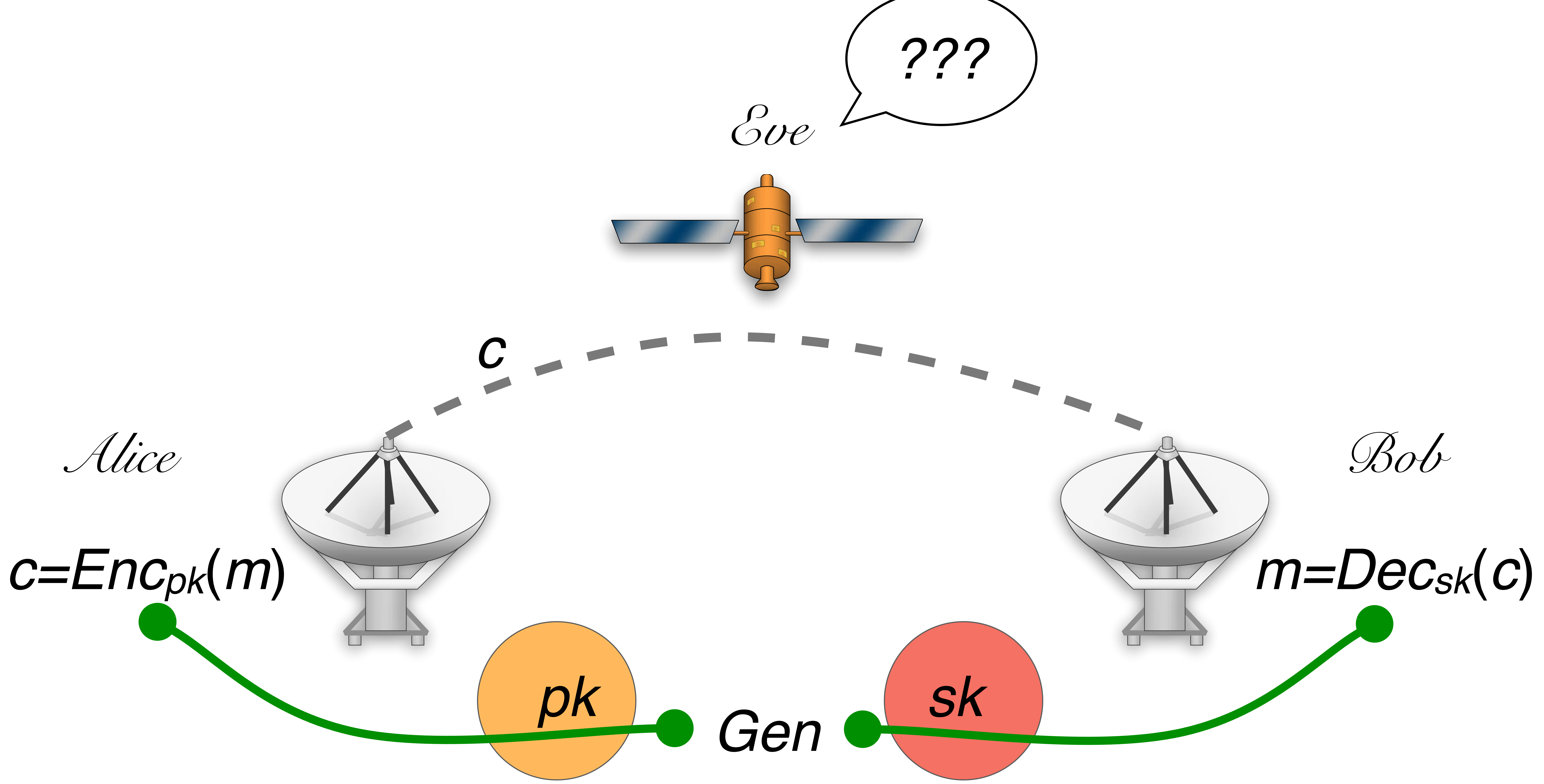
$O(n^2)$ keys to manage!

Evan
$k_{ea}, k_{eb}, k_{ec}, k_{ed}, k_{ef}, k_{eg}$

George
$k_{ga}, k_{gb}, k_{gc}, k_{gd}, k_{ge}, k_{gf}$

Francis
$k_{fa}, k_{fb}, k_{fc}, k_{fd}, k_{fe}, k_{fg}$


???
Eve
c
Alice
Bob
c=Enck(m)
m=Deck(c)
k
Gen
k



Pk can be used to encrypt.

sk can be used to decrypt.

PKC key enc

sk_b

Bob

sk_c

Carol

sk_d

Dave

Alice

sk_a

$pk_a, pk_b, pk_c, pk_d, pk_e, pk_f, pk_g$

Are publicly posted

Evan

sk_e

George

sk_g

Francis

sk_f

Public key encryption

Gen Enc Dec

3 algorithms

Gen (key generation)

$$(pk, sk) \leftarrow \mathsf{Gen}(1^n)$$

Enc (encryption)

$$c \leftarrow \mathsf{Enc}_{pk}(m) \text{ for } pk \in \mathcal{K}, m \in \mathcal{M}$$

Dec (decryption)

Public key encryption

Gen Enc Dec

3 algorithms

Gen (key generation)

$$(pk, sk) \leftarrow \mathsf{Gen}(1^n)$$

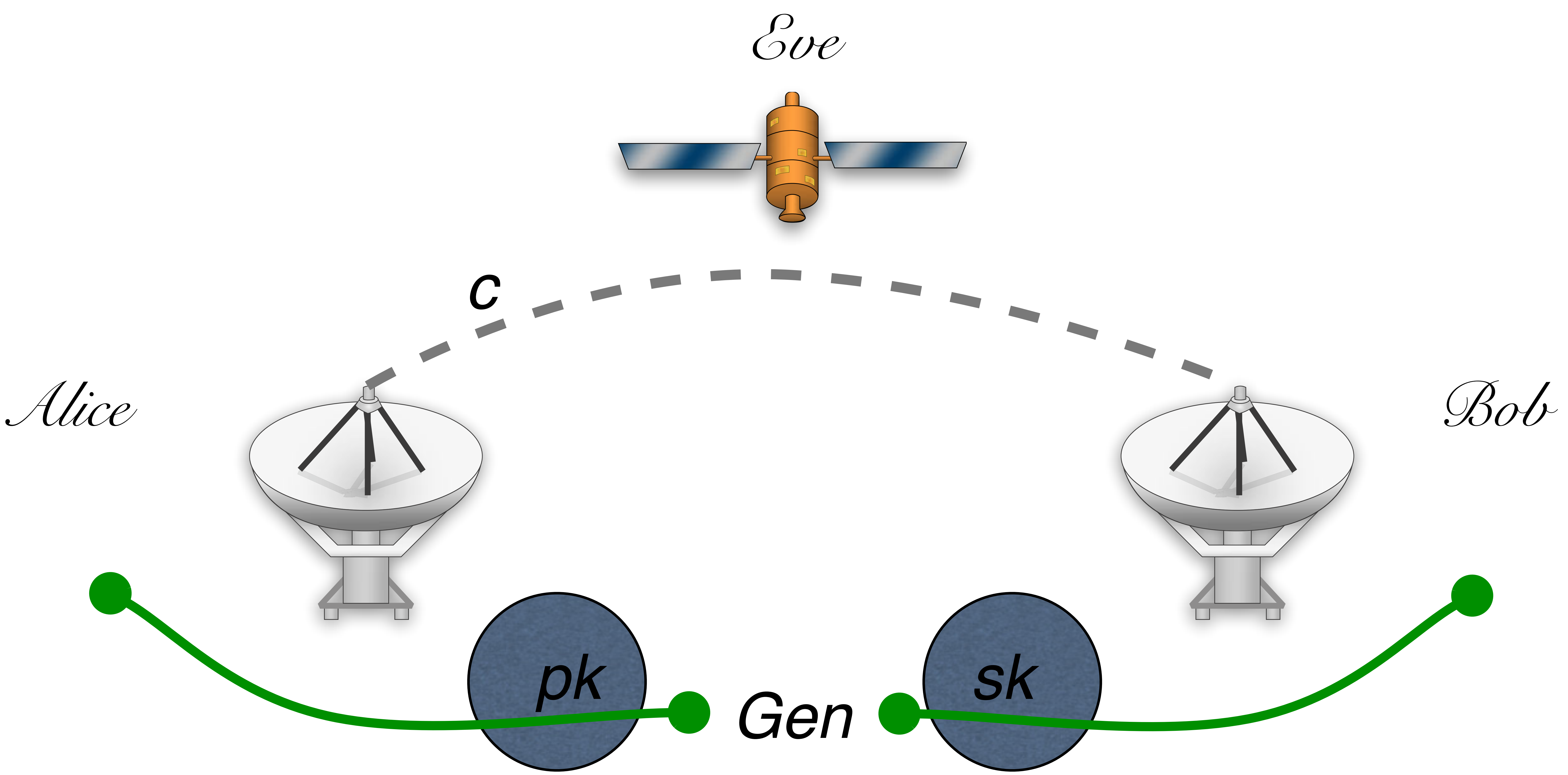
Enc (encryption)

$$c \leftarrow \mathsf{Enc}_{pk}(m) \text{ for } pk \in \mathcal{K}, m \in \mathcal{M}$$

Dec (decryption)

$$\forall m \in \mathcal{M}, (pk, sk) \leftarrow \mathsf{Gen}(1^n)$$

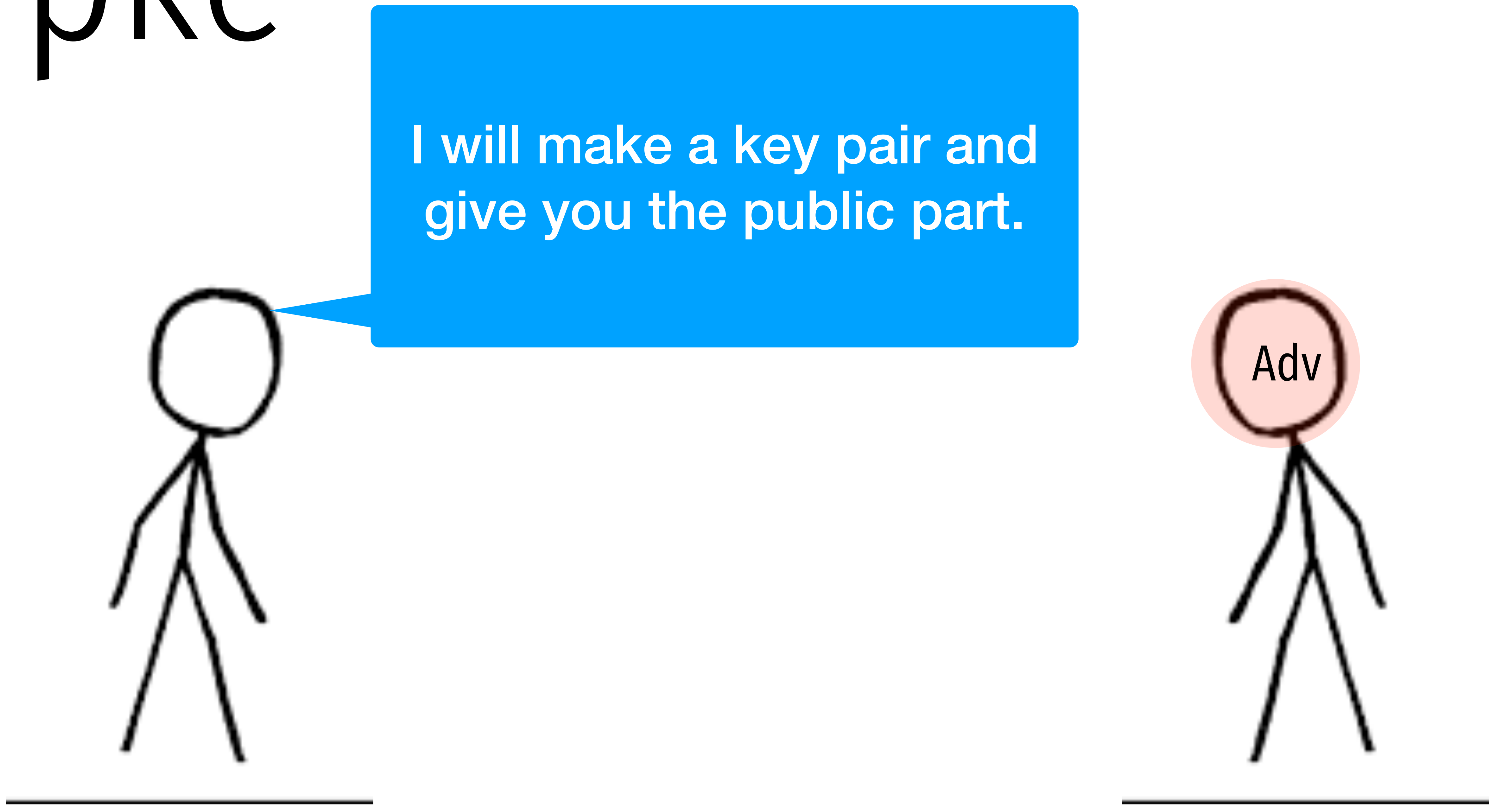
$$\Pr[\mathsf{Dec}_{sk}(\mathsf{Enc}_{pk}(m)) = m] = 1$$



"for any pair of messages m_1,m_2,
Eve cannot tell whether $c = Enc_{pk}(m_i)$."

IND-CPA security for pke

(weakest notion of security)



$pk, sk \leftarrow \mathsf{gen}(1^n)$

pk

IND-CPA security for pke

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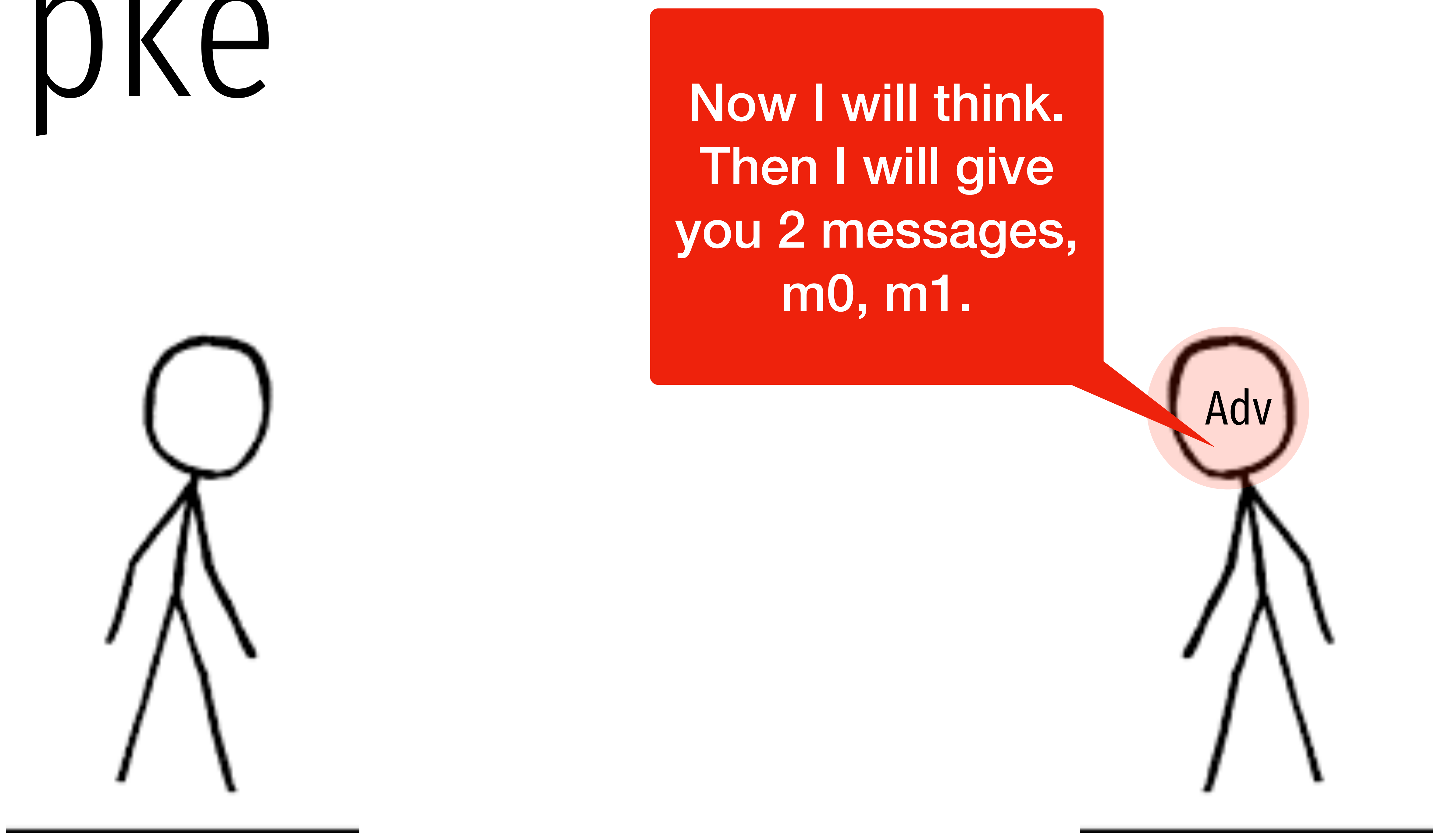


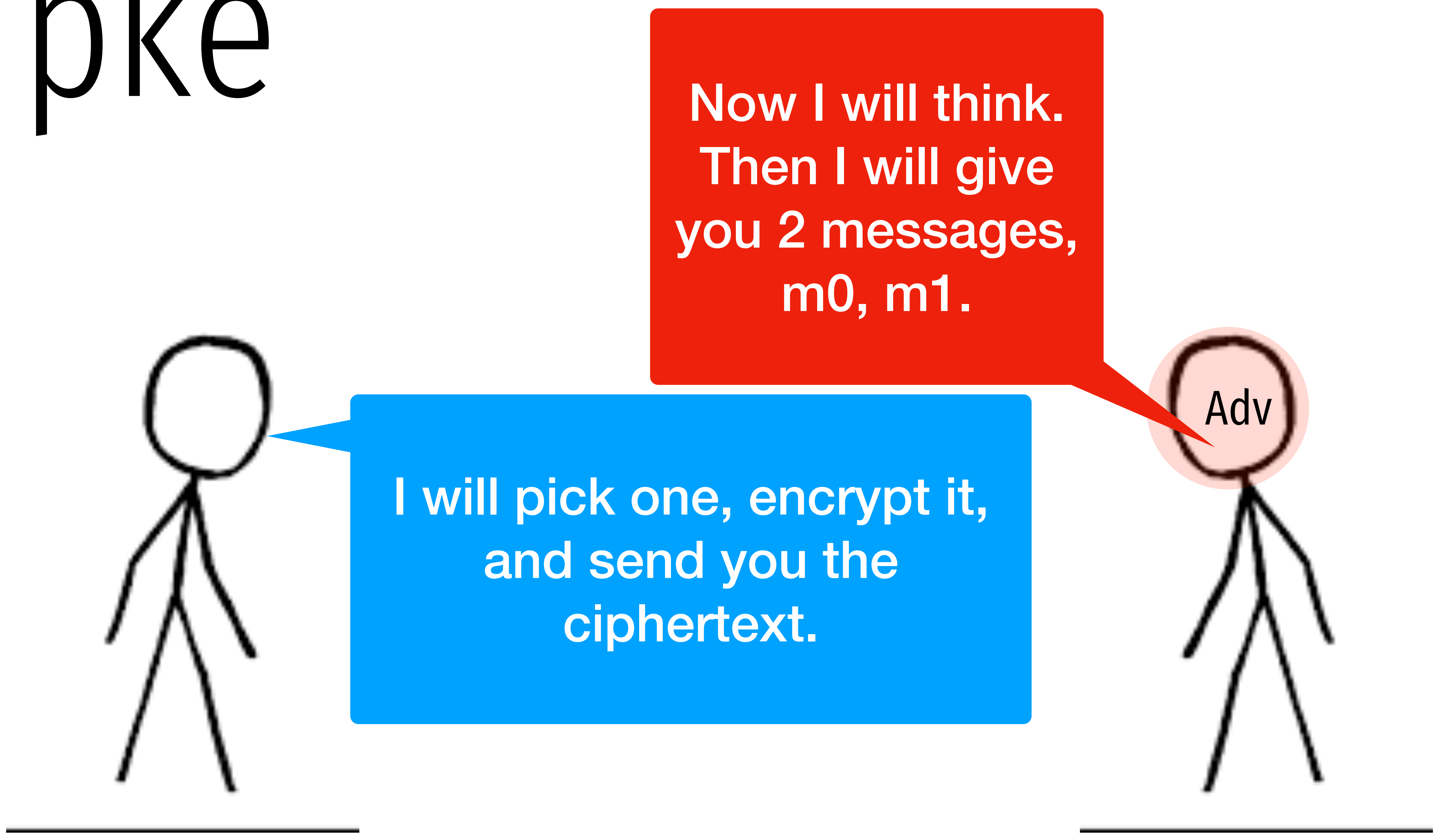
$pk, sk \leftarrow \mathsf{gen}(1^n)$

pk

$m_0, m_1 \leftarrow A(pk)$

IND-CPA security for pke

(weakest notion of security)



$pk, sk \leftarrow \mathsf{gen}(1^n)$

pk

$m_0, m_1 \leftarrow A(pk)$

$b \leftarrow \{0,1\}$

$c \leftarrow \mathsf{enc}_{pk}(m_b)$

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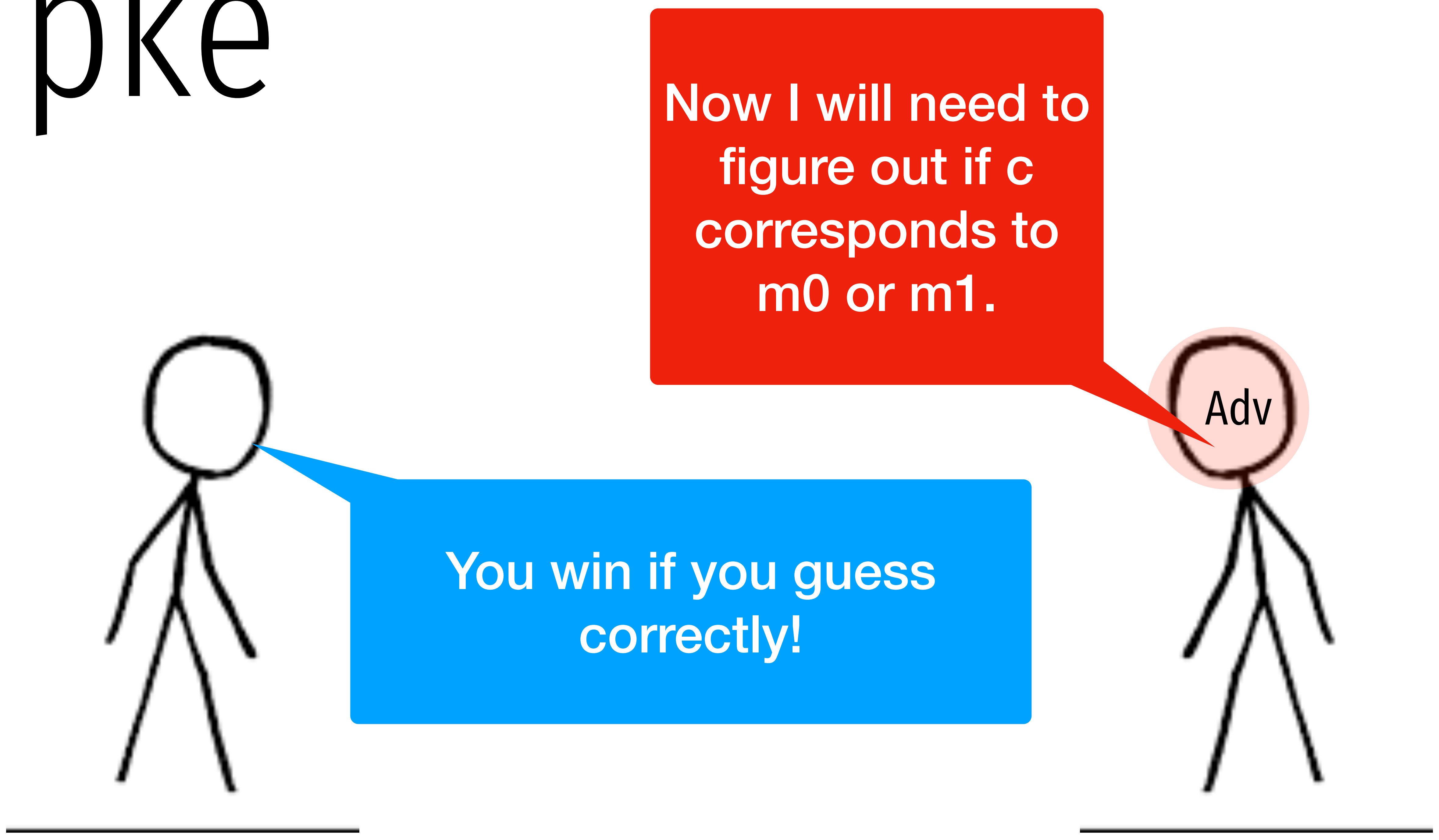

$pk, sk \leftarrow \mathsf{gen}(1^n)$

pk

$m_0, m_1 \leftarrow A(pk)$

$b \leftarrow \{0,1\}$

$c \leftarrow \mathsf{enc}_{pk}(m_b)$

$b' \leftarrow A(pk, m_0, m_1, c)$

IND-CPA security for pke

(weakest notion of security)

$$
\begin{aligned}
pk, sk &\leftarrow \mathsf{gen}(1^n) \\
m_0, m_1 &\leftarrow A(pk) \\
b &\leftarrow \{0,1\} \\
c &\leftarrow \mathsf{enc}_{pk}(m_b) \\
b' &\leftarrow A(pk, m_0, m_1, c)
\end{aligned}
$$

$$\Pr[b = b'] = 1/2 + \epsilon(n)$$

How to build public key encryption?

Basic Number theory

Modular Exponentiation

$$(a, x, n) \to a^x \bmod n$$

$$7^{19} \bmod 31$$

$$\underbrace{7 \cdot 7 \cdots 7}_{19 \text{ times}}$$

Modular Exponentiation

$(a, x, n) \to a^x \bmod n$

$$7^{19} \bmod 31$$

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$$7^{19} \bmod 31$$

Modular Exponentiation $(a, x, n) \to a^x \bmod n$

$$7^{19} \bmod 31$$

$7^1 \quad 7^2 \quad 7^4 \quad 7^8 \quad 7^{16}$ (mod 31)

Modular Exponentiation $(a, x, n) \to a^x \bmod n$

$$7^{19} \bmod 31$$

7^1	7^2	7^4	7^8	7^{16}	(mod 31)
7	18	14	10	7	

Modular Exponentiation

$$(a, x, n) \to a^x \bmod n$$

Algorithm 2: ModularExponentiation(a, x, n)

Input: $a, x \in [1, n]$

```
r ← 1
while x > 0 do
    if x is odd then
        r ← r · a mod n
    x ← ⌊x/2⌋
    a ← a² mod n
Return r
```

Modular Exponentiation

$$(a, x, n) \to a^x \bmod n$$

$$a^x \bmod n = \prod_{i=0}^{\ell} x_i a^{2^i} \bmod n$$

Algorithm 2: ModularExponentiation(a, x, n)

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    a ← a^2 mod n
Return r
```

Greatest Common Divisor

```
GCD(A,B) = GCD(                )
```

Greatest Common Divisor

`GCD(A,B) = GCD(B, A mod B)`

Greatest Common Divisor

```
GCD(6809,1641)
```

Greatest Common Divisor

GCD(6809,1641)

GCD(1641, 245)

GCD(245, 171)

GCD(171, 74)

GCD(74, 23)

GCD(23, 5)

GCD(5, 3)

GCD(3, 2)

GCD(2, 1)

$6809 = 4 \cdot 1641 + 245$

$1641 = 6 \cdot 245 + 171$

$245 = 1 \cdot 171 + 74$

$171 = 2 \cdot 74 + 23$

$74 = 3 \cdot 23 + 5$

$23 = 4 \cdot 5 + 3$

$5 = 1 \cdot 3 + 2$

$3 = \boxed{1} 2 + 1$

$2 = 2 \cdot 1 + 0$

given (a,b), finds (x,y) s.t.
ax + by = gcd(a,b)

Algorithm 1: ExtendedEuclid(a, b)

Input: (a, b) s.t $a > b \geq 0$
Output: (x, y) s.t. $ax + by = \gcd(a, b)$

```
if a mod b = 0 then
    Return (0, 1)
else
    (x, y) ← ExtendedEuclid (b, a mod b)
    Return (y, x − y(⌊a/b⌋))
```

Greatest Common Divisor

GCD(6809,1641)

		$6809 = 4 \cdot 1641 + 245$	$(-643, 2668)$
GCD(1641, 245)		$1641 = 6 \cdot 245 + 171$	$(96, -643)$
GCD(245, 171)		$245 = 1 \cdot 171 + 74$	$(-67, 96)$
GCD(171, 74)		$171 = 2 \cdot 74 + 23$	$(29, -67)$
GCD(74, 23)		$74 = 3 \cdot 23 + 5$	$(-9, 29)$
GCD(23, 5)		$23 = 4 \cdot 5 + 3$	$(2, -9)$
GCD(5, 3)		$5 = 1 \cdot 3 + 2$	$(-1, 2)$
GCD(3, 2)	`(1, 0 - 1*1 )`	$3 = \boxed{1} \cdot 2 + 1$	$(1, -1)$
GCD(2, 1)	`(0,1)`	$2 = 2 \cdot 1 + 0$	$(0, 1)$

Greatest Common Divisor

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Greatest Common Divisor

```
GCD(6809,1641)
```

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GCD(23, 5)

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GCD(23, 5)		$23 = 4 \cdot 5 + 3$
GCD(5, 3)		$5 = 1 \cdot 3 + 2$
GCD(3, 2)	(1,-1)	$3 = \boxed{1} \cdot 2 + 1$
GCD(2, 1)	(0,1)	$2 = 2 \cdot 1 + 0$

Greatest Common Divisor

GCD(6809,1641)

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GCD(5, 3)	(-1,2)	$5 = 1 \cdot 3 + 2$
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Greatest Common Divisor

GCD allows us to compute modular inverses.

`GCD(6809,1641)`

`6809*(-643) + 1641*2668 =`

Greatest Common Divisor

```
GCD(6809,1641)
```

GCD allows us to compute modular inverses.

```
6809*(-643) + 1641*2668 =   1

   -437818      437818
```

Euler totient

Euler totient

$\phi(15) =$

$$1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15$$

Euler totient

prime $\Phi(p) = p - 1$

product of 2 primes $\Phi(n) = (p-1)(q-1)$

Example of groups

$$(\mathbb{Z}_n, \star)$$

$$\{a \mid \gcd(a, n) = 1\}$$

multiplicative group, mod n

$$\mathbb{Z}_n^\star$$

Z*_15 = {1,2,4,7,8,11,13,14}

$$|\mathbb{Z}_n^\star| = \Phi(n)$$

Euler theorem

$$\forall a \in \mathbb{Z}_n^\star,\ a^{\Phi(n)} = 1 \bmod n$$

Examples

$$7^{30} \bmod 31 =$$

1	2	4	8	16
7	18	14	10	7

Examples

$$2^8 \bmod 15 =$$

Implications of Euler

$$a^{10\phi(N)} \bmod N =$$

$$a^{k\phi(N)+1} \bmod N =$$

compute

$$11^{30^{2021}} \bmod 23$$

(show your work)

“Textbook” RSA (insecure)

Pick N = p*q where p,q are primes.

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Pick e,d such that $e \cdot d = 1 \bmod \phi(N)$

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$\text{Enc}_{N,e}(m) = m^e \mod N$

$\text{Dec}_{N,d}(c) = c^d \mod N$

“Textbook” RSA (insecure)

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Pick e,d such that $e \cdot d = 1 \bmod \phi(N)$

Enc$_{N,e}(m) = m^e \mod N$

Dec$_{N,d}(c) = c^d \mod N$

$$(m^e)^d \bmod N =$$

Example of Textbook RSA

$m=5$

$PK = (N=143,\ e=7)$ $SK = (d=103)$

“Textbook” RSA (insecure)

Pick N = p*q where p,q are primes.

Pick e,d such that $e \cdot d = 1 \bmod \phi(N)$

$\text{Enc}_{N,e}(m) = m^e \mod N$

$\text{Dec}_{N,d}(c) = c^d \mod N$

Why is it insecure
against IND-CPA attack?

pkcs1.5

ENC$_{pk}$(m)

PICK r AS A RANDOM STRING WITH NO 0s

(TYPICALLY 8 BYTES)

$$c \leftarrow (0||2||r||0||m)^e \bmod N$$

"PADDING ORACLE" ATTACK AGAINST THIS SCHEME

RSA-OAEP+

GEN(1^n)

$f, f^{-1} \leftarrow$ TRAPDOOR OWP()

ENC$_{pk}$(m)

$r \leftarrow U_n$

$s \leftarrow R_1(r) \oplus m \;||\; R_2(r||m)$

$t \leftarrow R_3(s) \oplus r$

$c \leftarrow f(s||t)$

$R_1 : \{0,1\}^{k_0} \rightarrow \{0,1\}^n$

$R_2 : \{0,1\}^{n+k_0} \rightarrow \{0,1\}^{k_1}$

$R_3 : \{0,1\}^{n+k_1} \rightarrow \{0,1\}^{k_0}$

DEC$_{sk}$(c)

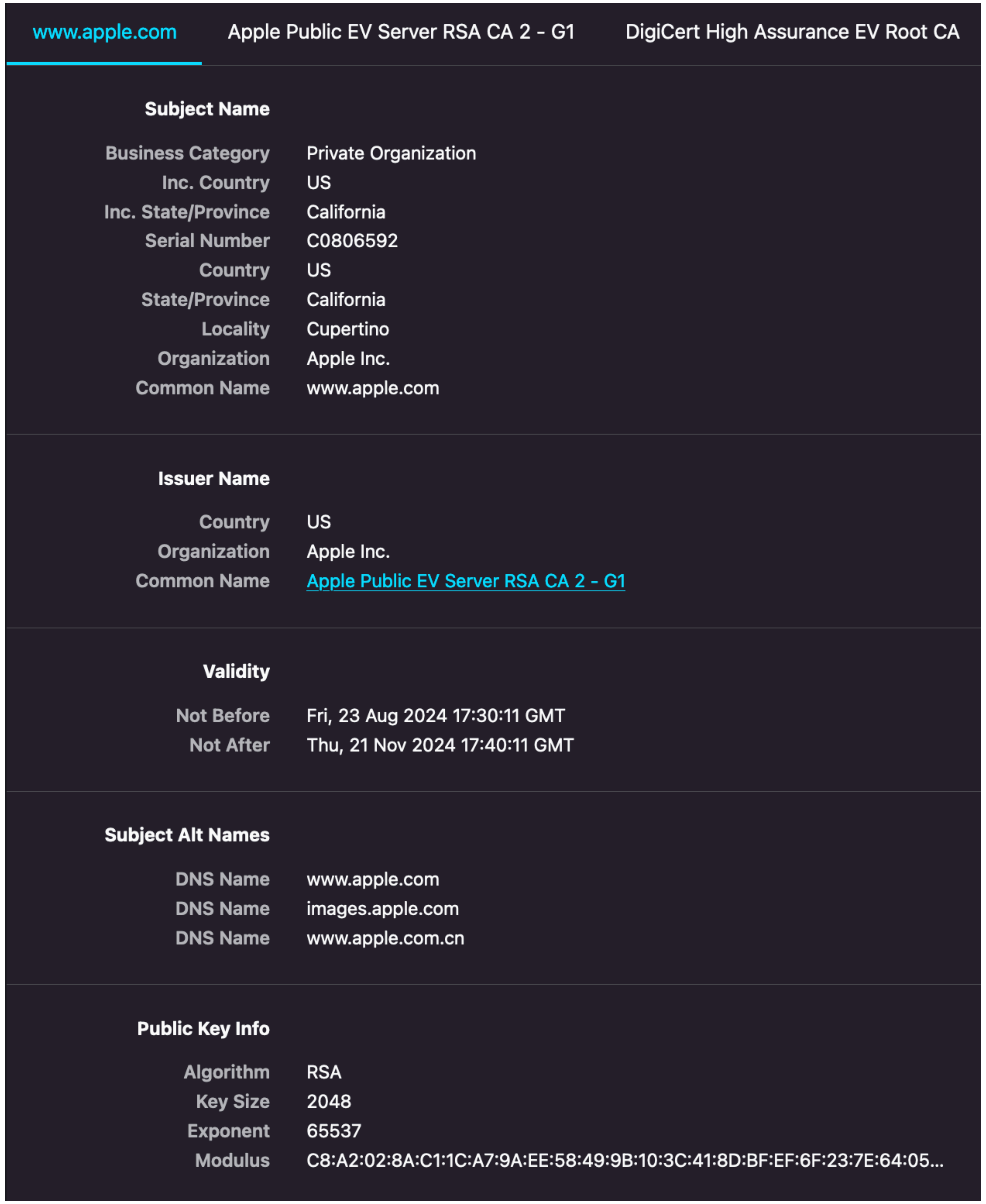
$(s = (s_1, s_2), t) \leftarrow f^{-1}(c)$

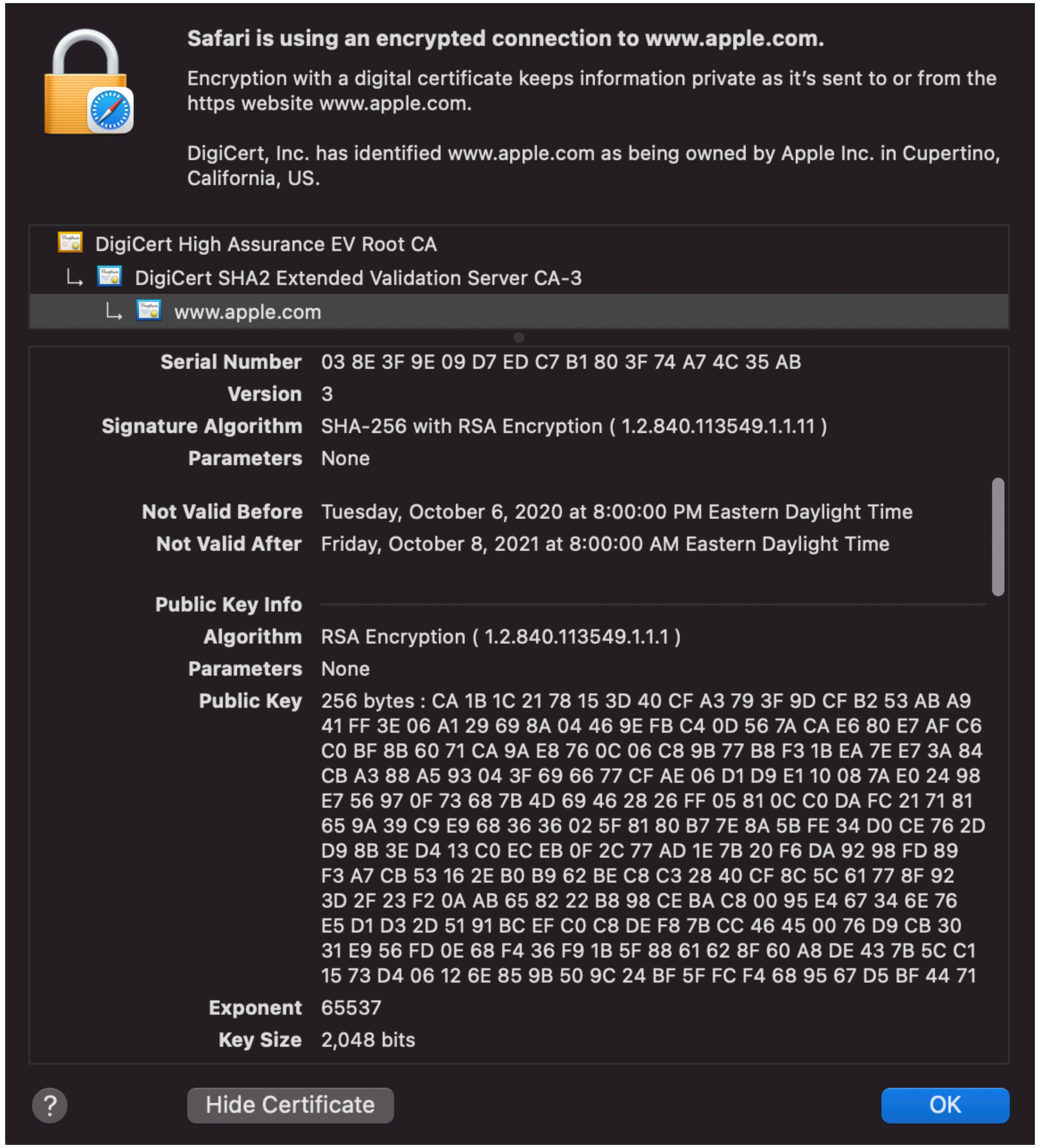
$r \leftarrow R_3(s) \oplus t$

$m \leftarrow R_1(r) \oplus s_1$

$R_2(r||m) \stackrel{?}{=} s_2$ OUTPUT m ELSE FAIL

Example: apple.com

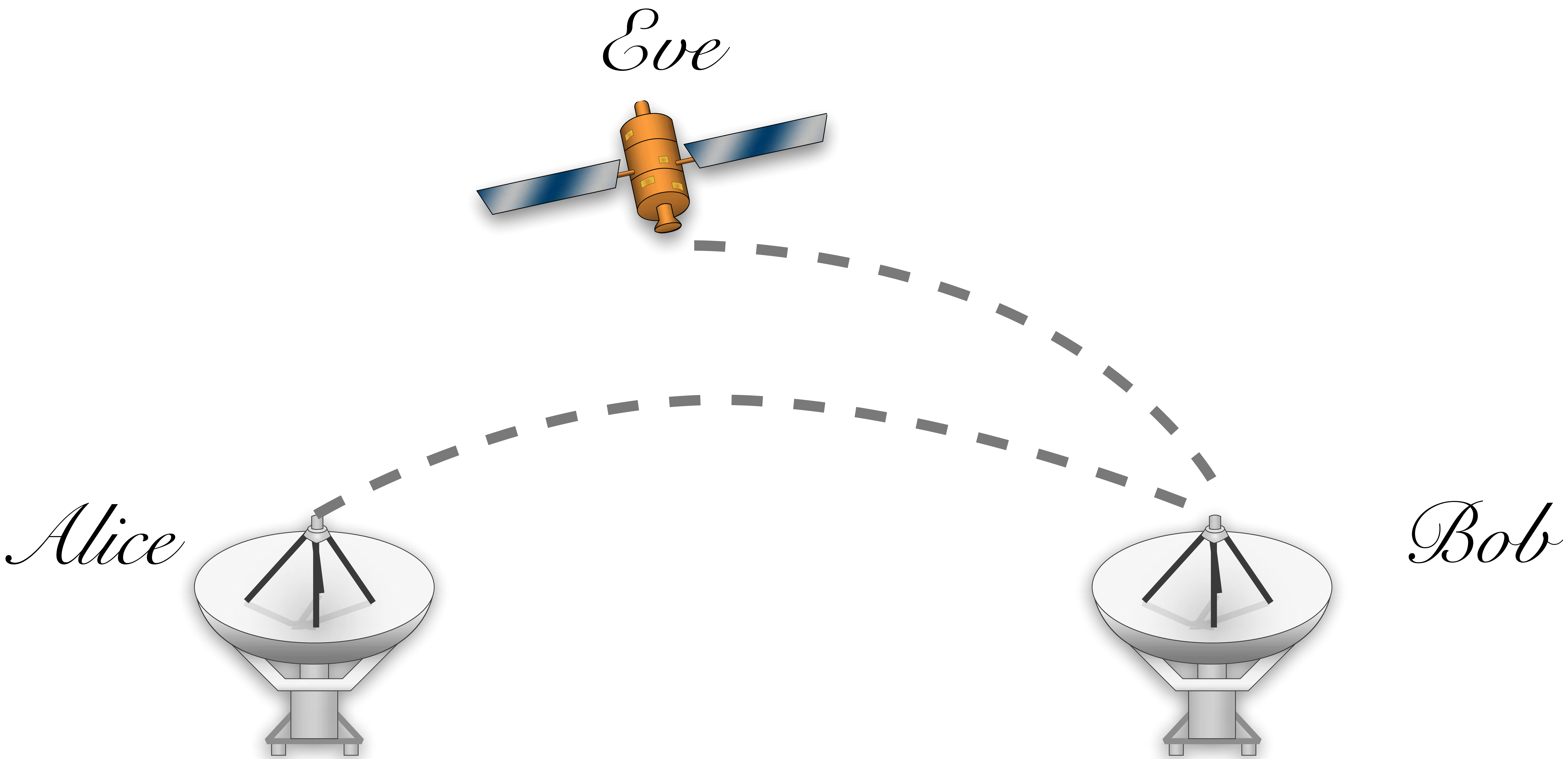




Very old problem

John Hancock

New Problem



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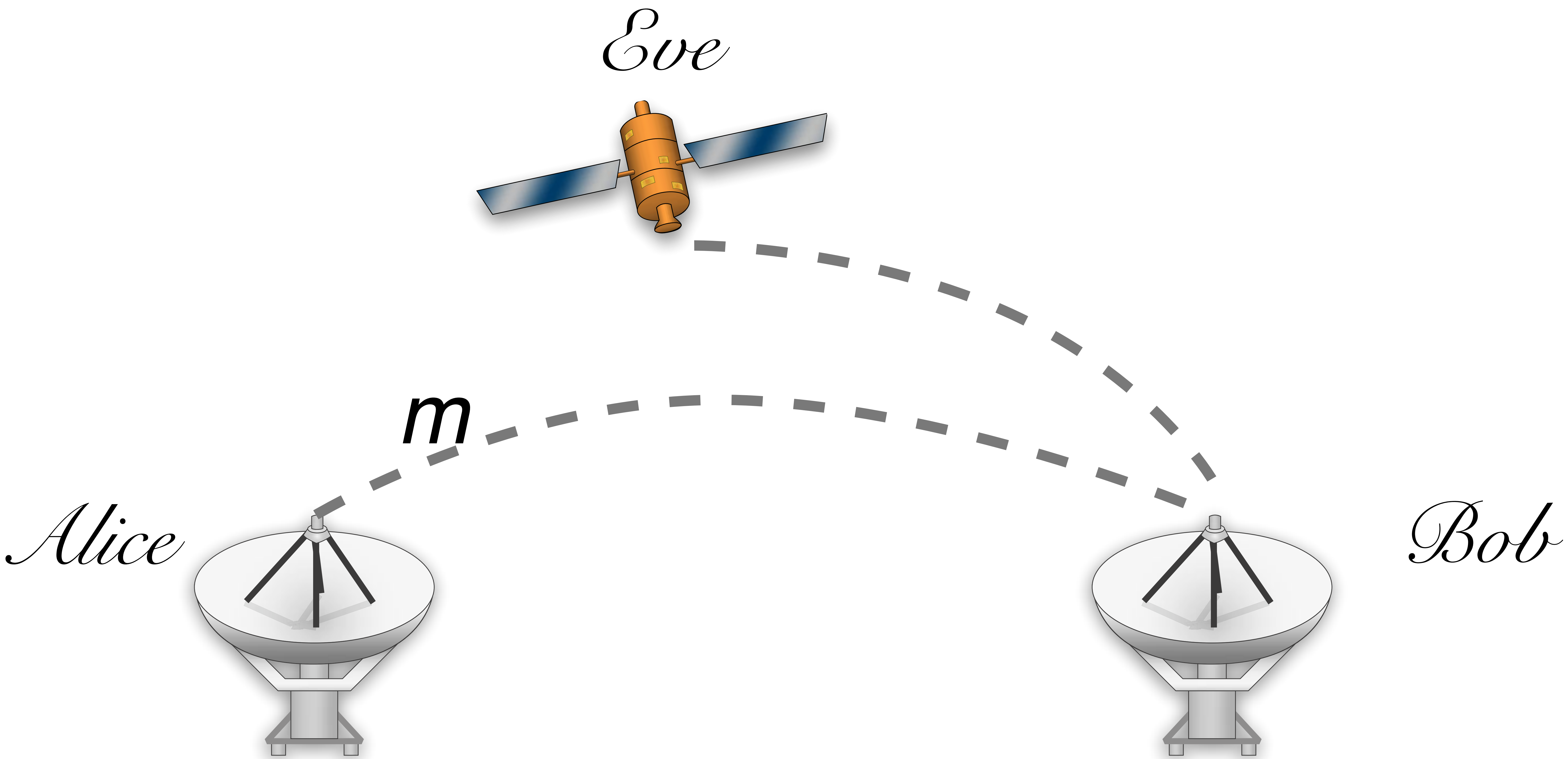

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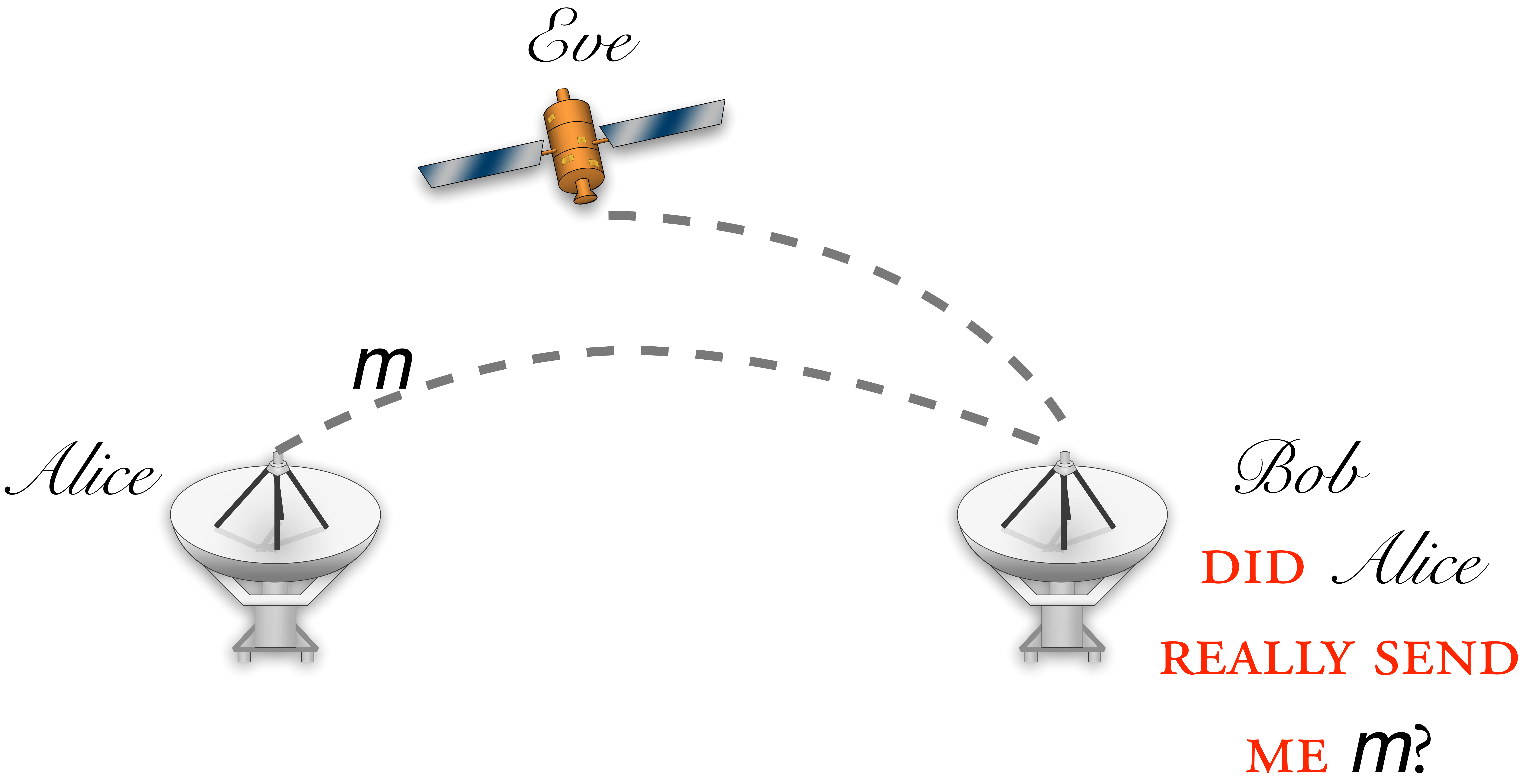
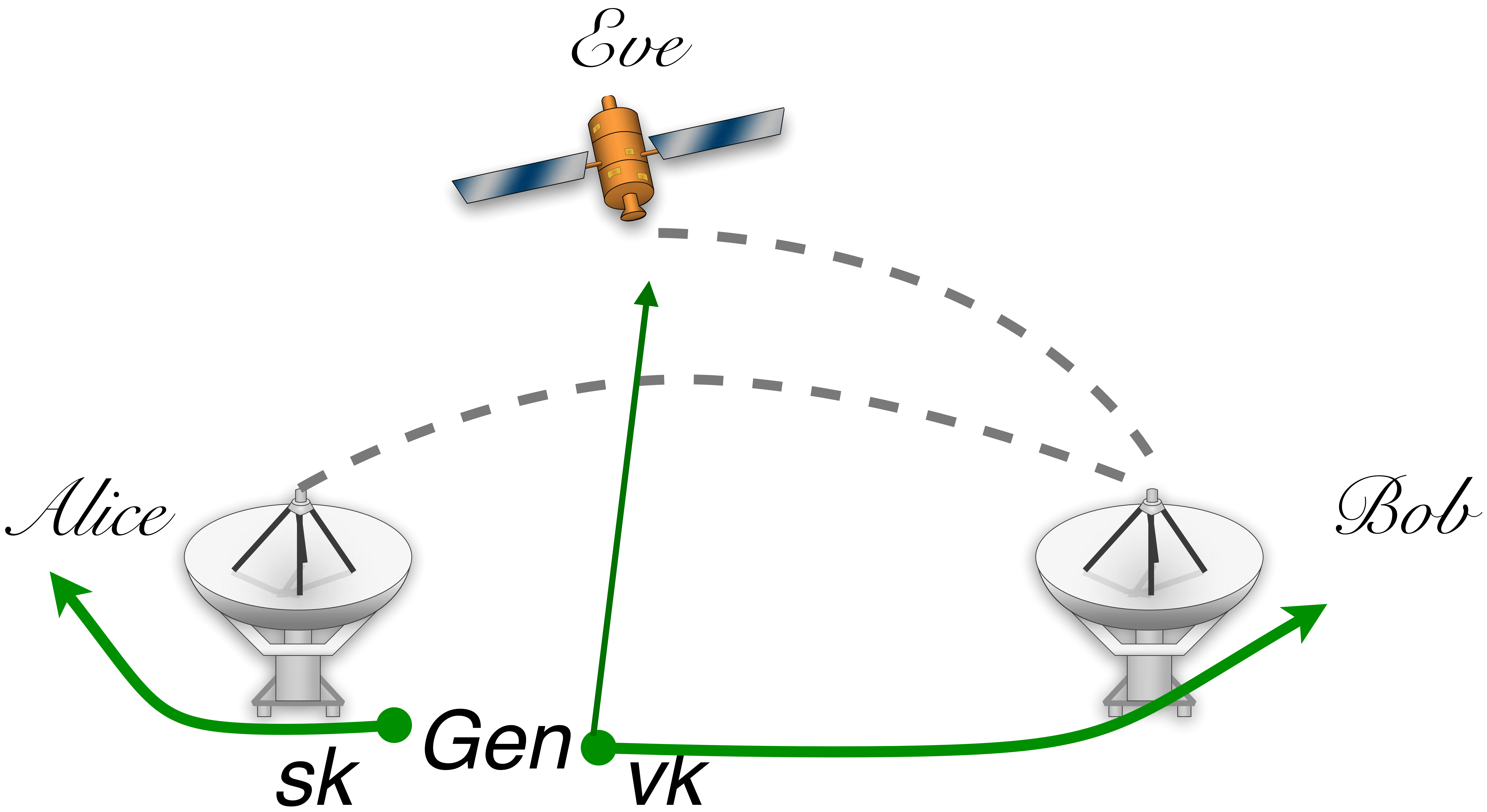


Public key digital signature



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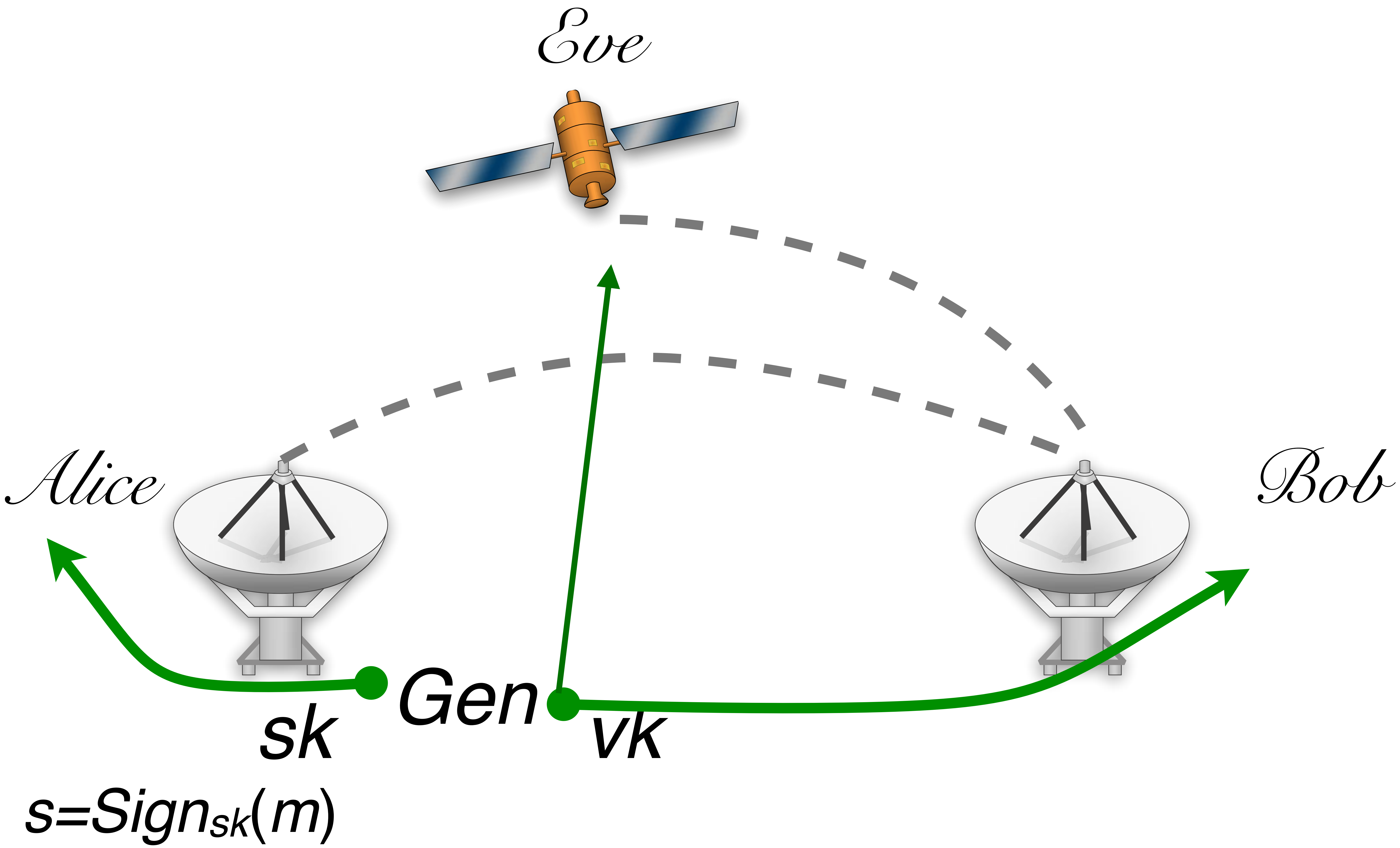

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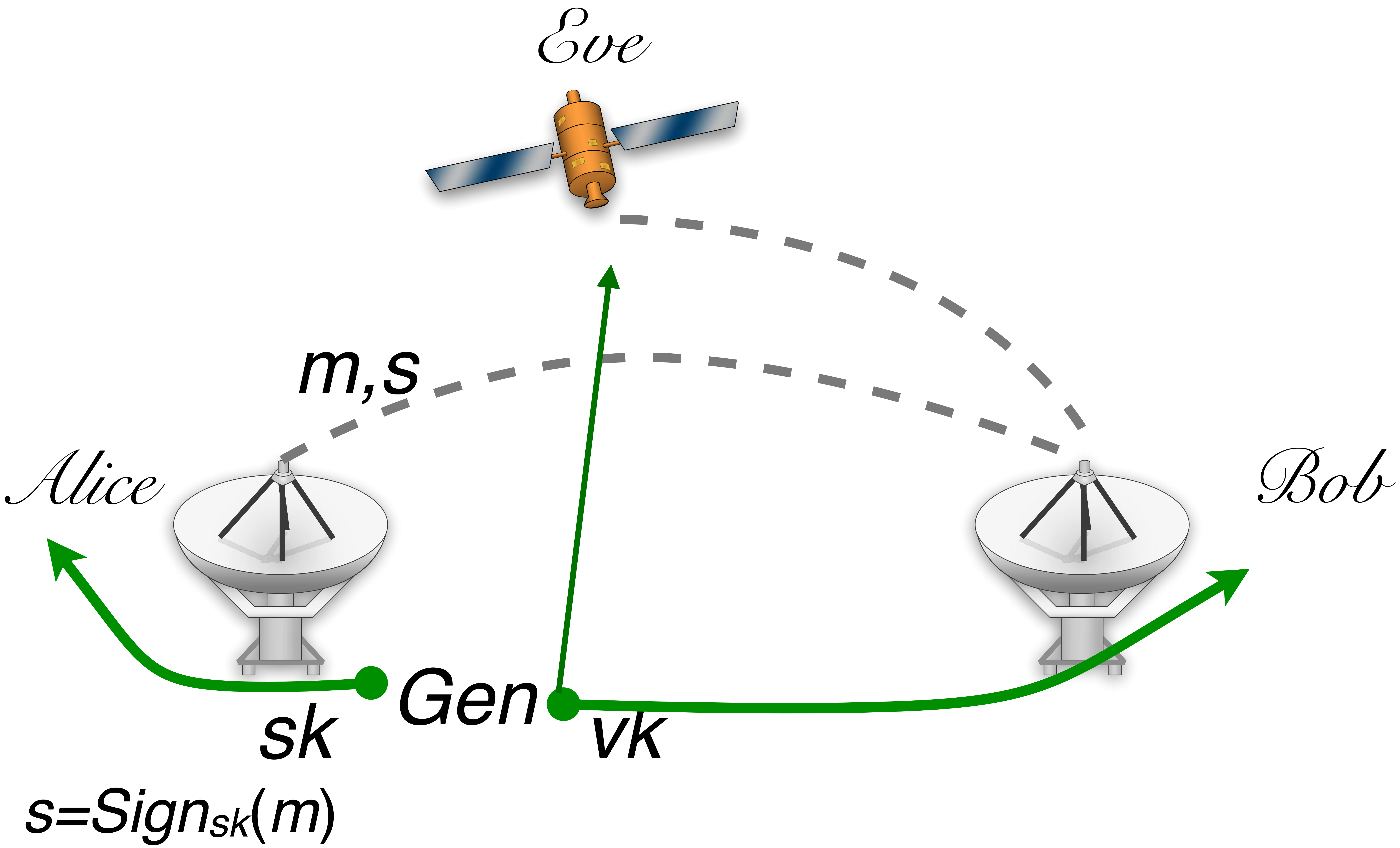

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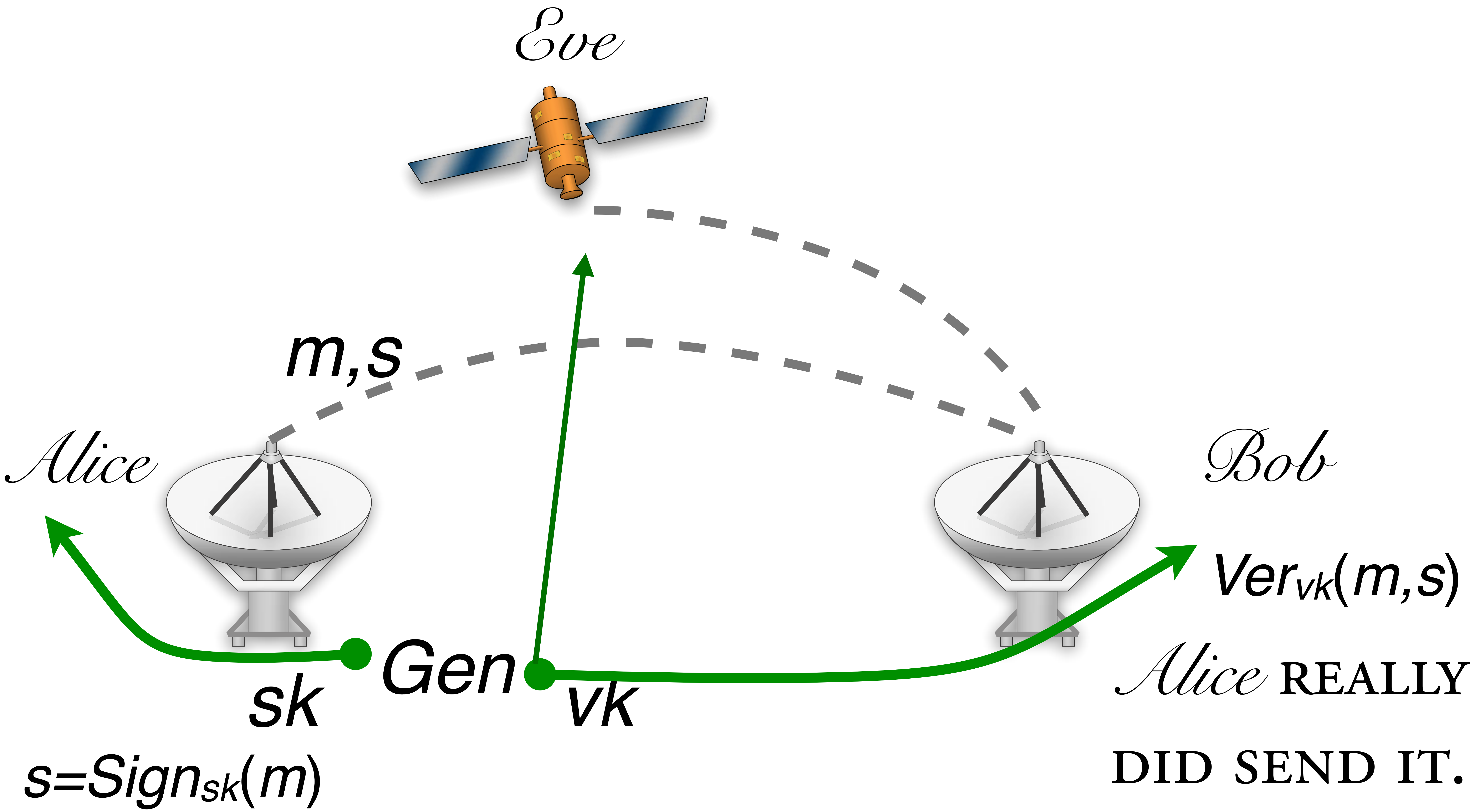


Public key digital signature

MESSAGE SPACE $\{\mathcal{M}\}_n$

$Gen(1^n)$

$Sign_{sk}(m)$

$Ver_{vk}(m,s)$

Public key digital signature

MESSAGE SPACE $\{\mathcal{M}\}_n$

$Gen(1^n)$ GENERATES A KEY PAIR sk, vk

$Sign_{sk}(m)$

$Ver_{vk}(m,s)$

Public key digital signature

MESSAGE SPACE $\{\mathcal{M}\}_n$

$Gen(1^n)$ GENERATES A KEY PAIR sk, vk

$Sign_{sk}(m)$ GENERATES A SIGNATURE s FOR $m \in \mathcal{M}_n$

$Ver_{vk}(m,s)$

Public key digital signature

MESSAGE SPACE $\{\mathcal{M}\}_n$

Gen(1^n) GENERATES A KEY PAIR *sk,vk*

Sign$_{sk}$(*m*) GENERATES A SIGNATURE *s* FOR $m \in \mathcal{M}_n$

Ver$_{vk}$(*m,s*) ACCEPTS OR REJECTS A MSG,SIG PAIR

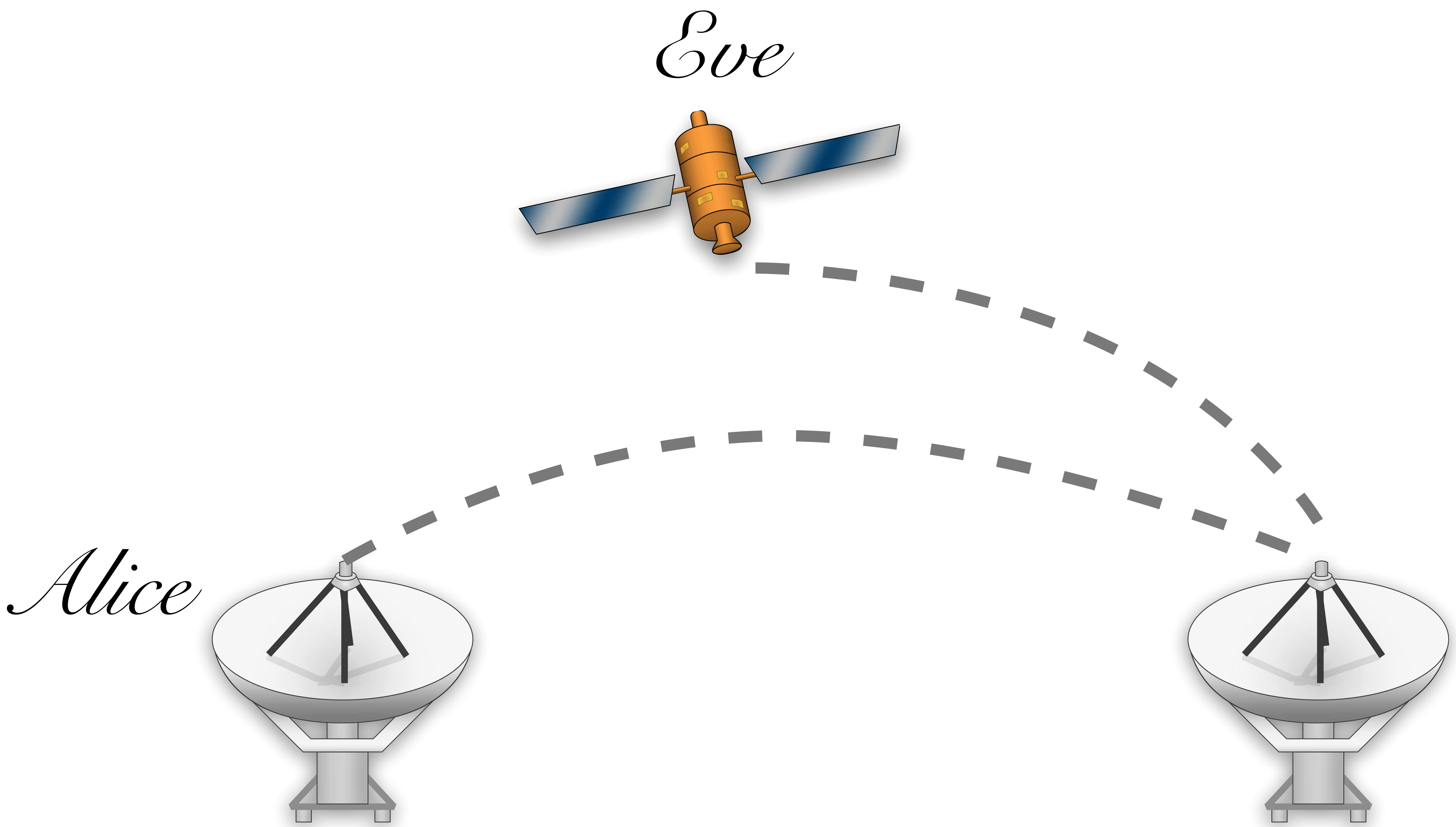
$$\Pr[k \leftarrow Gen(1^n) : Ver_{vk}(m, Sign_{sk}(m)) = 1] = 1$$

existential unforgability

“EVEN WHEN GIVEN A SIGNING ORACLE,

AN ADVERSARY CANNOT FORGE A SIGNATURE FOR

ANY MESSAGE OF ITS CHOOSING”



existential unforgability

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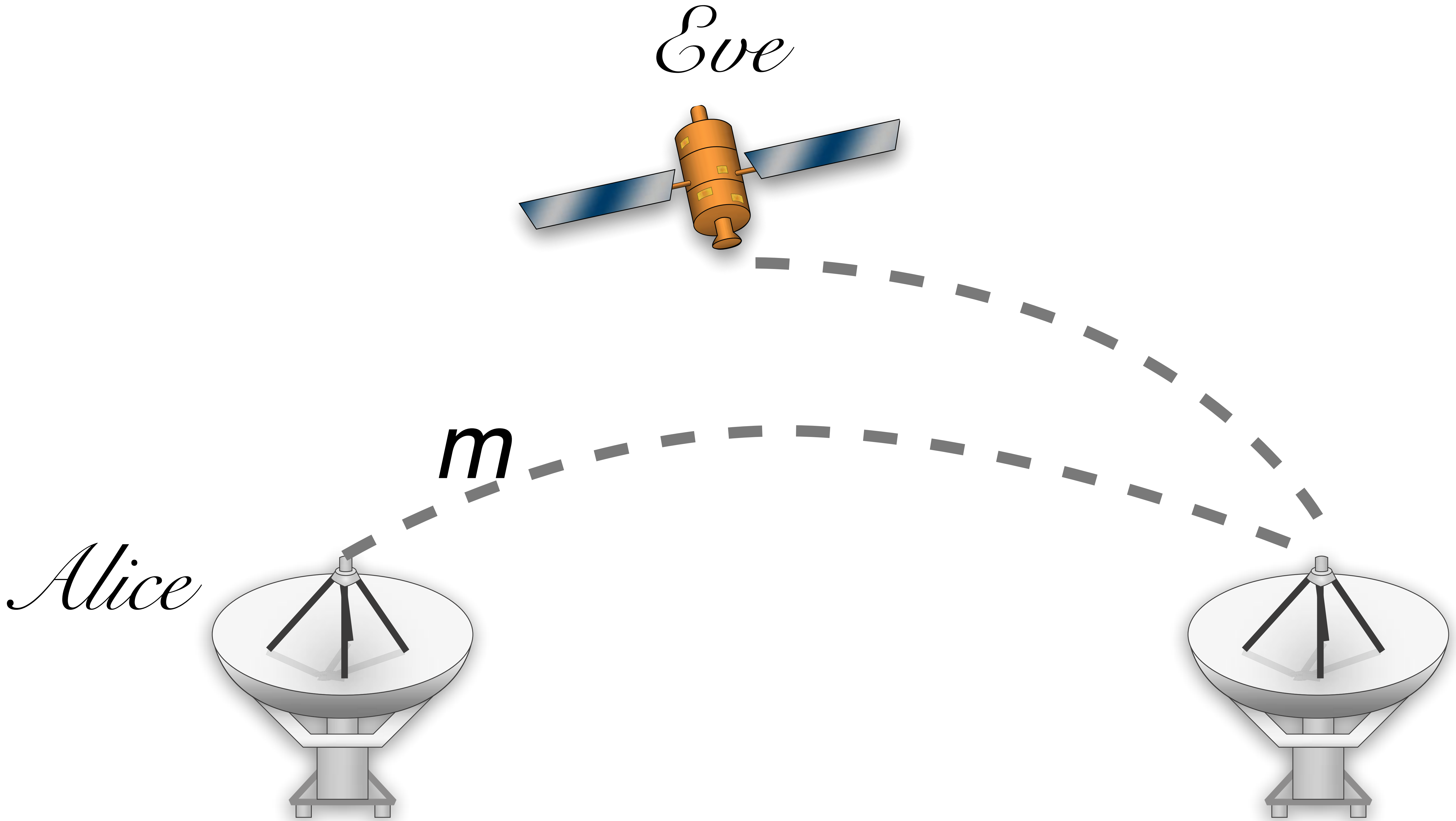
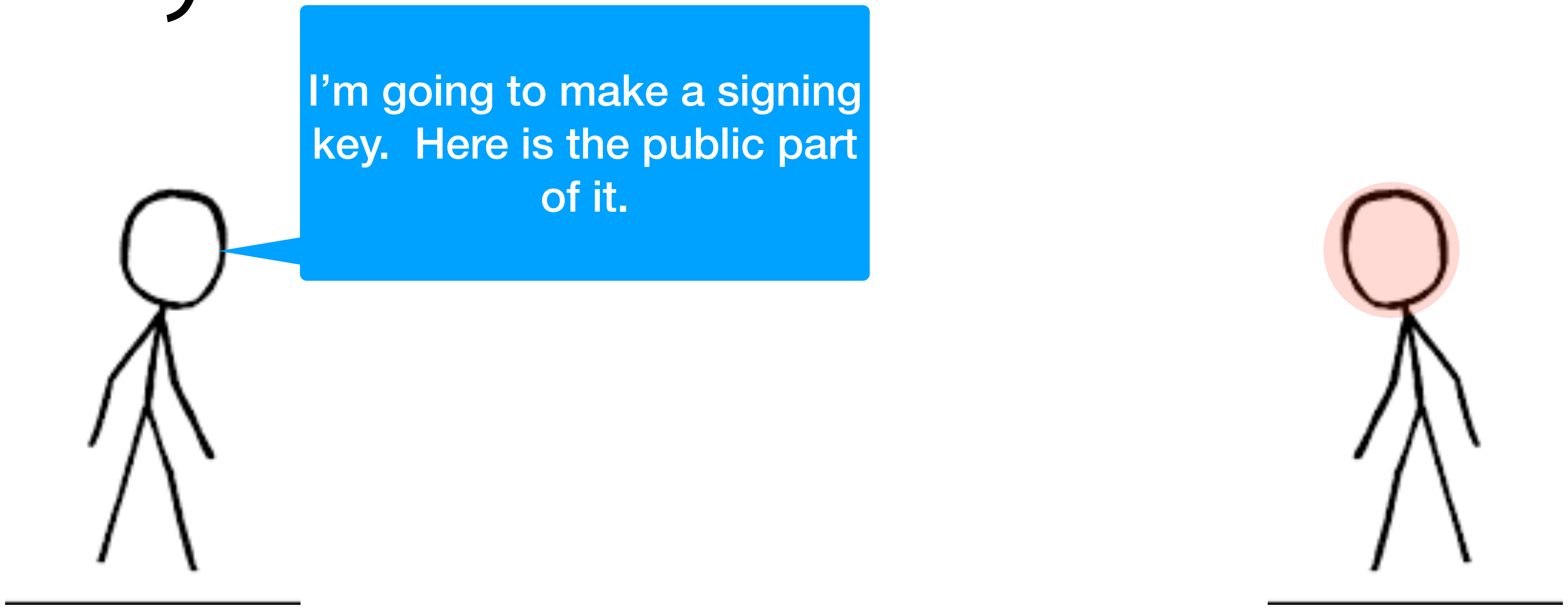


Signature security



$$(vk, sk) \leftarrow \text{Gen}(1^n)$$

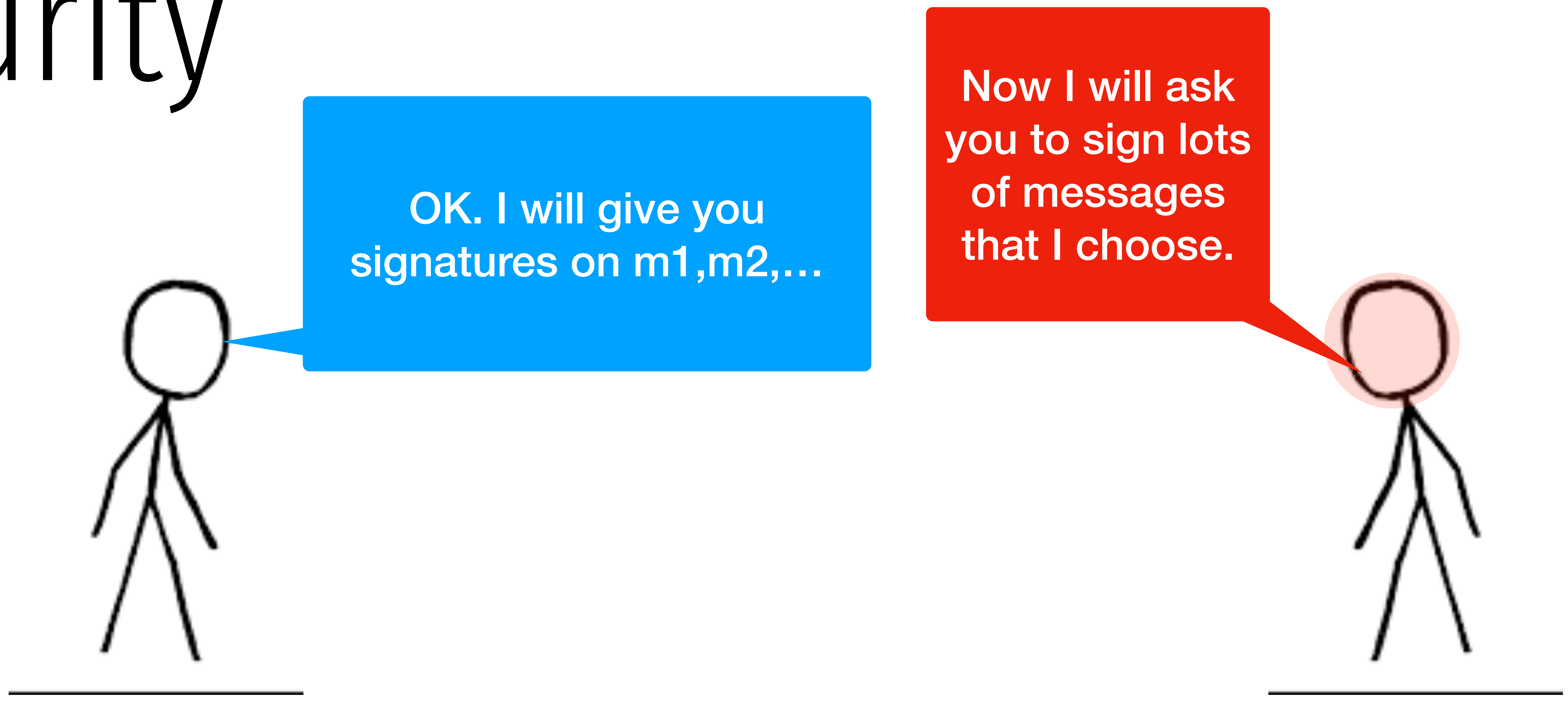
Signature security

$(vk, sk) \leftarrow \mathrm{Gen}(1^n)$

$m_0, m_1, \ldots$

vk

Signature security

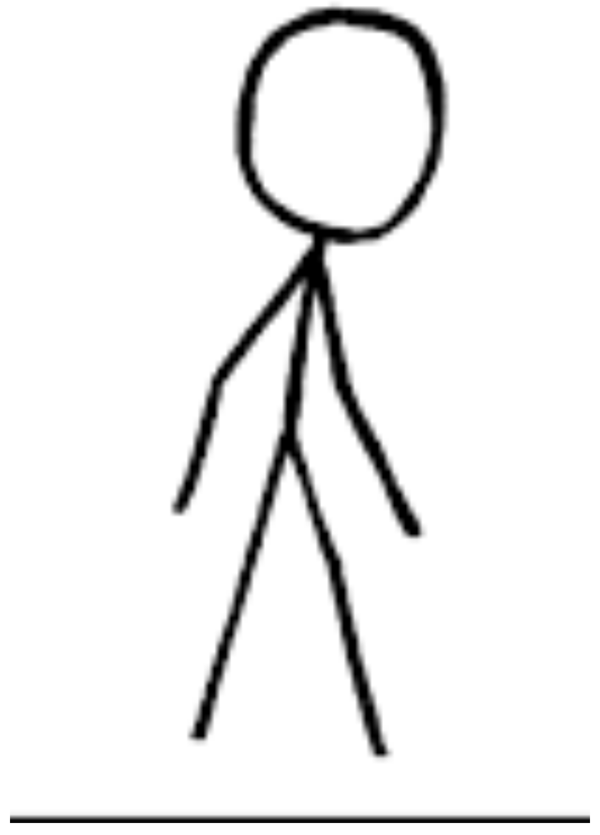


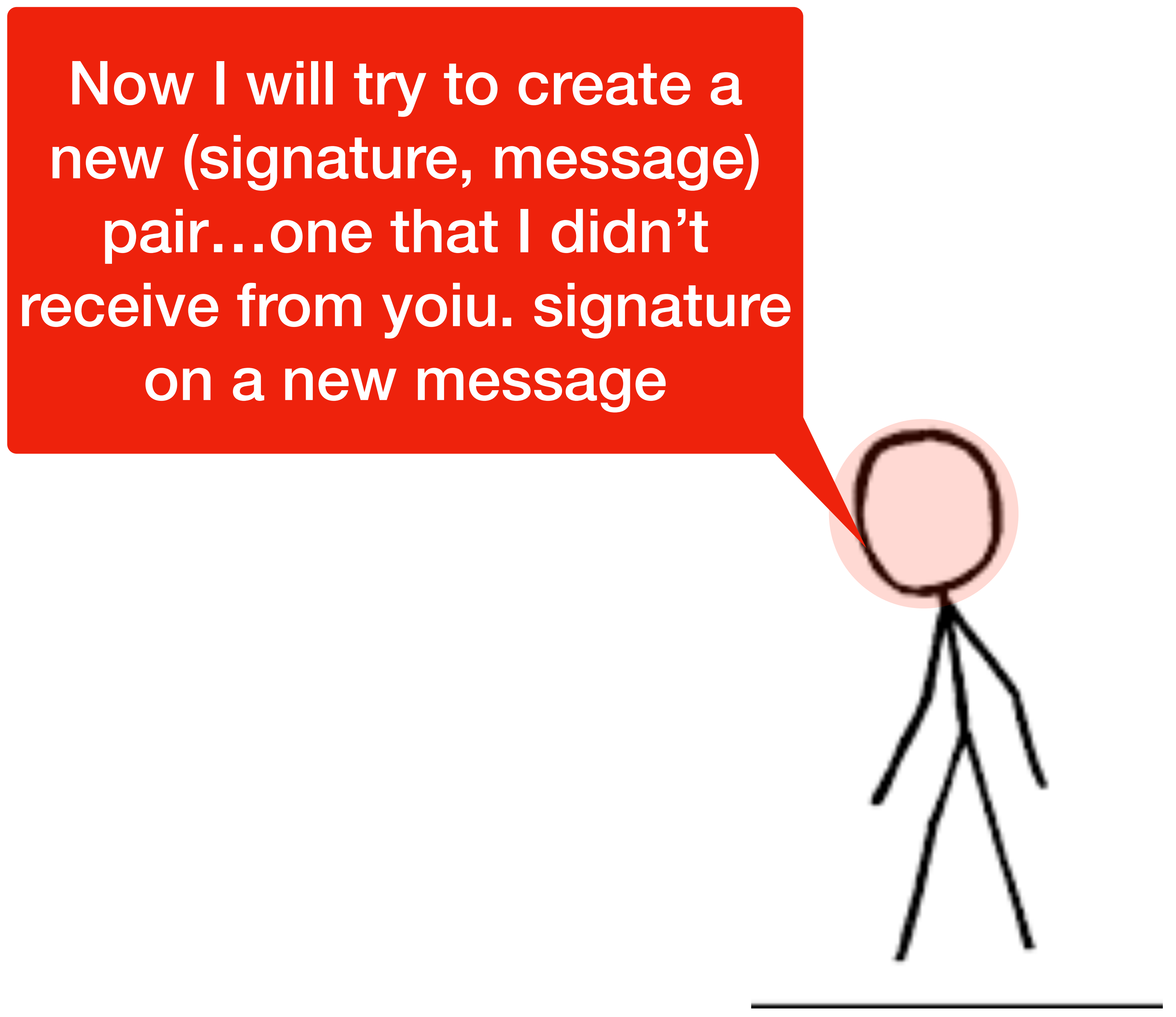
$$(vk, sk) \leftarrow \mathrm{Gen}(1^n)$$

$$vk$$

$$s_i \leftarrow \mathsf{Sign}_{sk}(m_i)$$

Signature security





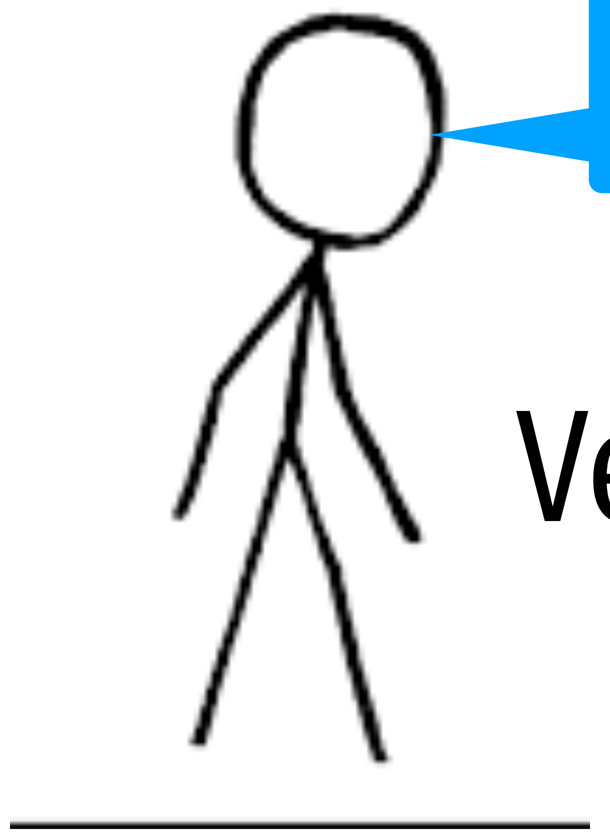
$(vk, sk) \leftarrow \text{Gen}(1^n)$

vk

$s_i \leftarrow \text{Sign}_{sk}(m_i)$

$s_1, s_2, \ldots$

Signature security



$$\mathsf{Ver}_{vk}(m^*, s^*) \stackrel{?}{=} 1$$

Textbook RSA Signatures (insecure)

Pick N = p*q where p,q are primes.

Pick e,d such that $e \cdot d = 1 \bmod \phi(N)$

Sign((sk=d, N) m):

Compute the signature: $\sigma \leftarrow m^d \bmod N$

Verify((pk=e, N), σ, m):

$$m \stackrel{?}{=} \sigma^e \bmod N$$

RSA Signatures in GPG

Sign((sk, N) m):

Compute the padding: $z \leftarrow 00 \cdot 01 \cdot FF \cdots FF \cdot 00 \cdot \mathsf{ID}_H \cdot H(m)$

Compute the signature: $\sigma \leftarrow z^{sk} \bmod N$